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How abstraction of a pre-service teacher in constructing relationships among quadrilaterals

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Abstract

Abstraction is essential to learning mathematics because the mathematical concepts obtained through abstraction will be more meaningful than directly receiving these concepts. This study aims to describe the pre-service teachers' abstraction in constructing relationships among quadrilaterals. This research method was explorative qualitative research with a purposive sampling technique. The subject of this research was a pre-service mathematics teacher who had taken a geometry course. The data analysis techniques used in this study were data condensation, data display, drawing and verifying conclusions. The research results showed that the participant used epistemic actions in an abstraction, such as recognising each quadrilateral type, building-with their properties, constructing relationships among them, and consolidating the abstract results made. Thus, the abstraction in constructing relationships among quadrilaterals can be observed from the epistemic actions: recognising, building-with, constructing, and consolidation, known as RBC+C.

Keywords: Abstraction, Consolidation, Constructing, RBC+C, Relationships among Quadrilaterals

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Abstraction is an essential aspect of learning mathematics (Ferrari, 2003; Hazzan & Zazkis, 2005; Kiliçoğlu & Kaplan, 2019; Memnun et al., 2017; Ozmantar & Monaghan, 2007) because some mathematical concepts are obtained through abstraction (Memnun et al., 2017) and through abstraction, students will find their concepts learned so that learning will be more meaningful for them because abstraction encourages students to connect previous knowledge with the knowledge being studied, as expressed by Agra et al. (2019) and Lemos (2007) that meaningful learning occurs when students connect previous knowledge with the knowledge being studied. The definition of abstraction is divided into classical and non-classical cognitive approaches. The main feature of this classical approach is the similarity of a set and its corresponding categorization as a primary form of abstraction (Hershkowitz et al., 2001; Rosch & Mervis, 1975). Soedjadi (2000) and Oers and Poland (2007) reinforced that abstraction focuses on the relationships among objects and ignores the differences in their quality. This means that abstraction occurs when we look at several objects. Then, we discard the properties of objects considered unimportant or unnecessary and finally only pay attention to or take the essential properties jointly owned.

Abstraction with a non-classical view is a cultural activity that creates new meaning in reorganizing and restructuring mathematical knowledge, known as a new structure (Bikner-Ahsbahs, 2004). The abstraction is a vertical reorganization activity of mathematics that was previously built into a new mathematical structure (Dreyfus, 2007; Hershkowitz et al., 2001). *Constructing* a new structure depends on understanding the concept and its relation to the previous construct (Guler & Arslan, 2015). This means that if someone wants to construct an object, he must have the knowledge, and their abstraction also depends on the construct or the knowledge that has been previously owned. If they do not have previous knowledge, they will not be able to construct the new structure correctly to be formed. For example, a pre-service teacher must know rectangles and parallelograms to construct a relationship between rectangles and parallelograms. If the properties of the parallelogram are added to one of the right angles, then the properties are rectangle characteristics. Furthermore, he can conclude that the rectangle is a parallelogram. This result is called the new mathematical structure. The result is more complex than the previous construct that he already has.

Hershkowitz et al. (2001) introduced Abstraction in Context (AiC). AiC represents a theoretical framework designed to explore the processes involved in developing and solidifying abstract mathematical understanding (Dreyfus et al., 2002; Hershkowitz et al., 2001; Tabach et al., 2017). A range of factors influences the process of abstraction, and these factors constitute the context in which the abstraction occurs (Dreyfus, 2012; Gilboa et al., 2018). The context in AiC can be a curricular context, such as a sequence of tasks designed; a historical context, like the students' previous learning experiences; a learning context, possibly including computerized environments with technological tools; and a social context, which may be an alternation of group work, individual work, and whole class work. In this research, tasks were designed to construct relationships among quadrilaterals. In addition, AiC represents a methodological framework based on action theory (Dreyfus, 2012; Giest, 2005). This theory proposes an adequate framework for considering fundamental cognitive processes while considering the mathematical, historical, social, and learning contexts in which these processes occur (Dreyfus, 2012). Therefore, the component of context cannot be separated from the activity. AiC incorporates a theoretical and methodological framework that posits a specific set of epistemic actions: Recognizing, building with, Constructing, and Consolidating, as essential for describing and analyzing the emergence and reinforcement of a new concept (Tabach et al., 2017).

Several studies examined abstraction by looking at epistemic actions (Dreyfus, 2007; 2015; Dreyfus et al., 2002; Gilboa et al., 2018; Sümen, 2019; Tsamir & Dreyfus, 2002). Epistemic action is a mental action used to construct a knowledge (Hershkowitz et al., 2001). These actions can be observed in the speech and actions of students with artifacts, gestures used, and others (Bikner-Ahsbahs, 2019). The epistemic actions are recognizing, building with, and constructing Field (Hershkowitz et al., 2001) known as RBC. The research was then developed independently by Dreyfus et al. (2015), which uncovered the essential elements of the three epistemic actions and added consolidation (+C) to become RBC+C epistemic actions.

Recognizing occurs when a person realizes that a previously constructed and stored memory structure is related to the current problem (Dreyfus et al., 2002). Recognizing can occur in at least two cases: (1) analogy with other objects with the same structure already known; (2) specialization, realizing that objects fit into more general groups (Dreyfus et al., 2002). *Building-with* combines previously recognized constructs to fulfill a purpose, such as solving a problem or justifying a statement (Dreyfus, 2015; Jirotková & Littler, 2005). A student has done this activity if he combines his knowledge to fulfill the goals, such as connecting the properties of parallelogram and rectangle. *Constructing*, the construction or restructuring of knowledge, is the goal (Dreyfus et al., 2002) or what processes of restructuring and reorganization are recognized and known to construct new meanings (Bikner-Ahsbahs, 2004). So, constructing is the primary goal of abstraction because new knowledge or structures will emerge from constructing. Consolidation directs students to how the abstraction that has been constructed makes it easier for subsequent activities (Breive, 2022; Güler & Arslan, 2017). An illustration of RBC+C can be seen in this example; a student remembers the definition or the properties of parallelogram and rectangle (recognizing). Next, a student connects these two properties (building-with) to make a construction; a rectangle is a parallelogram (constructing). The construction result is applied in solving problems so that it is more straightforward for students. This means that to find the area of a rectangle, a student can use the parallelogram formula. Thus, the results of the abstraction do not stop with the construction results but rather how they can be followed up (Ahlam & Michal, 2015; Dreyfus et al., 2015), especially in problem-solving. Therefore, researchers use RBC+C to analyze students' abstraction processes in constructing relationships among quadrilaterals.

The quadrilateral and relationships among quadrilaterals are significant to master because quadrilaterals are one of the requirements for studying geometry. The difficulties experienced by them in understanding the relationships among quadrilaterals need to be followed up, mainly in studying their abstractions in constructing relationships with them. Several studies examined the relationship among quadrilaterals (Avcu, 2022; Çontay & Paksu, 2012; Fujita, 2012; Zeybek, 2018; Žilková, 2015). However, these studies only examine students' understanding, misconceptions, and levels in quadrilaterals and the relationships among quadrilaterals. In addition, there is also research related to the construction of relationships among quadrilaterals (Agustan, 2015; Budiarto et al., 2017; Turnuklu, 2014). Research conducted by Turnuklu (2014) and Agustan (2015) examined pre-service mathematics teachers define and classify quadrilaterals, and they did not study their abstractions using epistemic actions. Budiarto et al. (2017) discuss the process of abstraction in constructing relationships among quadrilaterals using RBC epistemic actions, but abstraction research in constructing relationships among quadrilaterals only uses side attributes. This study only examined how students make relationships among quadrilaterals using side attributes. Therefore, it is necessary to carry out further research to see students' abstraction in constructing relationships among quadrilaterals by using the two definitions' side, diagonal, and equivalence attributes. In addition, the epistemic actions in constructing relationships among quadrilaterals in previous studies only used RBC, even though consolidation is significant in abstraction to strengthen the knowledge that has been constructed by students (Memnun et al., 2017). Therefore, this research needs to be continued to examine how student consolidation is produced after the construction so that RBC+C epistemic actions are used in this research to describe and analyze students' abstraction processes.

Apart from that, the results of this research can provide advice to lecturers, mathematics education study programs or curriculum planners in designing learning and curriculum development, especially in constructing relationships among quadrilaterals. Lecturers can design learning well if they understand the student's abstraction process (Dreyfus, 2012). For curriculum planners, the results of this research can serve as a guide in designing the geometry curriculum. This study aims to describe the pre-service teachers' abstraction in constructing relationships among quadrilaterals. Abstraction in constructing relationships among quadrilaterals will be studied from the epistemic action of RBC+C.

METHODS

The research method used in this research was explorative qualitative research because the researcher wanted to explore the participants' abstraction process in constructing relationships among quadrilaterals in depth through interviews. In the context of explorative research, the emphasis is on exploring a relatively unknown or less understood area. Explorative research is qualitative, so it is flexible and open-ended, allowing researchers to adapt their methods and questions as they delve deeper into the subject matter (Creswell & Creswell, 2018).

Participants and Data Collection Tools

The technique for taking this subject used a purposive sampling technique. Purposeful sampling (often termed purposive or judgment sampling) operates on the idea that selecting the most suitable cases for research leads to optimal data, with study outcomes directly tied to the chosen cases (Patton, 2014). This study's subject was a pre-service mathematics education teacher who has taken geometry courses and was willing to be interviewed. The researchers only take one subject to deepen the abstraction process because before constructing the relationship among quadrilaterals, the researchers also need to examine the abstraction process in relation to each quadrilateral. So, researchers will study the abstraction of relationships between parallelograms and rectangles and others before finally obtaining the construction of relationships among quadrilaterals. In addition, the abstraction in this research is described and analyzed based on the epistemic actions of RBC+C. This subject had high abilities (in the range of 90-100 of the plane geometry value). The main instruments in this study were the researchers and auxiliary instruments: tasks and task-based semi-structured interviews. The tasks were compiled based on indicators of epistemic action: recognizing, building-with, constructing, and consolidation in Table 1. The development of these abstraction tasks has been made into a paper. It was presented at the 9th SEA-DR IC in Unsri 2023 (Hodiyanto et al., 2023). Readers can read the paper if they want to see the tasks used in this research. Examples of tasks given to subjects are:

1. Make connections among parallelograms, rectangles, rhombuses, squares, kites, and trapezoids in chart or Venn diagram!
2. Show that to determine the area of the shape below can be solved by using the trapezoidal formula! (Hodiyanto et al., 2023)

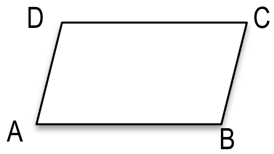

Table 1. Definitions of Epistemic Action Used in this Research

No	Epistemic Action	Operational Definition
1	*Recognising* (**R**)	Remembering the previous construct (properties/definition of each quadrilateral) related to the task given (relationships among quadrilaterals)
2	*Building-with* (**B**)	Combining existing elements (properties/definitions of each quadrilateral) to develop a new concept (relationships among quadrilaterals).
3	*Constructing* (**C**)	Reorganising previous constructs (properties/definitions of each quadrilateral) into a new structure (relationships among quadrilaterals).
4	*Consolidation* (**+C**)	Strengthen the knowledge that has been formed and provide convenience in further activities.

Procedures and Analysis

This research began with assigning tasks to the subjects, and after the subjects completed the tasks, the researcher interviewed them based on their work results. The data analysis techniques used in this study were data condensation, data display, drawing and verifying conclusions (Miles et al., 2014) and combined with RBC+C. Data condensation is reducing and refining large amounts of raw data into manageable and coherent chunks or themes without losing their essence. Data display is resenting condensed data visually or organized to help see patterns, relationships, or themes. Researchers begin understanding deeper meanings, patterns, and relationships based on the condensed and displayed data. At the data condensation stage, the researcher takes the recorded interviews, codes the data, and reduces it. To simplify the analysis in this study, researchers used the codes R for recognizing, B for building-with, C for constructing, and +C for consolidation. Next, at the data display stage, the researcher looks at the relationship between the interview results and the subject's work results. At the verifying conclusions stage, the researcher provides the meaning of the relationship.

RESULTS AND DISCUSSION

The data in this research is work and interview results. The results of interviews are displayed more in this research than the results of the subject's work because this research aims to describe the subject's abstraction in constructing relationships among quadrilaterals. The complete results of interviews between a researcher (Q) and a pre-service teacher (S) and figures can be seen in the Appendix. Only a few interview excerpts are shown.

1. Q: What do you know about quadrilaterals?
2. S: A quadrilateral formed of four intersecting lines where no three lines intersect at one point → (**R**)
7. Q: Try defining a quadrilateral using diagonals!
8. S: A quadrilateral is a polygon that has intersectional diagonals → (**B**)

Based on the interview excerpt above, a pre-service teacher tries to define a quadrilateral by defining genesis or genetics [2] (how the object is formed) (Molland, 1976; Sinclair et al., 2012; Soedjadi, 2000), but the genetic definition made is still not precise. According to Slaught and Lennes (1911), a quadrilateral can be defined if there are no three points A, B, C, and D on the same straight line, then four segments $\overline{AB}, \overline{BC}, \overline{CD}, \text{and } \overline{DA}$ will be formed. This definition is the genetic definition. But when reminded of polygons, S precisely defines a quadrilateral, as stated by Alexander and Koeberlein (2020) and Sanders (1908), where a quadrilateral is a polygon with four sides. At this stage, he already remembers previous knowledge related to quadrilaterals [2], so the activity is recognizing. Furthermore, after he recognizes the quadrilateral definition, S combines the definition of a quadrilateral with a diagonal to define the quadrilateral based on the diagonal attribute. The diagonal of a polygon is a line segment that connects two non-consecutive corner points (Alexander & Koeberlein, 2020). S has done the building-with activity because S combines the quadrilateral definition with a diagonal.

11. Q: Is there another definition of the parallelogram?
12. S: A parallelogram is a quadrilateral whose diagonals bisect each other ⟶ (**R**)
13. Q: Can you define a parallelogram with the side attribute?
14. S: A parallelogram is a quadrilateral whose opposite sides are parallel ⟶ (**R**)
15. Q: Can you prove that the two definitions (using side and diagonal attributes) are equivalent?
16. S: Yes, I can (he draws a parallelogram) (can be seen in Appendix)
17. Q: So, what do you want to prove?
18. S: Two congruent triangles ⟶ (**R**)
19. Q: To prove the two triangles, what theorem do you use?
20. S: Two congruent triangles can be proved by congruence theorems (side, angle, side; angle, side angle; side, side, side) ⟶ (**R**) and I can use side, angle, side (he shows that corresponding sides are equal and vertical angles are equal in Figure 1, $\overline{AE} = \overline{CE}$, $\overline{BE} = \overline{DE}$, $\angle AEB = \angle DEC$) ⟶ (**B**)
21. Q: Then what are the implications?
22. S: Yes, alternate interior angles are congruent (S points to angles of $\angle DCA = \angle BAC$ and $\angle BDC = \angle ABD$) → (**C**)

S can show the equivalence of the two definitions of a parallelogram using the side and diagonal attributes [16]. When S performs the equivalence proof of the two definitions, S draws a parallelogram so that the image helps him in his proof. When S proves that the two definitions are equivalent, S first draws a parallelogram because the image will help S in the proof process. This is supported by the findings Haj-Yahya (2020; 2021) that images or visualization of objects can affect the construction of evidence. The visualizations during the abstraction greatly enhanced the comprehension of the concept of congruent figures (Yilmaz & Argun, 2018). S can prove that the two definitions are equivalent using the triangle congruence theorem (side, angle, side) [20]. However, S memorizes the definition of a parallelogram using both side and diagonal attributes before S concludes that the two definitions are equivalent. After that, S recognizes the congruence theorem of triangles [18] and uses the theorem (side, angle, side) to prove the congruence of the two definitions. S connects relevant sides and angles that are equal in length and measure. The process of proving the equivalence of the two definitions is as follows:
Suppose q is a quadrilateral ABCD whose diagonals intersect and bisect each other, and p is a quadrilateral ABCD with parallel opposite sides. In that case, he can show that q is equivalent to p by showing that $q \leftrightarrow p$. The complete proof explanations are shown in the answers below, as shown in Figure 1.

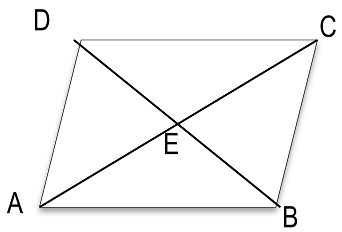

Figure 1. The Parallelogram

(i) $q \to p$

Quadrilateral ABCD, $\overline{AC}$, and $\overline{BD}$ intersect at E, $\overline{AE} = \overline{CE}$, and $\overline{CE} = \overline{DE}$. It will be proved that $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$.

Step	Statement	Response
1	ABCD with $\overline{AC}$ and $\overline{BD}$ intersecting at E	Given
2	$\overline{AE} = \overline{CE}$ and $\overline{CE} = \overline{DE}$	Given
3	$\angle AEB = \angle DEC$	Vertical angle
4	$\triangle$ ABE $\cong \triangle CDE$	Congruence Theorem (side, angle, side) (2nd & 3rd step results)
4	$\angle BAC = \angle DCA$ $\angle ABD = \angle CDB$	Alternate interior angles (3rd step result)
5	$\overline{AB} \parallel \overline{DC}$	4th step result
6	$\angle AED = \angle BEC$	Vertical angle
7	$\triangle$ AED $\cong \triangle BEC$	Congruence theorem (side, angle, side) (2nd & 6th step results)
8	$\angle DBC = \angle BDA$ $\angle BAC = \angle ACD$	Alternate interior angles (7th step result)
9	$\overline{AD} \parallel \overline{BC}$	8th step result

Thus, it is proved that if the quadrilateral ABCD, $\overline{AC}$, and $\overline{BD}$ intersect at E, $\overline{AE} = \overline{CE}$, and $\overline{BE} = \overline{DE}$, then $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$.

(ii) $p \rightarrow q$

Given the quadrilateral ABCD with $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$

It will be shown that $\overline{AC}$ and $\overline{BD}$ intersect at E, $\overline{AE} = \overline{CE}$, and $\overline{BE} = \overline{DE}$.

Step	Statement	Reason
1	ABCD with $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$	Given
2	Opposite sides are the same length	1st step result
3	$\overline{AC}$ and $\overline{BD}$ intersect at E	2nd step result
4	$\angle BAC = \angle DCA$	Alternate interior angles
5	$\angle ABD = \angle CDB$	Alternate interior angles
6	$\overline{AB} = \overline{CD}$	2nd step result
7	$\triangle$ ABE $\cong \triangle CDE$	Congruence Theorem (angle, side, angle)
8	$\overline{AE} = \overline{CE}$ $\overline{BE} = \overline{DE}$	7th step result

Thus, if the quadrilateral ABCD is with $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{DC}$, then $\overline{AC}$ and $\overline{BD}$ intersect at E, $\overline{AE} = \overline{CE}$, and $\overline{BE} = \overline{DE}$.

From (i) and (ii), it is evident that the two definitions are equivalent. The process of proving the equivalence of the two definitions submitted by students is different from the focus of this research. Still, the equivalence proof only wants to give confidence to students that the definitions they make have the same truth. The researchers obtained the results of this evidence based on the results of interviews with the subjects. The complete interview can be seen in the Appendix [17-24].

Several research results show that students or teachers still have difficulty proving the two equivalent definitions (Haj-Yahya, 2019; Haj-Yahya et al., 2022; Harel et al., 2006; Leikin & Winicki-Landman, 2001), but S can show that the two definitions conveyed are equivalent because S is a pre-service teacher classified as high ability in geometry, so it is easy to show the two equivalent definitions.

In the interview excerpt, S recalls the concept of a parallelogram by defining a parallelogram as a quadrilateral whose opposite sides are parallel. This definition is supported by Alexander and Koeberlein

(2020) and Usiskin (2008), who state that a parallelogram is a quadrilateral in which the two pairs of opposite sides are parallel. In addition, S also recalls another definition of a parallelogram using the side attribute. This remembering activity is included in the epistemic action of recognizing. Furthermore, S also recognizes the concept of triangle congruence to prove two congruent triangles [18,20]. Then S combines (building-with) the congruence concepts, triangle, quadrilateral, and parallelogram, to prove that these definitions are equivalent. Thus, S has been able to construct two equivalent definitions. Although showing the two definitions are equivalent is not easy, S can do it because S has a high ability in class compared to others in learning geometry. The epistemic activities in constructing that the two definitions are equivalent are *recognising→ building-with→constructing*. Several research results have shown that pre-service teachers need help defining geometry concepts (Haj-Yahya et al., 2022; Marchis, 2012; Pickreign, 2007; Selden & Selden, 2017), but S could provide the definition correctly. Teachers or pre-service teachers must have an understanding of definitions because definitions are the basis for their mathematics teaching, especially in teaching geometry (Johnson et al., 2014).

27. Q: What do you know about a rectangle?
28. S: A rectangle is a parallelogram with one right angle → (**R**)
29. Q: Why do you conclude that rectangles are parallelograms?
30. S: Because all the properties of a parallelogram exist in a rectangle (he draws a rectangle) → (**B**)
31. Q: Can you state the properties of a rectangle?
32. S: $\overline{AB} = \overline{CD}; \overline{AD} = \overline{CD}; \angle ABC = \angle BCD = \angle CDA = \angle BAD$ → (**R**)
33. Q: Can you define a rectangle using diagonals?
34. S: A rectangle is a parallelogram whose diagonals are congruent → (**C**)

In the second interview, S recognizes that a rectangle is a parallelogram with one right angle because it is his remembered knowledge. S has formed a new meaning when S concludes that a rectangle is a parallelogram [28]. Furthermore, S combines properties of rectangles and parallelograms [30]. S also recognizes the sketch and the characteristics of the rectangle [30,32]. When S mentions the characteristics of the rectangle, S also draws the diagonal of the rectangle to make it easier to remember the properties of the rectangle (it can be seen in the Appendix). The image or visualization of objects can influence the construction Fields (Haj-Yahya, 2020; Haj-Yahya et al., 2022). The picture will help S remember the characteristics of the rectangle. Then, S constructs a new definition based on the relationship of properties of both [34]. *Constructing* is the restructuring of knowledge that is the goal (Dreyfus et al., 2002; Haj-Yahya et al., 2016) or what processes of restructuring and reorganization are recognized and known to construct new meanings (Bikner-Ahsbahs, 2004). Before S constructs that rectangles are parallelograms whose diagonals are congruent, S first remembers the definition of a rectangle. Furthermore, S connects the properties of rectangles and parallelograms [30] and recognizes the properties of a rectangle [32] again. Before S performs *the constructing* activity, S performs recognizing and *building-with* activities. The epistemic activities in constructing that a rectangle is a parallelogram with one right angle are *recognizing→ building-with → recognizing→ constructing.*

The definition put forward by S is also by Alexander and Koeberlein (2020) that a rectangle is a parallelogram with one right angle. In addition, S can also define a quadrilateral by using the definition of genesis or genetics (how the object was formed) (Molland, 1976; Sinclair et al., 2012; Soedjadi, 2000). In addition, S also constructs that the set of rectangles is a subset of the parallelogram [36].

37. Q: Next, what do you know about squares?
38. S: A square is a rectangle with all sides congruent → (**R**)
45. Q: Can you define it using a diagonal?
46. S: The properties of a square are the same as a rectangle → (**B)**. Then a square is a rectangle whose diagonals are perpendicular → (**C**)
51. Q: Why?
52. S: Because all the properties of a rectangle exist in the square, but there are properties of the square that do not exist in the rectangle → (**B**)

The interview snippet above shows that S has recognized that a square is a rectangle with all sides congruent [38] because it is his initial knowledge. This activity is the same as the previous activity when S constructs that a rectangle is a parallelogram with one right angle. S also remembers the properties of a square [42,44]. Furthermore, S connects the properties of a square and a rectangle and finds a new construct: a square is a rectangle whose diagonals are perpendicular [46]. S constructs it after S remembers the definition and properties of a square [42,44] and connects square and rectangle properties [46]. Then, S combines the definition and properties of a rectangle with a square diagonal to define a square using the diagonal attribute [46]. These activities are also sequentially involved in *constructing and building-with* activities. This definition is submitted by Usiskin (2008) and Alexander and Koeberlein (2020), that a square is a rectangle whose sides are congruent. Furthermore, S constructs that the rectangle is not a square, but a square is a rectangle. After S has constructed, S tries to explain why a square is a rectangle. S says that all the rectangle properties are in the square, but not all square properties are in a rectangle [52]. At this stage, S did the building-with activity. The epistemic activities in constructing that a square is a rectangle with all sides congruent are *recognising→building-with →constructing.*

53. P: Next, we go to the rhombus. Can you define a rhombus?
54. S: A rhombus is a parallelogram with all sides congruent → (**R**)
63. Q: How do you define a rhombus with a diagonal?
64. S: A rhombus is a parallelogram with perpendicular diagonals and bisects the same length because the properties of a parallelogram exist in the rhombus, but there are properties of the rhombus that do not exist in the parallelogram. → (**B,C**)

The process of constructing relationships of rhombus and parallelogram carried out by S is the same as when constructing a relationship of square and rectangular, *recognising→building-with→constructing*. The thinking flow in constructing these two cases is the same. S constructs it after he remembers the definition of a rhombus [54]. S connects the properties of a rhombus and parallelogram and combines the definition and properties of a parallelogram with a rhombus diagonal to define a rhombus using the diagonal attribute [64]. Next, S constructs that a rhombus is a parallelogram with perpendicular diagonals and bisects the same length [64]. The definition conveyed is the same as one of the definitions submitted by Usiskin (2008) and Alexander and Koeberlein (2020), that a rhombus is a parallelogram in which all sides are congruent.

65. Q: What do you know about a kite?
66. S: A kite is a quadrilateral whose adjacent sides are congruent (S draws a kite) → (**R**)
67. Q: How do you define it using a diagonal?
68. S: A kite is a quadrilateral whose diagonals are perpendicular → (**B,C**)
69. Q: Is a rhombus a kite?
70. S: Yes, that is right, a rhombus is a kite, but a kite is not necessarily a rhombus → (**C**)

The interview results show that S recognizes the kite's sketch, definition, and properties. S also combined the definition of a kite with a diagonal to define a kite using the diagonal attribute [68]. These two activities are also sequentially involved in *building-with* and *constructing* activities. This definition is submitted by Usiskin (2008) and Alexander and Koeberlein (2020) that a kite is a quadrilateral with two pairs of adjacent sides of the same length. Furthermore, S constructs that a rhombus is a kite [70], but a kite is not necessarily a rhombus, and he also constructs again that a square is a kite. S gives the reason why a square is a kite. S says the adjacent sides are congruent in a square, and the diagonals are perpendicular, so S concludes that the square is a kite [72]. At this stage, S did the *constructing* activity. S remembers square, rhombus, and kite properties and combines them to construct that a rhombus is a kite and a square is a kite before S constructs them. The epistemic activities in constructing them are *recognising→ building-with →constructing.*

73. Q: What is a trapezoid?
74. S: A trapezoid is a quadrilateral with one of the opposite sides parallel (he draws a trapezoid) → (**R**)
75. Q: Is a parallelogram a trapezoid?
76. S: Yes → (**C**)
77. Q: Why?
78. S: Because all the properties of a trapezoid are in a parallelogram → (**B**)

The interview results indicate that S recalls the trapezoid's sketches, definitions, and properties. From the proposed trapezoid definition, S makes a construction that parallelogram, rectangle, and rhombus are trapezoid [76-82]. However, S has yet to find another trapezoid definition. Furthermore, S constructs the relationship among the quadrilaterals, as shown in Figure 1. This definition is almost the same as the definition expressed by Usiskin (2008): a trapezoid is a quadrilateral with at least one pair of parallel sides. Nevertheless, Usiskin (2008) and Alexander and Koeberlein (2020) have another definition of a trapezoid, a quadrilateral with exactly one pair of parallel sides. S has yet to learn this definition. This definition has yet to be discovered by S. This shows that it is not easy to construct a definition, as some research results show that students or pre-service teachers need help constructing geometric definitions (Haj-Yahya et al., 2022; Marchis, 2012; Pickreign, 2007; Selden & Selden, 2017).

87. Q: Is a parallelogram a trapezoid?
88. S: Yes, this does not seem right → (**R**)

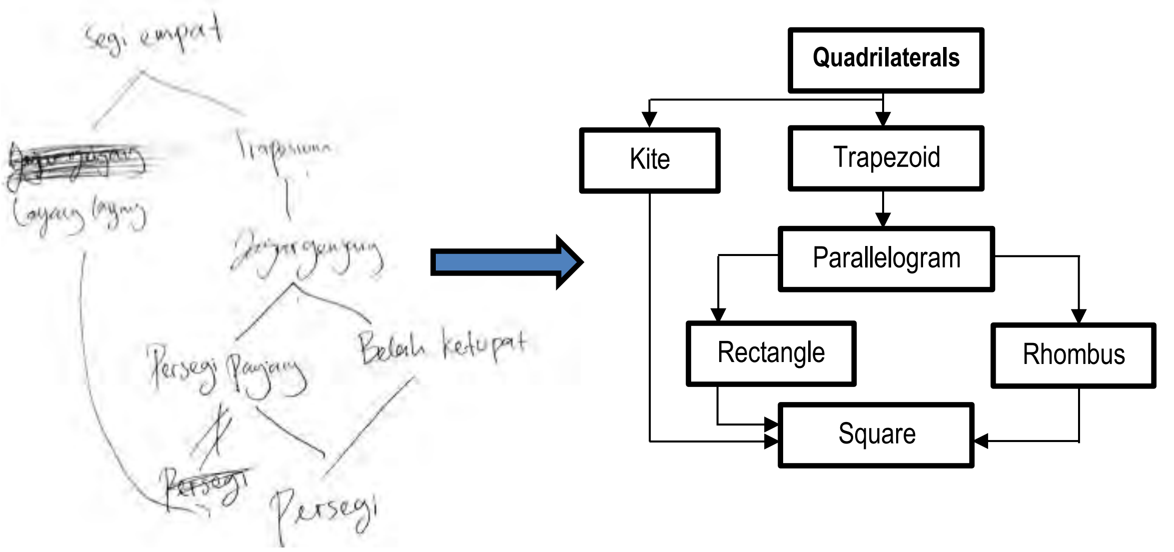

Figure 2. The First Construction Result of Relationships among Quadrilaterals

In Figure 2, S made mistakes and crossed them out. S writes a parallelogram below the quadrilateral that should be below a trapezoid and then S corrects it as shown in the Figure 3.

95. Q: Is a rhombus a kite?
96. S: Yes, yes… let me draw again (Figure 3) → (**C**)
97. Q: What are you thinking when creating this figure (Figure 3)?
98. S: I use conclusions that have been made previously (**R**). After that, I make a connection (**B**).

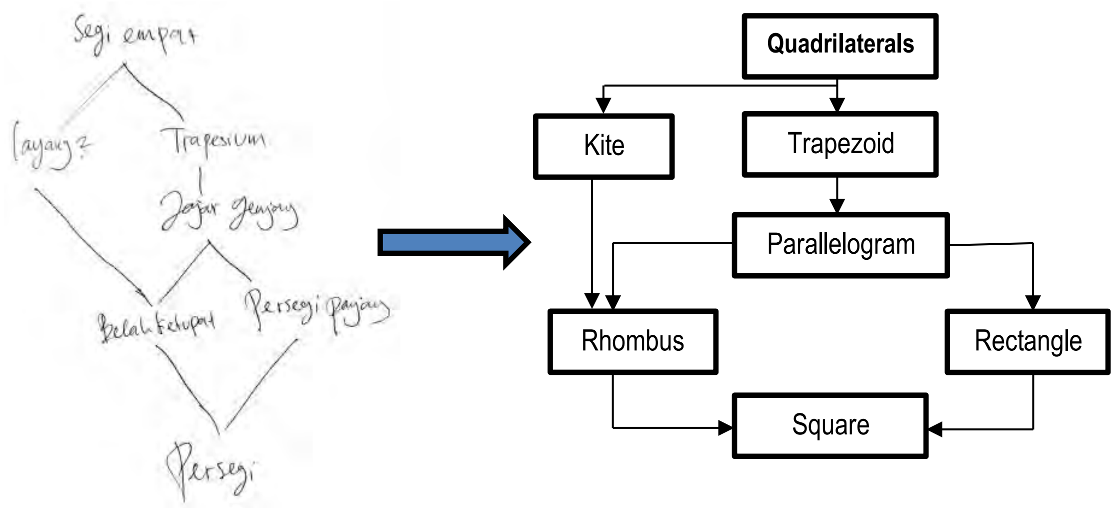

Figure 3. The Second Construction Result of Relationships among Quadrilaterals

Furthermore, S also performs *recognizing* activities by saying that a parallelogram is a trapezoid [88]. S remembers the previous construction results disclosed when constructing the relationship between the parallelogram and the trapezoid. In Figure 2, S does not combine the rhombus and the kite, the kite and the square, so the construction needs to be corrected. The exact construction results made by S can be seen in Figure 3. The construction results obtained in Figure 3 show the relationship structure among quadrilaterals if the definition of a trapezoid used is a quadrilateral with a pair of parallel opposite sides. The construction of the relationship among quadrilaterals in Figure 3 is a new structure created by S using the previous constructs. Thus, the *constructing* actions that produce these constructs are nested in more complex constructing actions (Hershkowitz et al., 2014). The abstraction is a vertical reorganization activity of mathematics that was previously built into a new mathematical structure (Dreyfus, 2007; Hershkowitz et al., 2001). So, the new structure is the relationship among quadrilaterals, as seen in Figure 3. The squares, rhombuses, rectangles, and parallelograms are trapezoids, so their area can be found using the area formulation of the trapezoid [102]. To produce the construction in Figure 3, S requires the previously formed constructions discussed earlier, such as the construction that a parallelogram is a trapezoid. S will not produce a construction like Figure 3 if he has no previous knowledge or constructs related to Figure 3. *Constructing* a new structure depends on understanding the concept and its relation to the last construct (Guler & Arslan, 2015).

103. Q: Does the formula for the area of a trapezoid apply to a parallelogram?
104. S: Yes, it does
105. Q: Can you show?
106. S: (Student draws and proves, and the result can be seen in Figure 4) → (**+C**)
107. Q: Do you hypothesize that all of the following apply?
108. S: Yes → (**+C**)

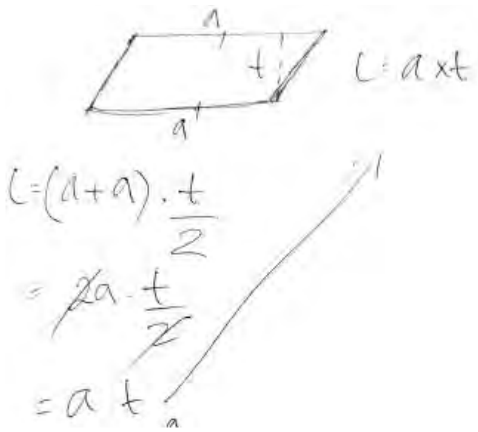

Figure 4. The Consolidation Result

After S constructs relationships among quadrilaterals, S can also apply the new structure to a problem as a follow-up to the results of the construction that has been made so that the results are more precise and strengthen what S has previously constructed. S says that a parallelogram is a trapezoid, and S can show that the area of a parallelogram can be found using a trapezoid [103-104], and the results can be seen in Figure 4. This activity is included in the consolidation. Consolidation is characterized by reorganizing previous constructs with higher confidence while utilizing earlier constructs in new activities (Güler & Arslan, 2017). The construct formed by students should provide an opportunity to consolidate knowledge (Güler & Arslan, 2017; Tabach et al., 2006). The construct results can later be used for further abstraction or solving problems. Based on the abstraction of relationships among quadrilaterals, S uses the result to solve the given problem. S shows that the area of a parallelogram can be found using a trapezoid formula based on the previous abstraction results.

Before S made the construction in Figure 3, S remembers previous constructions. After that, S makes several connections among them. S always looks back at the results of the last construction to combine them. Therefore, S will return to the recognizing activity after S carries out the building-with activity to create the construction result. The process of constructing relationships among quadrilateral is *recognising→building-with→ recognising→constructing,* and an illustrates the process can be seen in Figure 5. Furthermore, the result of the construction is continued in consolidation. This result reinforces the previous findings that these three epistemic actions of knowledge acquisition are nested (Bikner-Ahsbahs, 2019). In the previous illustration, it was not combined with consolidation. This research emphasizes three nested epistemic actions based on the research results obtained. Figure 5 shows that construction activities include recognizing and building-with while building-with activities include recognizing, but consolidation will be carried out after construction.

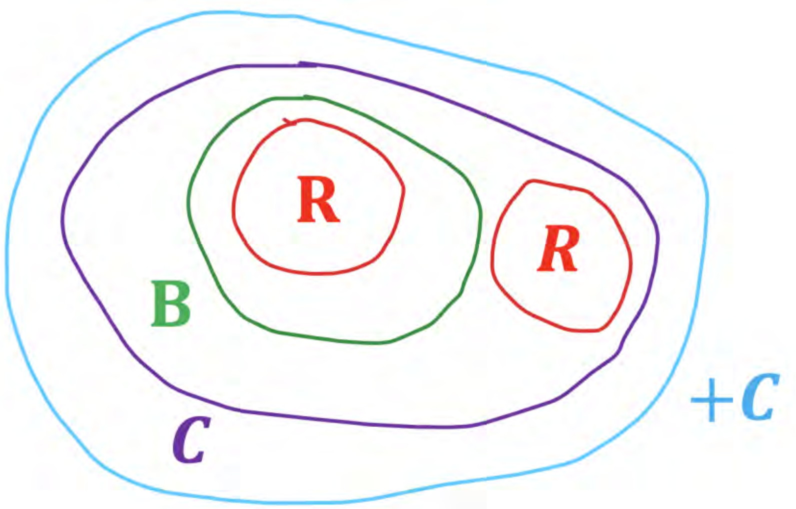

Figure 5. Illustration of the Nested Structure of the RBC Adapted from Bikner-Ahsbahs (2019)

109. Q: If there is a trapezoid definition, it is a quadrilateral with exactly one pair of parallel opposite sides. What do you think? Is a parallelogram a trapezoid?
110. S: If there is only one pair, then the parallelogram is not a trapezoid → (**C**)
115. Q: Can you make a construction if a trapezoid is defined as a quadrilateral that has exactly one pair of parallel opposite sides?
116. S: (*He draws)*

Based on the interview above, if a trapezoid is a quadrilateral with exactly one pair of parallel opposite sides, S concludes that a parallelogram is not a trapezoid. Hence, squares and rectangles are not trapezoids either. Furthermore, S constructs a relationship among quadrilaterals if the definition of a trapezoid used is a quadrilateral with precisely a pair of parallel opposite sides. The results of S construction can be seen in Figure 6. S also says the construction results are the same as in Figure 6 when viewed based on the diagonal.

In addition, these findings show that S can consolidate (Figure 4) based on the results of his construction (Figures 3 and 6) even though previous research only covered construction (Budiarto et al., 2017). Thus, these findings can be used as a study for mathematics education study programs and curriculum planners in preparing the curriculum, especially in preparing Courses Learning Outcomes (CLO) referring to Learning Outcomes (LO), especially in consolidating a concept or solution from a solved problem. Consolidation is essential in CLO to strengthen the student's understanding of what has been constructed or the problem solved (Memnun et al., 2017).

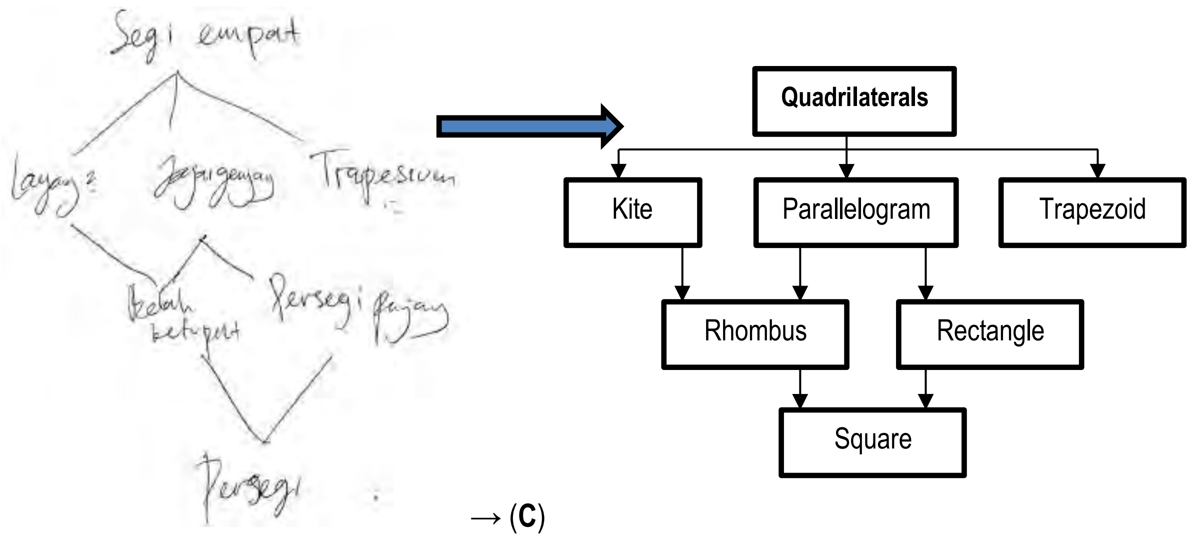

Figure 6. Results of the Third Construction of Relationships among Quadrilaterals

These findings also suggest that mathematics education lecturers must design learning and tasks that encourage students to construct and consolidate to make learning more meaningful. Students need previous knowledge in constructing and consolidating. The learning is meaningful if students connect prior knowledge with the inside being studied (Agra et al., 2019; Lemos, 2007).

CONCLUSION

In this research, AiC is a theoretical and methodological framework that posits a specific set of epistemic actions: *Recognizing, Building-with, Constructing*, and *Consolidation* (RBC+C), as essential for describing and analyzing the emergence and reinforcement of a new concept (Dreyfus, 2012; Tabach et al., 2017). Therefore, to construct relationships among quadrilaterals, one can describe and analyse them with RBC+C. The results of this study also show that the RBC epistemic actions are not a tiered action that must start from *recognising→building-with→constructing*. Still, it is possible that after the building-with activity, they will return to recognizing activity, *recognising→building-with→recognising→constructing→consolidation*. Students must learn the concept of each quadrilateral to construct relationships among quadrilaterals, which can be constructed to become more complex.

Further research is needed on abstraction for students with high, medium, and low geometry abilities. Future researchers can also examine abstractions on different subjects, such as students with the same material or other mathematical concepts, using RBC+C. In addition, further research is also necessary to pay attention to the subject's gestures because S often uses his hand, arm and finger gestures during the interview. Moreover, gestures can help and support students in thinking (Arzarello et al., 2009), especially in abstraction because epistemic actions can be observed from speech, student actions with artifacts, gestures, and others (Bikner-Ahsbahs, 2019). Furthermore, researchers can study abstractions in constructing relationships among quadrilaterals in collaboration or group discussions of three to five students. This research result also provides recommendations to lecturers, mathematics education study programs, and curriculum planners in designing geometry learning strategies and geometry curricula, especially for constructing concepts and solving problems of geometry.

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Declarations

Author Contribution	:	H: Conceptualisation, Writing-Original Draft, Visualisation, Data Curation, Methodology, and Investigation. MTB: Conceptualisation, Investigation, Methodology, and Supervision. RE: Conceptualisation, Writing-Original Draft, Data Curation, Formal analysis, and Methodology. GS: Writing-Review & Editing, Investigation, Visualisation, Resources, and Validation. JK: Validation and Formal Analysis. EB: Writing-Review & Editing, Resources, and Validation.
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Conflict of Interest	:	The authors declare no conflict of interest.

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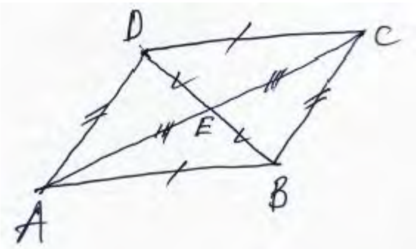
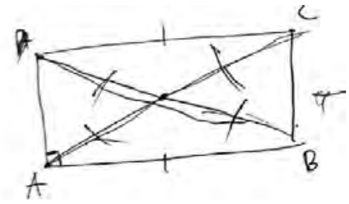
APPENDIX

1. Q: What do you know about quadrilaterals?
2. S: A quadrilateral formed of four intersecting lines where no three lines intersect at one point ⟶ (**R**)
3. Q: Have you ever heard of polygons?
4. S: Yes
5. Q: Can you define it using polygons?
6. S: A quadrilateral is a polygon that has four sides ⟶ (**R**)
7. Q: Try defining a quadrilateral using diagonals!
8. S: A quadrilateral is a polygon that has intersectional diagonals ⟶ (**B**)
9. Q: What do you know about parallelograms?
10. S: A parallelogram is a quadrilateral whose opposite sides are the same length (**R**)
11. Q: Is there another definition of the parallelogram?
12. S: A parallelogram is a quadrilateral whose diagonals bisect each other ⟶ (**R**)
13. Q: Can you define a parallelogram with the side attribute?
14. S: A parallelogram is a quadrilateral whose opposite sides are parallel ⟶ (**R**)
15. Q: Can you prove that the two definitions (using side and diagonal attributes) are equivalent?
16. S: Yes, I can (he draws a parallelogram)

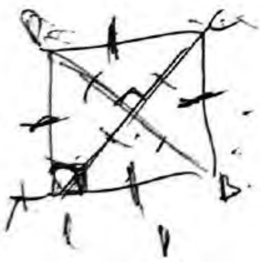


17. Q: So, what do you want to prove?
18. S: Two congruent triangles ⟶ (**C**)
19. Q: To prove the two triangles, what theorem do you use?
20. S: Two congruent triangles can be proved by congruence theorems (side, angle, side; angle, side angle; side, side, side) ⟶ (**R**) and I can use side, angle, side (he shows that corresponding sides are equal and vertical angles are equal in Figure 1, $\overline{AE} = \overline{CE}$, $\overline{BE} = \overline{DE}, \angle AEB = \angle DEC$) ⟶ (**B**)
21. Q: Then what are the implications?
22. S: Yes, alternate interior angles are congruent (S points to angles of $\angle DCA = \angle BAC$ and $\angle BDC = \angle ABD$) ⟶ (**C**)
23. Q: So what are the consequences?
24. S: These two lines are parallel (while pointing to the line referred to in Figure 1, $\overline{AB} \parallel \overline{CD}$) ⟶ (**C**)
25. Q: Let's now discuss a rectangle!
26. S: Yes, Sir.
27. Q: What do you know about a rectangle?
28. S: A rectangle is a parallelogram with one right angle ⟶ (**R**)

29. Q: Why do you conclude that rectangles are parallelograms?
30. S: Because all the properties of a parallelogram exist in a rectangle (he draws a rectangle) → (**B**)

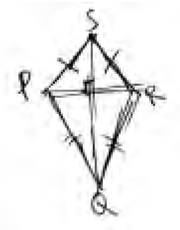


31. Q: Can you state the properties of a rectangle?
32. S: $\overline{AB} = \overline{CD}; \overline{AD} = \overline{CD}; \angle ABC = \angle BCD = \angle CDA = \angle BAD$ → (**R**)
33. Q: Can you define a rectangle using diagonals?
34. S: A rectangle is a parallelogram whose diagonals are congruent → (**C**)
35. Q: What can you conclude if there is a parallelogram and rectangle set?
36. S: Then the set of rectangles is a subset of the parallelogram → (**C**)
37. Q: Next, what do you know about squares?
38. S: A square is a rectangle with all sides congruent → (**R**)
39. Q: Is that all?
40. S: Yes (he draws it)



41. Q: What about the angles of the square?
42. S: Right angle → (**R**)
43. Q: Can you mention the properties of a square?
44. S: $AB = CD = BC = AD, \angle BAD = \angle BCD = \angle CDA = \angle ABC \rightarrow$ (**R**)
45. Q: Can you define it using a diagonal?
46. S: The properties of a square are the same as a rectangle → (**B**). Then a square is a rectangle whose diagonals are perpendicular → (**C**)
47. Q: Can a rectangle be considered a square?
48. S: No → (**C**)
49. Q: Can a square be considered a rectangle?
50. S: Yes → (**C**)
51. Q: Why?
52. S: Because all the properties of a rectangle exist in the square, but there are properties of the square that do not exist in the rectangle → (**B**)
53. P: Next, we go to the rhombus. Can you define a rhombus?
54. S: A rhombus is a parallelogram with all sides congruent → (**R**)
55. Q: What is different from a square?

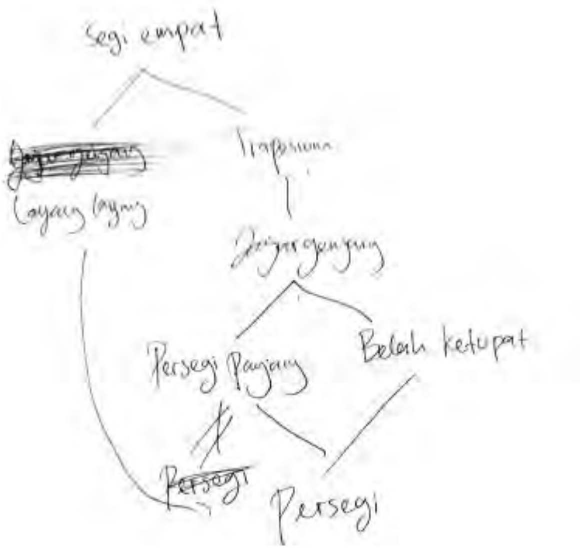
56. S: A square has a right angle, but the rhombus does not have to be → (**B**)
57. Q: Is a rhombus a square?
58. S: No → (**C**)
59. Q: Is a square a rhombus?
60. S: Yes
61. Q: Why?
62. S: The properties of a rhombus are in a square, but there are properties of a square that are not in a rhombus (one of the angles is a right angle) → (**B**)
63. Q: How do you define a rhombus with a diagonal?
64. S: A rhombus is a parallelogram with perpendicular diagonals and bisects the same length becouse the properties of a parallelogram exist in the rhombus, but there are properties of the rhombus that do not exist in the parallelogram→ (**B**,**C**)
65. Q: What do you know about a kite?
66. S: A kite is a quadrilateral whose adjacent sides are congruent (S draws a kite) → (**R**)



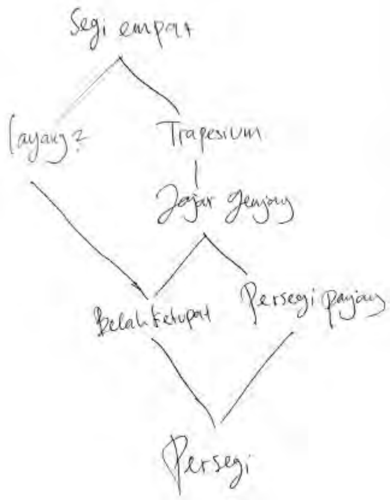
67. Q: How do you define it using a diagonal?
68. S: A kite is a quadrilateral whose diagonals are perpendicular → (**B**,**C**)
69. Q: Is a rhombus a kite?
70. S: Yes, that is right, a rhombus is a kite, but a kite is not necessarily a rhombus → (**C**)
71. Q: Is a square a kite?
72. S: Yes, because adjacent sides are the same length, and the diagonals are perpendicular → (**C**).
73. Q: What is a trapezoid?
74. S: A trapezoid is a quadrilateral with one of the opposite sides parallel (he draws a trapezoid) → (**R**)

75. Q: Is a parallelogram a trapezoid?
76. S: Yes → (**C**)
77. Q: Why?
78. S: Because all the properties of a trapezoid are in a parallelogram → (**B**)
79. Q: Is a rectangle a trapezoid?
80. S: Yes → (**C**)

81. Q: Is a trapezoidal a rhombus?
82. S: Yes
83. Q: Are there other definitions of a trapezoid that you know?
84. S: (no answer)
85. Q: Can you make a relationship among the quadrilaterals?
86. S: (He draws)
87. Q: Is a parallelogram a trapezoid?
88. S: Yes, this does not seem right → (**R**)
89. P: Cross it out
90. S: Maybe this is a kite (*he replaces the parallelogram with a kite*) → (**R**)

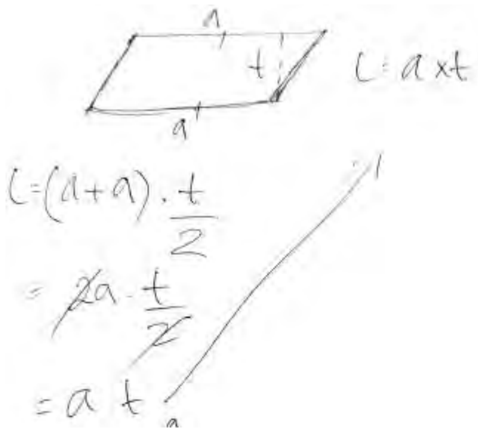


91. Q: Is a square a rhombus?
92. S: Yes (C)
93. Q: Is a square a kite?
94. S: Yes (C)
95. Q: Is a rhombus a kite?
96. S: Yes, yes… let me draw again → (**C**)

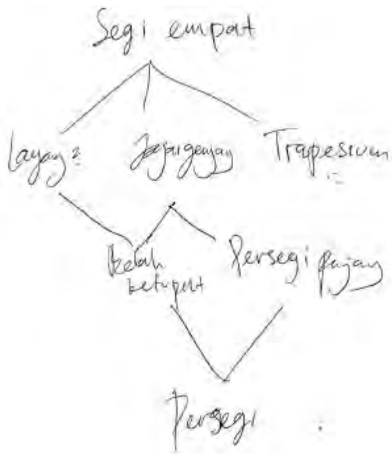


97. Q: What are you thinking when creating this figure?
98. S: I use conclusions that have been made previously (**R**). After that, I make a connection (**B**).

99. Q: So, in your opinion, is the downward parallelogram a trapezoid?
100. S: Yes
101. Q: Do you still remember the trapezoidal formula?
102. S: Sum of opposite sides multiplied by the height divided by 2.
103. Q: Does the formula for the area of a trapezoid apply to a parallelogram?
104. S: Yes, it does
105. Q: Can you show?
106. S: (Student draws and proves, and the result can be seen below) → (**+C**)
107. Q: Do you hypothesise that all of the following apply?
108. S: Yes → (**+C**)



109. Q: If there is a trapezoid definition, it is a quadrilateral with exactly one pair of parallel opposite sides. What do you think? Is a parallelogram a trapezoid?
110. S: If there is only one pair, then the parallelogram is not a trapezoid → (**C**)
111. Q: What about rectangles?
112. S: Same → (**C**)
113. Q: Square?
114. S: Same
115. Q: Can you make a construction if a trapezoid is defined as a quadrilateral that has exactly one pair of parallel opposite sides?
116. S: (*He draws)*



→ (**C**)