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Promoting conceptual change regarding infinity in high school mathematics teachers through a workshop

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Abstract

This report delineates the outcomes of an intervention conducted with in-service high school educators, focusing on elucidating three distinct scenarios within geometric and arithmetic domains: the infinitely large, infinitely numerous, and infinitesimally close. Grounded in the theoretical framework of conceptual change, it is posited that when an individual exhibits entrenched conceptions, it signifies a misclassification of the pertinent concept, necessitating a categorical shift to effectuate a transformation in their cognitive schema, particularly concerning the notion of infinity. Thus, the principal objective of this investigation was to ameliorate the entrenched conceptions held by educators pertaining to infinity through a workshop-based intervention. Preceding the workshop, educators predominantly exhibited conceptions aligned with natural and potential infinities. However, after the workshop, a discernible transition was observed, with educators engendering an actual conception of infinity or an omega-epsilon position, exemplified by their acceptance of equivalences such as 0.999…=1 and the parity in the cardinality of sets comprising natural numbers, even numbers, and perfect squares. Nonetheless, notwithstanding this progress, confident educators evinced resistance to embracing the concept of actual infinity, particularly in instances such as the hypothetical scenario depicted in Hilbert's Grand Hotel. Consequently, drawing upon the framework of conceptual change theory, it can be postulated that a complete categorical shift was not universally realized among educators due to their reluctance to revise entrenched beliefs concerning natural or potential infinity.

Keywords: Conceptual Change, High School Teachers, Infinity, Workshop

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The concept of infinity is theoretically intertwined with almost all branches of mathematics (Tall, 2001). From an educational point of view, this concept has been of interest among researchers who wish to delve into how teachers teach it (Montes et al., 2014). One example of this interest is the particular volume 48 dedicated to this concept in *Educational Studies in Mathematics*. Some teachers' difficulties with the meaning of the concept of infinity are related to the fact that they are reluctant to accept equality $0.999 \ldots = 1$. Faced with the inability of an infinite process, the teacher limits himself to rounding off and accepting the previous equality (Díaz-Espinoza et al., 2023). Likewise, investigations among students and in-training teachers have revealed the existence of incomplete understandings of expressions of the type $0.999 \ldots = 1$ (Díaz-Espinoza et al., 2023; Dubinsky et al., 2005b; Juter, 2019; Kattou et al., 2010; Schwarzenberger & Tall, 1978; Vinner & Kidron, 1985; Wistedt & Martinsson, 1996;

Yopp et al., 2011). Another difficulty is reported by Krátká et al. (2021), who affirm that "when a teacher with a conception of infinity formulates certain statements during the course of teaching, a student with a different conception of infinity can attach a different meaning to them" (p. 23).

These findings underscore the importance of in-service teachers: (i) achieving a deep understanding of the concept of infinity (Date-Huxtable et al., 2018), (ii) being familiar with the stages through which students go to understand infinity, and (iii) determining how best to introduce this concept in their classes when the curriculum requires it (Holton & Symons, 2021; Montes et al., 2014). With respect to the stages mentioned by Montes et al. (2014), one recent investigation proposes four phases through which subjects pass as they seek to conceive infinity: natural, potential, omega-epsilon position, and actual (Krátká et al., 2021).

From the 1980s to the present, numerous investigations have been presented about infinity based on work with students and teachers, either in-training or in-service. Key research topics related to infinity can be classified as follows: study of paradoxes involving infinity (Dubinsky et al., 2005a; Mamolo & Zazkis, 2008; Medina Ibarra et al., 2019; Roa-Fuentes & Oktaç, 2014; Waldegg, 2005), equivalence between infinite decimal numbers (Ángeles-Navarro & Pérez-Carreras, 2010; Díaz-Espinoza et al., 2023; Eisenmann, 2008; Hannula et al., 2006; Mena-Lorca et al., 2015; Wistedt & Martinsson, 1996; Yopp et al., 2011), problems of series and sequences to infinity (Kidron & Tall, 2015; Monaghan, 2001; Schwarzenberger & Tall, 1978), comparison of infinite sets (Homaeinejad et al., 2021; Kattou et al., 2010; Singer & Voica, 2003; Tsamir, 1999), geometric problems involving infinity (Cihlář et al., 2009; Fischbein et al., 1979; Krátká, 2013; Tall, 1980; Tall & Tirosh, 2001), infinity in differential and integral calculus (Villabona Millán et al., 2022; Wijeratne & Zazkis, 2015), and combinations of these (Belmonte & Sierra, 2011; Cihlář et al., 2015; Juter, 2019; Krátká et al., 2021; Manfreda Kolar & Čadež, 2012; Moreno Armella & Waldegg, 1991; Zippin, 1962).

Although there is a large corpus of literature on conceptions of infinity among students and teachers, the general aim of the present investigation is to analyze a process of conceptual change of the conceptions present in in-service high school teachers and to link this objective to the following research question: *In what way can a workshop promote conceptual changes in the misconceptions present among in-service high school mathematics teachers concerning the concept of infinity?* Additionally, for this research, the specific aims are:

1. Identify the conceptions that in-service mathematics teachers have about the concept of infinity.
2. Analyze the conceptions that in-service mathematics teachers have about the concept of infinity.
3. Categorize the conceptions of in-service mathematics teachers about infinity based on the natural, potential, omega-epsilon position, and actual infinity.
4. Test a workshop as an instruction to foster conceptual change in in-service mathematics teachers regarding the concept of infinity.

The study by Kattou et al. (2010) found that close to 42% of teachers believe that the equality $0.333\ldots = \frac{1}{3}$ is correct, while approximately 5% of those same teachers accept the equality $0.999\ldots = 1$ as correct (p. 1777). This can be explained as "it may be easier to accept, for example that $\frac{1}{3}$ and $0.333\ldots$ are equal since neither one is an integer, and one representation does not seem to be smaller than the other, as $0.999\ldots$ does when compared to 1" (Juter, 2019, p. 84). In other words, "the fact that the representations $0.999\ldots$ and 1 differ so much in appearance may add to the cognitive conflict" (Juter, 2019, pp. 83-84). As Yopp et al. (2011) mention, the idea that "they believe that one is a whole, can be expressed in only one way, and its value is explicitly tied to notions of units" (p. 310) makes it even more difficult to overcome the conception of potential infinity through actual infinity.

Likewise, Juter (2019) found that students need to consider that $0.999\ldots$ and 1 are representations of the same number. In contrast, a minority of the students in that study disagreed that $0.999\ldots$ and 1 are different numbers, a finding that revealed inconsistent beliefs among the other students, since almost all the ones stated that there are no numbers between $0.999\ldots$ and 1 and that these numbers are different. Moreover, these same students stated that there are numbers between any two different numbers, thus contradicting their arguments. Therefore, the discrete nature of natural numbers of conflicts with new representations for numbers of this kind, that is, misconceptions about the existence of unique representations for numbers emerge in students.

In addition, in the research by Cihlář et al. (2015), where they conducted interviews with 20 university students, when asked about the equality $0.999\ldots = 1$, they reported that only two students (10% of respondents) have an actual conception of infinity, responding that the numbers are the same. On the other hand, Krátká et al. (2021) concluded that the concept of natural infinity disappears with age, and the concept of actual infinity is rarely present in young students (approximately 15 years of age). However, the works by Díaz-Espinoza et al. (2023) and Díaz-Espinoza and Juárez-López (2023) show that naturalistic conceptions do not necessarily disappear in teachers, perhaps due to the tendency of misconceptions to resist disappearing or the dearth of situations where the concept of infinity emerges in the classroom. In another investigation about the infinite sum of functions, 28% of students had a conception of infinity in the omega-epsilon position; that is, "the infinite sum of functions is perceived as a legitimate object but not clearly as the formal limit definition" (Kidron & Tall, 2015, p. 194).

In the present study, a 'misconception' is understood as the kind of conception "that occurs as a result of students' repeated experiences with phenomena of their everyday world" (von Aufschnaiter & Rogge, 2010, p. 12). In other words, the abuse of everyday references when communicating an idea to students creates misconceptions. For example, when the teacher tells students that *infinity is a very large number that cannot be represented with digits and uses the symbol* ∞, or when working with limits $\lim_{x\to\infty} \frac{1}{x}$, it's said to be zero, because it's like dividing one by a very large number, resulting in *almost zero*. Abusing these ways of conceiving infinity leads to misconceptions in students' understanding in the future. So, "if one wishes to ensure that high school students and others achieve a proper understanding of infinity, it is necessary to modify the strategies used and explicitly consider that teachers may also encounter difficulties regarding that concept" (Mena-Lorca et al., 2015, p. 353). On this topic, Smith III et al. (1994) argue that misconceptions need to be reconceived as flawed extensions of productive knowledge, insisting that they are not always resistant to change. In this sense, the theory of conceptual change claims that subjects have knowledge that conflicts with correct knowledge; therefore, learning consists of changing the subject's conceptualization to the correct one.

According to Carey (1991), conceptual change requires reassigning a concept to a distinct ontological category or creating new ontological categories. Chi (2008), meanwhile, affirms that "categorizing is the process of identifying or assigning a concept to a category to which it belongs" (p. 62). Thus, categorization is an important learning mechanism because, once categorized, a concept can inherit features from its parent category. Chi (2008) proceeds to distinguish three types of conceptual change: *revision of beliefs*, when previous knowledge conflicts with new information if the conflict occurs in a single idea; *mental model transformation*, where the subject's mental model conflicts with the correct model when it fails; and *categorical shift*, when false beliefs or faulty mental models show resistance because they have been miscategorized.

Previous studies like that of Díaz-Espinoza et al. (2023) show that teachers exhibit a '*rounding effect*' in accepting the expression $0.999\ldots = 1$. During the interview conducted by the researchers with the teacher, he highlights that to accept the expression, '*one must round* $0.999\ldots$ *to make it equal to 1*'. The research indicates that such misconceptions were resistant to disappearing in the subject. In another subsequent study, Díaz-Espinoza and Juárez-López (2023) administer a questionnaire to nine in-service teachers pursuing a semester of Master's in Mathematics Education. They demonstrate that teachers hold resilient misconceptions of the type '*two-line segments of different lengths have a different number of points or infinities of different sizes*'. This, in turn, evidence in the subjects natural or potential conceptions of infinity. Thus, according to conceptual change theory, the concept has been miscategorized, and a categorical shift change will be necessary to alleviate the conception of infinity in the subjects.

Manfreda Kolar and Čadež (2012) observe that "potential infinity is related to an ongoing process without an end (...) [while on the other hand] (...) actual infinity attributes a finite entity to this infinite process" (p. 390). Krátká (2013) adds that "natural infinity is the simplest, (...) it is a subjective phenomenon, (...) a set can appear naturally infinite if it extends within its horizon" (p. 98). Finally, "the omega[-epsilon] position is a transitional development phase between potential and actual infinity, created predominantly utilizing primary intuition" (Cihlář et al., 2015, p. 70). Similarly, Manfreda Kolar and Čadež (2012) defined three types of situations in which the infinite emerges. These are related to the subjects' perception of the infinite:

> *The concept of 'infinitely large' [view into the distance according to Krátká et al. (2021)] (...) in the sense of expansion, (...) 'infinitely many' (...) closely related to the density of elements within a bounded set. (...) [and] 'infinitely close' [view into the depth according to Krátká et al. (2021)] (...) [with] approaching a given object as close as possible (p. 399).*

METHODS

As mentioned above, this paper reports on an intervention study: a workshop for high school mathematics teachers designed to develop a conceptual change regarding the concept of infinity. Chi (2008), Mena-Lorca et al. (2015), and Vosniadou et al. (2008) mention that for an appropriate construction of the infinite, situations must be considered in which subjects confront their limitations, notions, and contradictions of their own conceptions with the new information to achieve a conceptual change regarding the concept of the infinite. In this regard, Roa-Fuentes and Oktaç (2014) reached the conclusion that realistic ideas persist even after students receive instruction. Similarly, Mamolo and Zazkis (2008) warn that students may not notice conflicting ideas between potential and actual infinity. This suggests that resistance to change may be due to an incorrect categorization of infinity among these subjects. In light of the foregoing, the situations to be addressed in our study are designed at a level of categorical shift.

The workshop consisted of nine sessions (three for each type of situation where infinity appears), each with a four-hour duration. Some authors mention that individuals "(…) need experiences that allow them to develop rich images of the topic, which will function as the basis for a formalization at a later stage" (Hannula et al., 2006, p. 335). The times for each session for each specific situation of the workshop are shown in Table 1. Participants were provided with the material necessary to work in each session. The situations considered for the workshop were taken and modified from various research studies that formed the literature review. For practical purposes and to simplify, the titles of the situations and the references from where they were taken are provided. To exemplify this, two situations of the infinitely large type are shown in the Appendix. A pre-test and post-test were applied to visualize the workshop's impact as an intervention for conceptual change among in-service math teachers.

Nine sessions were designed in which infinity emerged: three for infinitely large, three for infinitely many, and three for infinitely close. Three of each type of conception have been considered following the sequence: an opening, a development, and a closing session. They were considered in that order since, according to the report by Manfreda Kolar and Čadež (2012), "respondents had the most success with tasks of the infinitely large type […] followed by tasks of the infinitely many types and, finally, the tasks of the infinitely close type" (p. 407). In the design of the situations to be addressed, the theory of conceptual change was similarly taken into account (Chi, 2008; Vosniadou et al., 2008) with the purpose of leading the teachers through each phase of the conception of infinity.

Finally, the choice of the number of sessions and their duration is associated with the theory of conceptual change:

> *An important limitation of this type of instruction is the assumption that conceptual change is something that can happen in a short period of time and involves a rational process of concept replacement. (...) On the contrary, (...) the process of conceptual change is a gradual and continuous process that involves many interrelated pieces of knowledge and requires a long time to be achieved. (Vosniadou et al., 2008, p. 26)*

Table 1. Distribution of specific content to be addressed during the workshop

	Session	Specific content	Reference	Time
Infinitely large	1	What definition of infinity does the professor have in mind?	(Homaeinejad et al., 2021)	30 min
		Distinguish between finite and infinite sets	(Belmonte & Sierra, 2011)	50 min
		Given the set of natural numbers. What is the largest number?	(Cihlář et al., 2015)	50 min
		Given the pictorial representation of two non-parallel straight lines, do they intersect at any point?	(Cihlář et al., 2009)	50 min
		Given a straight line b and a point A outside it, draw the longest possible segment with point B on the line	(Cihlář et al., 2015)	50 min
	2	Given the sets of natural numbers and perfect squares, analyze if one considers a set larger than the other	(Juter, 2019)	60 min
		Establish a bijective relationship between natural numbers and perfect squares to compare infinite sets	(Moreno Armella & Waldegg, 1991)	60 min
		Given a straight line p and two points A and B outside it opposite each other by the line. Is it possible to join them?	(Cihlář et al., 2009)	60 min
	3	Given different infinite sets, determine if they can be compared as done in the previous activity	(Homaeinejad et al., 2021)	60 min
		Given two infinite sets, analyze that they are infinite, but their elements cannot be compared through a bijective relationship	(Kattou et al., 2010)	50 min
		Given two parallel half-lines, where one pictorial representation begins before the other, is one greater?	(Krátká, 2013)	50 min

		Hilbert's Grand Hotel Paradox	(Roa-Fuentes & Oktaç, 2014)	60 min
Infinitely many	4	Given a closed interval $[a, b]$, how many numbers does it contain?	(Juter, 2019)	50 min
		Given a line segment, how many points does it contain?	(Manfreda Kolar & Čadež, 2012)	60 min
		Offer a justification of the infinity of points in a line segment from partitions of a segment	(Fischbein et al., 1979)	90 min
	5	Given a closed intervals $[0,1]$ and $[0,10]$, analyze the responses to the question: Are there more points in any?	(Moreno Armella & Waldegg, 1991)	50 min
		Given a closed intervals $[0,1]$ and $[0,2]$, find a bijective relationship to compare the sets	(Moreno Armella & Waldegg, 1991)	60 min
		Offer a geometric representation of the bijective relationship given two-line segments of different lengths	(Moreno Armella & Waldegg, 1991)	60 min
		Generalize the bijective relationship given two arbitrary line segments	(Fischbein et al., 1979)	60 min
	6	Identify recursive processes from partitions in a unit square	(Eisenmann, 2008)	60 min
		Identify from partitions in a unit equilateral triangle the infinite sum of areas as a representation of the total area of the unit triangle	(Fischbein et al., 1979)	60 min
		Given a circumference c with center at O and a straight line p outside the circumference, is there a line for each point X' on the circumference passing through some point X on the line p?	(Zippin, 1962)	90 min
Infinitely close	7	Systematically construct a sequence of decimal numbers, each with a longer period than the previous one	(Zippin, 1962)	40 min
		Analyze the expression $\frac{2}{0}$	(Cihlář et al., 2015)	40 min
		Analyze the closeness to zero of decreasing sequences	(Cihlář et al., 2015)	90 min
		Analyze the expression $0.\overline{3} = \frac{1}{3}$	(Kattou et al., 2010)	60 min
	8	Analyze the expression $0.\overline{9} = 1$	(Juter, 2019)	30 min
		From the midpoint between 0.9 and 1, 0.95 and 1, 0.975 and 1, and so on, analyze the expression $0.\overline{9} = 1$		40 min
		Compare the representations $0.\overline{1}$ and $\frac{1}{9}$ to validate the expression $0.\overline{9} = 1$		40 min
		Construct a decreasing sequence converging to 1 from unit equilateral triangles	(Zippin, 1962)	60 min
		Given a unit square, find point X on side BC that triangle ABX has the smallest possible area	(Cihlář et al., 2015)	60 min
	9	Is there a rectangle of unit area such that one of its sides is zero?	(Zippin, 1962)	30 min
		Identify recursive processes from partitions in a unit square converging to the total area of the square	(Eisenmann, 2008)	60 min
		Tennis Ball Paradox	(Roa-Fuentes & Oktaç, 2014)	90 min

Participants

The sample consisted of 15 in-service high school math teachers (6 men and 9 women) from both the public (5/15) and private (10/15) educational systems in the state of Tlaxcala, Mexico. Some of the participants (4/15) were also working in secondary school. The following criteria were taken into account to select the participants: 1) at least one year of work as a high school teacher; 2) a background in mathematics (pure or applied) or in related careers in engineering; 3) most of their class hours spent on math subjects (e.g., algebra or differential and integral calculus); and 4) employed in educational institutions or residing in Tlaxcala.

Instruments

First, for the pre-test and post-test, five items were considered (in arithmetic and geometric contexts) from the instrument shown in Krátká et al. (2021), modified so that teachers could argue their responses openly. For example, items 1 and 3, which posed situations of the infinitely large type, and the question to be answered was, *what is the largest number?*, have been analyzed in other investigations (Belmonte & Sierra, 2011; Cihlář et al., 2015; Hannula et al., 2006; Homaeinejad et al., 2021; Manfreda Kolar & Čadež, 2012). In Hannula et al. (2006), a high percentage (>50%) of students between the ages of 11 and 14 responded with a large finite number (999 999 999 999 999 999). In the words of Krátká et al. (2021), this revealed a conception of natural infinity. In contrast, Belmonte and Sierra's study (2011) found that a smaller percentage (just over 5%) of students aged 11 to 13 responded, for example, in the following way: the number of grains of sand on Earth and the number of cells in the human body cannot be ordered because they are infinite. This, once again, shows a conception of natural infinity. These percentages are also reflected in Manfreda Kolar and Čadež (2012) and Homainejad et al. (2021).

Items 2 and 5, which posed infinitely large and infinitely close situations, and the questions to be answered were, *what is the last point on a straight line?*, and *what is the smallest triangle that can be built?*, respectively, have been used previously with the same study objective (Belmonte & Sierra, 2011; Cihlář et al., 2009, 2015; Fischbein et al., 1979; Manfreda Kolar & Čadež, 2012). For example, item 2 appears in Cihlář et al. (2009), where it was analyzed qualitatively with the goal of overcoming the obstacles between the image displayed of a straight line and the abstract concept in the context of an interview. The authors concluded that students overcome this obstacle, realizing that no such point item 5 appeared in the study by Manfreda Kolar and Čadež (2012), who observed that a very low percentage of in-training teachers (only 2%) note that this point does not exist, while a much higher percentage (54%) draw such a point above the straight line to form the triangle.

Finally, item 4 posed a situation of the infinitely close type, where the question to be answered was, *what is the smallest number?* Has been explored in Belmonte and Sierra (2011), Cihlář et al. (2015), Hannula et al. (2006), and Monaghan (2001). Cihlář et al. (2015), for example, observed that less than a third of students aged 12 to 18 could overcome the conception of natural infinity. On average, only 6% of students surpassed the conception of natural infinity to locate the actual infinity. Belmonte and Sierra's (2011) observations showed that approximately 30% of students aged 12 to 16 reflected a conception of infinite potential. Similarly, in Monaghan (2001), approximately 50% of students conceived of infinite potential. Second, given the extensive study and results obtained on similar items in other investigations, the authors could identify, categorize, and analyze in depth the teachers' responses in this study for each item, type of situation, and context.

RESULTS AND DISCUSSION

For both the pre- and post-tests, teachers were allowed 50 minutes to complete their responses, which were subsequently analyzed. The pre-test was applied to all the teachers before the workshop intervention began; however, the results of only 11 teachers are shown because some still needed to complete the intervention or were absent from one or more sessions. The post-test was applied to the same teachers at the end of the workshop after a one-week interval between the final session and the application. The results obtained were organized in relation to the type of situation and the context. In addition, they were classified according to the conception of infinity that the teachers presented. Figure 1 shows the number of teachers who presented each phase of the conception of infinity before and after the workshop.

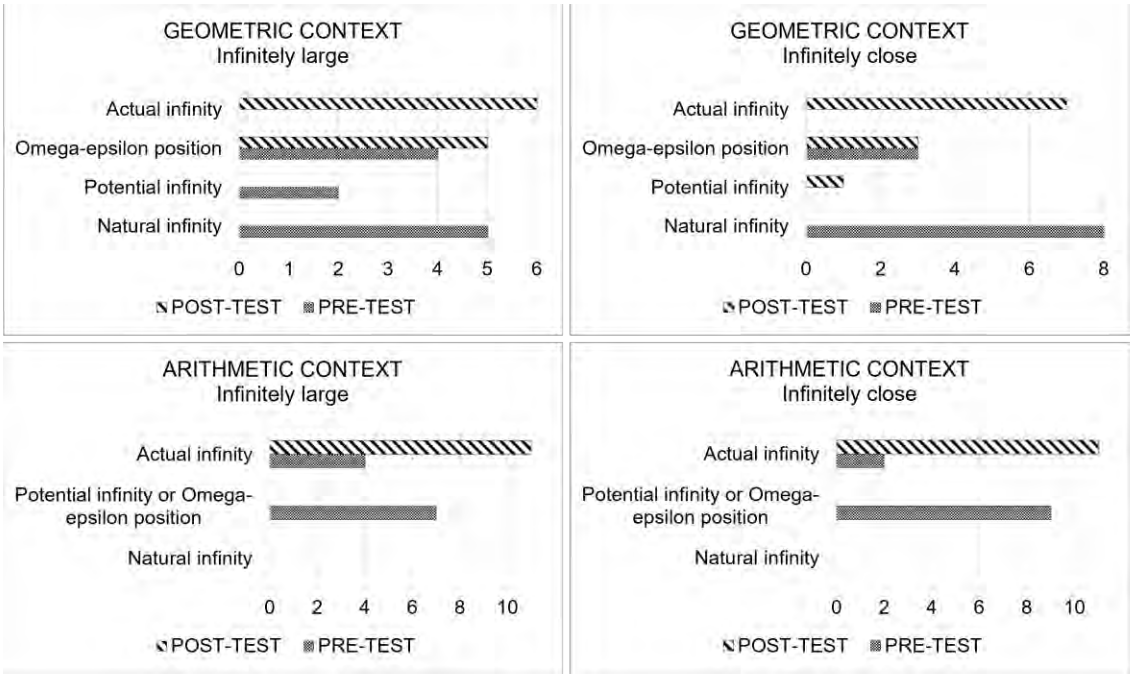


Figure 1. Number of teachers who presented the conception of infinity in the context and type of situation before and after the workshop

For example, before the workshop, in the geometric context of the infinitely large type (see Figure 2), most of the teachers drew the line segments $\overline{AY}$ and $\overline{BZ}$ with the extreme point at the end of the line p in the illustration.

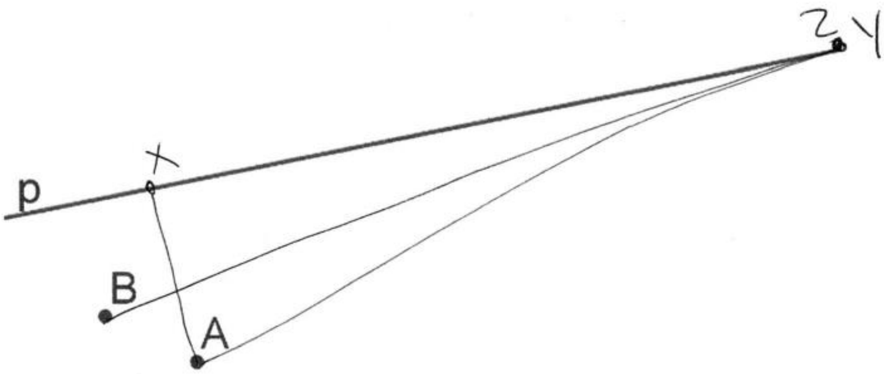


Figure 2. Strokes made by Professor 11 on item (2) before the workshop

They answered that one of them is more extraordinary, thereby evidencing a natural conception of infinity (see Figure 3). From the perspective of the theory of conceptual change, these teachers need to transform their mental model of the straight-line concept since the image they have of a straight line generates a conflict that, in turn, influences how they conceive the infinite.

> Ambos segmentos tienen el punto Y y Z en el lado más largo de la recta p y estos coinciden. El punto A tiene una posición adelantada por lo q' es mascorto comparado con el segmento B

> Both segments have the point Y and Z on the longest side of the line p, and they coincide. Point A has a forward position so it is shorter compared to segment B.

Figure 3. Teacher 11's response to item (2e) – *Which segment is larger?* – before the workshop

In the infinitely large and infinitely close tasks in arithmetic contexts, most of the teachers found themselves in the omega-epsilon position of infinity, or potential infinity, as they considered the word infinity or the symbol ∞ to be the answer (see Figure 4).

> 3.00...1 Un número que tienda a 3 pero que no sea 3 de tal manera que tenga decimas que indiquen q'ese número es mayor que 2 al restarle el uno.

> $3.00 \ldots 1$ a number that tends to 3 but is not 3 in such a way that it has tenths that indicate that number is greater than 2 by subtracting one.

Figure 4. Teacher 11's response on item (1d) – *What is the closest number to three?* – before the workshop

It is important to note that these teachers presented all the different phases of the conception of infinity in the arithmetic and geometric contexts before, during, and after the workshop. For example, in the preworkshop, teacher 11 presented potential notions in arithmetic but natural ones in geometric contexts. This shows that in the former case, teachers have conceptions of stages higher than the natural ones compared to geometric contexts.

In general, in a geometric context, teachers presented an actual conception of the omega-epsilon position after the workshop. In fact, only one presented a potential conception of infinity in the situation of the infinitely close type. In the arithmetic context, regardless of the type of situation, all teachers presented a conception of actual infinity. This suggests that it is easier to overcome conceptions of infinity in arithmetic contexts than in geometric contexts.

> No podemos crear las rectas $\overline{AY}$ ni $\overline{BZ}$ ya que no hay un punto Y, Z que sean lo más largos posibles ya que las rectas son infinitas.

> We cannot create the lines $\overline{AY}$ or $\overline{BZ}$, since there is no point Y, Z that makes them as long as possible, since the lines are infinite.

Figure 5. Teacher 10's response on item (2e) – *Which segment is larger?* – after the workshop

For example, in a geometric context of the infinitely large type, the teachers' answers on one item (see Figure 2) were divided into two classes: one in which the segments cannot be compared because they do not exist and the second because they are endless. An example of the conception of actual infinity is present in teacher 10's response (see Figure 5), where the transformation of the mental model helped her/him to consider that segments cannot be built.

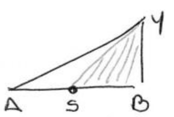
In arithmetic contexts of both the infinitely large and infinitely close types, all the teachers presented an actual conception of infinity by responding that there is no larger number or number closer to zero, respectively (see Figure 6).

No se puede representar, Por que siempre habra uno más y no se terminaría de establecer quien sería

It cannot be represented, because there will always be one more and one would not be able to establish who it would be.

Figure 6. Teacher 11's response on item (1c) – *What is the largest number?* – after the workshop

After responding to the situations presented in the workshop, these math teachers' conceptions in arithmetic and geometric contexts were improved. For example, teacher 3 presented actual conceptions in an arithmetic context and a geometric context of the infinitely close type (see Figure 7),



El área es más pequeña pues uno de los catetos esta segmentado a la mitad.

$d = \overline{AB}$ $A(x) = dy$

$A(x) = \frac{dy}{2}$

Before: The area is smaller because one of the legs is segmented in two.

No podemos comparar algo que no existe por definicion de triangulo, por tal motivo no existen ambos triangulos.

After: We cannot compare something that does not exist, by the definition of the triangle. For this reason, neither triangle exists.

Figure 7. Teacher 3's response to item (5f) – *Which triangle is smaller?* – before and after the workshop

while the infinitely large type induced the presentation of an omega-epsilon position conception (see Figure 8).

Las distancias son equivalentes pues el punto de referencia Y es fijo para ambos puntos y si la implicacion de la distancia de $\overline{B}X$ se satisface no importando donde se ubique A y B son equidistantes.

Before: The distances are equivalent since the reference point Y is fixed for both points and its implication of the distance $\overline{BX}$ is satisfied no matter where it is located; A and B are equidistant.

Siempre podemos construir un segmento más grande que otro por tal motivo por muy lejanos que esten los puntos Z y Y de A y B no existe dicho segmento.

After: We can always build a segment larger than another; for this reason, no matter how far the points Z and Y are from A and B, there is no such segment.

Figure 8. Teacher 3's response to item (2e) – *Which segment is larger?* – before and after the workshop

For purposes of the discussion, the transcribed responses of teacher four and teacher 15 – referred to by the pseudonyms 'Cathy' and 'John' – are presented. Their results allowed the authors to analyze the evolution of their conceptions of infinity during the instruction due to their ability to expound on their comments, questions, and feedback during each workshop session. The responses transcribed here are taken from the material covered during the workshop.

One of the first situations addressed during the infinitely large type workshop exercises involved comparing finite and infinite sets (Belmonte & Sierra, 2011). It was entitled, *Numbers that are very large*. What we expected from the teachers was, first, that they would distinguish between finite and infinite sets. For example, a teacher with a conception of natural infinity would think that all sets are infinite because the number of grains of sand on the beach is infinite since, according to Krátká et al. (2021), this conception moves within his horizons. From the theory of conceptual change, a revision of beliefs is necessary for the individual to modify their initial conception of an infinite set.

This occurred in the case of Cathy, who, when faced with the situation of ordering the sets, mentioned that:

> *I believe all sets are infinite because they have no end, so we can't classify which set has the least or the greatest number of elements.*

Faced with the questioning by her colleagues as to why she considered that every set is infinite, she affirmed that:

> *It's just that I can't count all the grains of sand on the beach, much less on Earth.*

Here, Cathy is familiar with the sets; that is, she "understands the set of atoms [here the grains of sand] as naturally infinite, the set exists, let's not create it" (Krátká, 2013, p. 93), but the inability to count a totality of elements limits her conception of the infinite to the natural. This finding concurs with the report in Cihlář et al. (2015), where when working with students, the sets were conceived as infinite, as long as they extended within their horizons.

A theory of conceptual change holds that for a conceptual change of infinity to occur, the individual must first review her/his beliefs. This means that the first step consists of modifying the belief that all sets are infinite. Thus, to generate a conflict between the belief and the information of the situation and, at the same time, foster this review of beliefs to allow a conceptual change (Chi, 2008), the instructor asked the teachers: *How many divisors do the number 24 have? How many divisors does the number 100 have?* and *how many divisors does* 10^{100} *have?* Some teachers calculated this, while others – Cathy included – intuited that although they are considerable in number (e.g., 101^2 the dividers for 10^{100}), they must be finite:

Well… if it's a number, then it's finite… maybe I exaggerated when I said it was infinite because I didn't know how to calculate it.

The instructor told her to think about the amount of sand in the following way: *will there always be more sand to count, or will it eventually run out?* Cathy was quick to reply that it has to run out, so this should be a finite number, just like the divisors of 10^{100}:

Well, it would end, in the end, there must be one last grain of sand… Ah! Therefore, it's a very, very large number… well, but it's still a number.

At the end of this situation, the instructor mentioned various other very large finite sets (e.g., number of stars, number of cells in the human body) to note whether the teachers (including Cathy) had changed the belief that not all sets are infinite. In the case of Cathy, at this point in the workshop, it sufficed to review her idea of infinity to point out that not all sets are infinite.

One unexpected response in this situation was from John, who proposed ordering the sets. For this teacher, the smallest of all sets is that of the natural ones. When the instructor asked how he could justify his argument, John replied:

Well, it's that they all have a certain amount, but I know that the smallest must be the natural ones because some authors cite the natural numbers as the primary elements up to 10 without counting zero.

Although it could be concluded that his initial conception is not of natural infinity because it does identify that not all sets are infinite, this is not completely overcome since John considered that the only natural numbers are the ones from 1-10. His conception of the set of natural numbers, $\mathbb{N}$, is influenced by the belief that $\mathbb{N} = \{1, 2, 3, \dots, 9, 10\}$.

The theory of conceptual change argues that modifying one's belief would also change one's conception of infinity. To do this, the teachers built the natural numbers – the difference of the base 10 (which could be the source of John's confusion) – and led John to notice that this set is infinite.

I don't know why I confused it. I thought the natural ones were just the ones we could count on our fingers; that's why they're natural. I've been explaining it wrong all these years.

The instructor then asked John to order the sets differently, considering the new construction of the natural numbers. John then stated that the last one would now be the natural numbers. Thus, for him, the conception of natural infinity when comparing sets was successfully superseded in this situation.

Second, once teachers reached the point in the workshop where they were able to discern finite from infinite sets, the instructor proceeded to compare infinite sets, specifically, natural numbers, even numbers, and perfect squares (see also Fischbein et al., 1979; Juter, 2019; Moreno & Waldegg, 1991; Monaghan, 2001). This exercise was called *Comparing infinite sets*. The expectation was that the teachers would identify that all three sets are infinite, establish a one-to-one relation between the natural and even numbers and between the natural numbers and the perfect squares, and, therefore, see that they have the same number of elements. A teacher with a natural or potential conception would think, for example, that there are twice as many natural numbers as even numbers or fewer perfect squares than natural ones. These types of arguments are also analyzed by Juter (2019), who reported that almost 40% of the students believed there are fewer even numbers than natural numbers, and in Moreno and Waldegg (1991), where approximately 40% of the student subjects believed that there are fewer perfect squares than natural ones. In contrast, Monaghan (2001) determined that 46% of the students believed that there are the same number of pairs as natural ones.

Both Cathy and John argued that there are as many natural numbers as there are even numbers, but on the one hand, Cathy claimed that one cannot tell if there are more natural or perfect squares because both are infinite, while John argued that there are fewer perfect squares than natural ones, stating that:

> *To be a perfect square, the condition must exist that the number multiplied by itself results in another number.*

At that juncture, the teacher considered the natural ones and observed that not all are perfect squares. Although it is true, the natural condition is that, given the set of perfect squares, a teacher can obtain the associated natural square within the set of natural numbers. To make this evident, the instructor asked John to organize the first ten perfect squares and try to determine the natural square that gives rise to them:

> *Since they are all perfect squares, [my way] would be to look for the [natural] number that multiplied by itself results in that perfect square; for example, from the perfect square 16, its associated natural number would be 4 because* $4 \times 4 = 16$.

At that point, the instructor asked John, who had constructed the relation between perfect and natural squares, to consider whether he could use this as an argument to say something about the equality of elements in both sets, as he did with the even and natural ones.

> *It's just that I was thinking about the gaps in the numbers that [exist] when there is no perfect square. However, if I think about it kike, I did with the pairs… well, if they are equal, there's the same thing because the relation is there for the natural ones; if I take out their square, I obtain the perfect squares, and vice versa, if I take the square root of the perfect square because I have the natural one from before.*

Cathy, in contrast, built a relation between natural numbers and perfect squares, going so far as to state that she will always have a perfect square of a natural. However, her answer to the question was:

Which are greater, natural, *or perfect squares?* Suggests that she does not use that relation, so an earlier conception resurfaces, one that consists of counting the total elements to see which is greater, as is done with finite sets:

> *Since perfect squares arise from the sequence of natural numbers, there are an infinite number of them; therefore, they cannot be compared.*

The theory of conceptual change holds that her tendency to count apparently refuses to disappear because her mental model needs to be transformed. The instructor asked her to build a one-to-one relationship, but this one between the perfect squares and the natural ones. Cathy answered that the relation would be the square root. Again, the instructor observed that her argument could be the same as the one she used with the even and natural sets by finding a double relation between one set and another. In this case, how does her argument change:

> *If it's the same, I can say that it would be the same if for each natural there is a perfect square and vice versa. I think I'm no longer worried about counting one-by-one, like I no longer pay attention to seeing each natural or even number, but as if I already had all of them, as a whole. The relation helps me compare sets, if it's there, it's because they both have an equal number of elements.*

This response shows that Cathy is overcoming the conception of potential infinity that she presented in this situation. This approach is in conflict with the process of counting each element because it would have no end, at least not from a potential conception. Cathy understands the process of counting elements in a finite set but encounters conflict when counting elements in an infinite set. From the theory of conceptual change, she has adjusted her mental model to compare infinite sets, thereby recategorizing the concept of infinity beyond potentiality. Thus, the new information suggests that counting is no longer necessary; it is only necessary to establish relations over the entire set and change the model utilized to compare infinite sets. Finally, the paradoxical situation of Hilbert's Grand Hotel (Mamolo & Zazkis, 2008; Roa-Fuentes & Oktaç, 2014) shows that Cathy, after working on different situations of the infinitely large type, came to build an actual conception of infinity.

The answer in Figure 9 suggests that Cathy delved into infinite set comparison and noted a function, $y = x$, that relates the number of guests, x, to room number, y, to find a new function, $y' = x + 1$, that relates the number of guests, x, to the next room, y'. Cathy then used the relations to compare sets, allowing her to 'move' elements to generate the 'space' for the new host.

Only two of the 15 teachers, such as Cathy's, presented a solution to this situation. Most of the teachers with a potential or natural conception (such as John) focused on placing the new host last, arguing that if the rooms are infinite, there must always be one more that is empty. After Cathy showed the others how the situation could be fixed, John commented:

> *It is not clear to me, I feel that it's the same as if you put it at the end, because if you're moving all the infinite room guests to the next room, at the end, the last one must be unoccupied in order to be able to put the guest in the last room there. If the previous room is empty, it's better to put the one who just arrived there and not move anyone else around.*

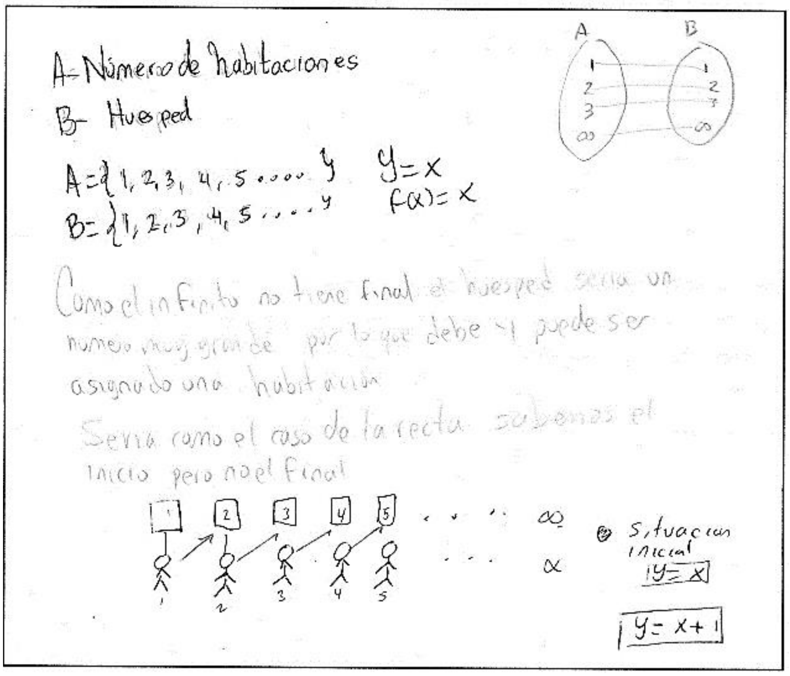


Figure 4. Cathy's response to the question: *How would you accommodate a new guest in a hotel with infinitely full rooms?*

Like John, the other teachers resisted Cathy's solution since, as evidenced in Mamolo and Zazkis (2008), students tend to resist the actual solution because they associate it with realistic meaning. Although John presented potential conceptions, natural ideas also emerged in his work, such as the fact of talking about a 'last' room when the number of rooms is infinite. To eliminate this idea, the instructor asked them if this fact would not violate the condition that all the rooms were full. Once again, John argued that Cathy's solution does not elucidate how guests would be accommodated if their number were infinite. This would not terminate the process. In this case, the resistance to accepting the solution is limited by a realistic meaning, that is an infinite rearrangement process that would never stop moving (Mamolo & Zazkis, 2008).

To help teachers change their conceptions, the instructor highlighted the various contradictions in their arguments. For example, John mentioned a 'last' guest or room when, in fact, the rooms are infinite. For John, the beliefs revision was insufficient and could not transform the model to compare infinite sets. According to the theory of conceptual change, this may occur because the original conception is miscategorized. Here, realistic arguments and making physical sense of the infinite process limit conceptual change.

Once again, Cathy showed conceptions of superior potential when she asked the instructor: *what would happen if infinite guests were to arrive?*

> *If we have an infinite set and [...] a bijective function, $y = x$, we multiply this by 2, $y = 2x$, allowing certain infinite rooms to stay empty, ready to house infinite guests. In this case, the sets of rooms are infinite, the number of guests is infinite, and a set of infinite empty rooms must be created for the set of guests.*

This shows that the situations considered while comparing infinite sets contributed to a better model of cardinality in infinite sets for Cathy; indeed, she has built an actual conception of infinity in situations of the infinitely large type.

Cathy's responses during the workshop's final sessions and in the post-test demonstrate that she has progressed beyond previous stages of infinity and has developed a current conception of infinity in most situations presented during the workshop. We believe that the situations proposed in sessions 2, 3, and 5 of the workshops regarding the cardinality of infinite sets and the comparison of infinite sets have gradually helped transform Cathy's conception of infinity. Of course, this does not mean that the other situations are not necessary; on the contrary, the conceptions of built-in infinite sets helped accept the fact that two different closed intervals or line segments contain the same number of infinite elements.

However, we cannot guarantee homogeneous results in other teachers. An example of this is John, who exhibited resistant misconceptions at certain moments during the workshop, reinforcing that the journey through the stages of infinity conception is not linear (Krátká et al., 2021) and that "as soon as the concept is required for further work, the obstacle will reappear" (Mena-Lorca et al., 2015, p. 353). Therefore, as several authors point out, situations where infinity emerges must be present in the curriculum when training teachers who will be in service in classrooms with students and who will partly inherit conceptions from their teachers when introducing this concept (Diaz-Espinoza et al., 2023; Hannula et al., 2006; Holton & Symons, 2021; Roa-Fuentes & Oktaç, 2014).

CONCLUSION

The literature review synthesis, the analysis of teachers' responses across pre and post-tests, and the workshop's scenario development collectively suggest that constructing current conceptions of infinity is comparatively more straightforward in arithmetic contexts than in geometric ones. This underscores the necessity for the workshop design to strike a balance between arithmetic and geometric scenarios, as illustrated in Table 1. The diverse range of conceptions exhibited by teachers like John underscores the intricacies of addressing misconceptions across varying contexts. As previously discussed, we contend that the diversity of scenarios presented in these two contexts and the gradual evolution of the concept of infinity throughout the workshop facilitated a substantial shift in most teachers' conceptions. This underscores that the conceptions articulated by mathematics educators regarding infinity are contingent not only upon the context—whether arithmetic or geometric—but also on the specific type of scenario presented.

The conceptions held by mathematics teachers before the workshop intervention align with those documented in existing literature. Nonetheless, the workshop facilitated the emergence of diverse scenarios, prompting alterations in these conceptions. As delineated by the findings of this investigation, the spectrum of conceptions developed by teachers ranged from those associated with natural infinity to those reflective of actual infinity, albeit following a non-linear trajectory, as highlighted earlier. For instance, upon encountering scenarios pertaining to the infinitely large and infinitely numerous types, Cathy articulated conceptions indicative of actual infinity. However, in instances involving the infinitely close type, her conceptions did not inherently encompass actual infinity.

This phenomenon was particularly pronounced in the case of John, who demonstrated resistance to arguments rooted in actual infinity at various junctures throughout the workshop, notably in the context of Hilbert's Grand Hotel scenario. Indeed, according to the theory of conceptual change, it can be contended that John did not realize a complete categorical shift due to his hesitance in reconsidering certain beliefs associated with natural or potential infinity.

While the post-test results indicated a collective shift towards conceptions aligning with omega-epsilon and actual infinity positions among these educators following the workshop, a closer examination revealed considerable variability in individual conceptions across the presented scenarios. Certain participants persisted in retaining their initial beliefs regarding the infinite, such as applying finite set comparison rules to infinite sets or grappling with the realistic interpretation of infinite processes. This inherent variability posed a limitation to the research endeavor. Despite the anticipation that such phenomena might manifest during the workshop, the instructional intervention proved inadequate in effectively addressing and overcoming these persistent beliefs.

In summary, the workshop has yielded valuable insights for informing the design of future interventions targeted at both in-service and pre-service educators. It underscores the importance of striking a balance between arithmetic and geometric contexts while aligning with the theory of conceptual change. The overarching objective is to systematically construct the concept of infinity within each scenario by juxtaposing novel information with existing beliefs, thereby engendering cognitive conflict. However, it is imperative for future research endeavors to delve deeper into the realm of resistant beliefs or flawed conceptual models, seeking innovative strategies to surmount them effectively.

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APPENDIX

An Infinitely Large Type Situation in an Arithmetic Context: Numbers that are Very Large

In the natural world, numbers play an important role in helping us count, from the days of the week to the amount of money we have in our bank accounts. However, sometimes those numbers can get out of hand, for example, the number of hairs on our head, or the number of fish in the ocean. With this in mind:

1. Order the following sets from smallest-to-largest and explain your answer in the box:

   A. The number of stars.

   B. The number of grains of sand on Earth.

   C. Natural numbers N = {1, 2, 3, 4, 5, …}.

   D. Divisor numbers of a 'googol' (10^{100}, a one followed by one hundred zeros).

   E. The number of cells that make up the human body.

A Situation of the Infinitely Large Type in an Arithmetic Context: Hilbert's Grand Hotel

Imagine that you are the manager of the best hotel in the world: *The Grand Hotel Hilbert*. This hotel has an infinite number of rooms (don't worry about the physical space needed for the hotel) where one rainy night, all the rooms are occupied (that's right, you heard right, absolutely all!). Just at that moment, a very famous guest arrives who cannot be denied service and asks for a room.

1. If only one guest is allowed per room, how could I fit the new guest into one? Take some time to reflect, make diagrams, representations, or drawings if necessary. Consider what you have done before with infinite sets and deduce a possible solution.

2. Worse yet, after a while (since the rain doesn't stop), an endless number of guests arrive. Could you place them all? If so, how? Take some time again, breathe and analyze the situation. Use everything you have seen and done in previous sessions. Keep your spirits up!