Journal on Mathematics Education
Volume 15, No. 2, 2024, pp. 639-660

Assessing knowledge to teach early algebra from the Mathematical Knowledge for Teaching (MKT) perspective: A support tool for primary school teachers

Nataly Pincheira*, Ángel Alsina

Department of Subject-Specific Didactics, University of Girona, Girona, Spain
*Correspondence: nataly.pincheira@udg.edu

Received: 4 December 2023 | Revised: 17 February 2024 | Accepted: 21 March 2024 | Published Online: 3 April 2024

Abstract

Research about the optimal methodologies for guiding the training of primary school instructors in the instruction of early algebra remains in the process of determination. This manuscript delineates the methodology involved in developing and validating an assessment tool to evaluate the mathematical acumen of prospective primary educators in the domain of early algebra during their initial pedagogical training, drawing upon the constructs delineated in the Mathematical Knowledge for Teaching (MKT) model. To this end, an instrumental inquiry comprising four distinct phases has been executed: an exhaustive review of extant literature concerning the mathematical proficiency of primary school instructors and the pedagogy of early algebra; formulation of the preliminary version of the assessment instrument; validation of said instrument via expert appraisal and a preliminary application involving ten pre-service primary educators enrolled in a Spanish academic institution; and subsequent refinement and finalization of the assessment tool. This endeavor's outcome culminated in the MKT-Early Algebra Questionnaire (6-12-year-olds), comprising six open-ended inquiries meticulously designed to explore diverse facets of early algebra while aligning with the various sub-domains delineated in the MKT model. It is deduced that the resultant instrument holds promise as an effective diagnostic apparatus, serving dual purposes: elucidating the mathematical proficiency of primary school educators in the context of early algebra and fostering introspection regarding pedagogical strategies conducive to the cultivation of algebraic cognition at this developmental juncture. By furnishing a comprehensive questionnaire that systematically addresses all facets of pedagogical knowledge requisite for the effective instruction of early algebra, this study furnishes invaluable insights into the specific components that should be integrated into teacher education curricula in the domain of algebraic didactics.

Keywords: Assessment Instrument, Early Algebra, Mathematical Knowledge for Teaching, Pre-Service Teachers, Primary Education

How to Cite: Pincheira, N., & Alsina, Á. (2024). Assessing knowledge to teach early algebra from the Mathematical Knowledge for Teaching (MKT) perspective: A support tool for primary school teachers. *Journal on Mathematics Education, 15*(2), 639-660. http://doi.org/10.22342/jme.v15i2.pp639-660

The incorporation of algebra only in the secondary education curriculum and the problems that this has generated in the development of mathematical abstraction and generalization in students has led to the idea of introducing knowledge of an algebraic nature and promoting algebraic thinking from the first levels of schooling (Schliemann et al., 2012).

This curricular transformation has given rise to Early Algebra as an approach that cultivates mental habits that focus on mathematics' deeper and underlying structure (Blanton & Kaput, 2005). According to Katz (2007), these mental habits consider two central characteristics: a) generalizing, identifying, expressing, and justifying mathematical structure, properties, and relationships, and b) promoting reasoning and actions based on forms of generalization. Likewise, several authors (e.g., Blanton et al., 2015; Kaput, 2008) suggest different contents that allow us to consider the treatment that should be given to the study of algebra in primary education: generalized arithmetic, functional thinking, equivalence, expressions, equations, inequalities, variables, and more.

Consequently, to address the characteristics proposed by Katz (2007) and considering the contributions of the literature (e.g., Blanton et al., 2015; Kaput, 2008), Pincheira and Alsina (2021a) establish an initial characterization of the knowledge involved in solving tasks that promote algebraic thinking in primary education as a dynamic dimension, which emerges from the review of the curricula of various countries that explicitly consider early algebra from the early stages of schooling: a) understand the different types of relationships and patterns; b) use algebraic symbols and mathematical models to represent situations; c) understand change; and d) use variables to determine a constant or unknown. This characterization is also supported by the contributions of various authors, who underscore how early algebra offers students the opportunity to analyze the relationships between quantities intuitively and informally, identify structures from the generalization of the properties of arithmetic, explore patterns, study change and functions (e.g., Kieran, 2004), express equivalences and equations from a relational understanding of the equal sign, and use variables (e.g., Carraher & Schliemann, 2019; Molina et al., 2009), which is related to different modes of algebraic thinking, primarily relational and functional thinking.

According to Strand and Mills (2014), teachers who introduce early algebra teaching are responsible for facilitating their students' ability to build their algebraic understanding. However, it is likely that in-service and pre-service teachers have yet to have opportunities to explore early algebra during their time in primary education. Therefore, the only teaching and learning experience they have in this regard is the one they receive during their training process.

Elsewhere, Ball et al. (2005) state that teachers who possess mathematical knowledge for teaching can better promote student learning, which makes it necessary to investigate the mathematical knowledge of early algebra of pre-service primary education teachers. Hohensee (2017) argues that there needs to be more research to guide how to train primary school teachers to teach early algebra. Therefore, "teacher training involves developing, in addition to appropriate teaching situations that promote reflection and evolution of their knowledge of elementary algebra, instruments for assessing the state of their knowledge" (Godino et al., 2015a, p.128). Consequently, assessment tools must be developed to characterize the mathematical knowledge possessed by primary school teachers to provide instruction in this content block.

From this perspective, our study aims to build and validate an instrument to assess the mathematical knowledge of early algebra of primary education teachers during their initial training. To do so, we relied on the Mathematical Knowledge for Teaching (MKT) model proposed by Ball et al. (2008).

Mathematical Knowledge for Teaching

The notion of knowledge for teaching was introduced by Shulman (1986) in response to what he called a blind spot regarding the subject of study of research on teaching training and knowledge. The contributions of Shulman (1986; 1987) gave rise to a solid field of research on what teachers know and how they think about specific content.

Based on these ideas, Ball et al. (2008) identified a domain map of mathematical comprehension and skill, forming the MKT model. This model emerges as the result of an attempt to redefine and empirically validate teachers' knowledge to teach mathematics.

The MKT is defined as “mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” (Ball et al., 2008, p.399) and considers two significant domains of knowledge (Figure 1): subject matter knowledge and pedagogical content knowledge, each made up of different subdomains.

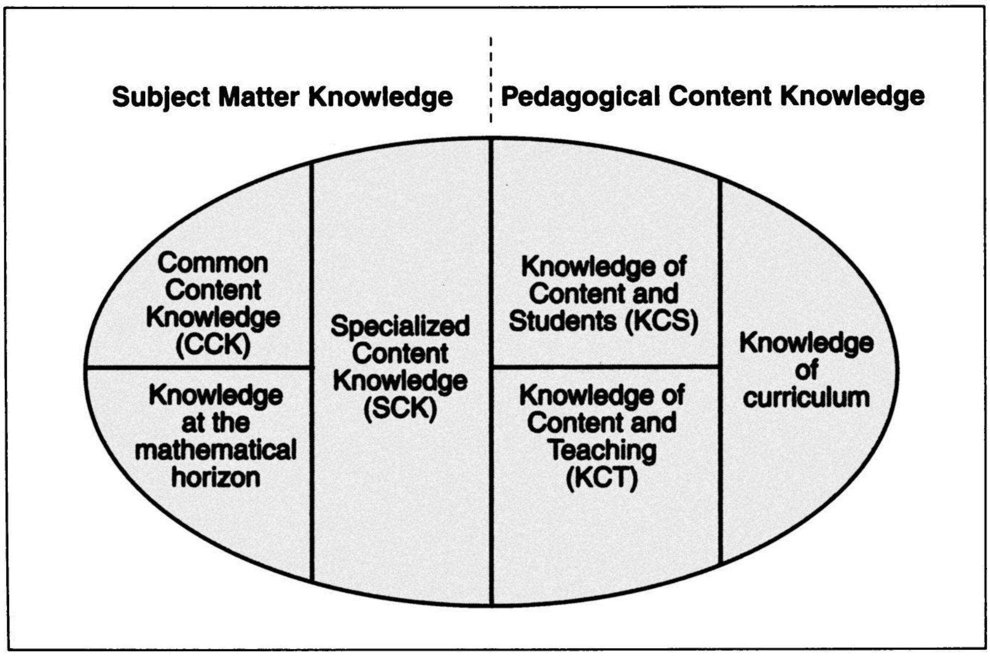


Figure 1. Mathematical Knowledge for Teaching (MKT) (Hill, Ball, & Schilling, 2008, p.377)

Subject matter knowledge considers: common content knowledge (CCK), defined as “mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p.399), meaning it corresponds to a type of knowledge that is used in a wide variety of settings to correctly solve mathematical problems; specialized content knowledge (SCK), which is “mathematical knowledge and skill unique to teaching” (Ball et al., 2008, p.400), meaning it is knowledge that is not normally used for purposes other than teaching and is related to how to accurately represent mathematical ideas, as well as to provide mathematical explanations, rules and procedures (Hill et al., 2008); and horizon content knowledge, which is related to understanding the connections between the mathematical topics that comprise the curriculum (Ball & Bass, 2009).

On the other hand, pedagogical content knowledge consists of knowledge of content and students (KCS), which is "knowledge that combines knowing about students and knowing about mathematics" (Ball et al., 2008, p.401); more specifically, it refers to the knowledge that teachers have about the conceptions and common misconceptions of students, as well as the specific difficulties they face in covering specific mathematical content; knowledge of content and teaching (KCT) is that which "combines knowledge about teaching and knowing about mathematics" (Ball et al., 2008, p.401), meaning it is the knowledge that teachers have about how to design instruction and make decisions about instruction, such as selecting examples, materials, methods or teaching techniques; and finally, knowledge of content and curriculum as it relates to knowledge of the content involving the study plan designed for each educational level in the area of mathematics.

The categorization of the subdomains of the MKT model can be used to examine the teachers' knowledge in the practical aspect (Ng, 2011). However, this categorization reflects various teaching styles or approaches, whether in a whole-class discussion, a written assignment, or a questionnaire (Ball et al., 2008).

Development of Assessment Tools and Previous Research on the Knowledge of Pre-Service Primary School Teachers and In-Service Teachers to Teach Early Algebra

Studies of the knowledge of primary school teachers about early algebra and its teaching are still being determined. These studies have contributed to the development of some tools that have been approached from different theoretical perspectives of the teacher's knowledge, providing evidence of the knowledge that pre-service and in-service teachers must handle tasks of an algebraic nature and guide their teaching.

Using the Onto-semiotic Approach to Cognition and Mathematical Instruction-*EOS* (Godino et al., 2007), for example, Castro (2011) evaluates the didactic analysis skills of pre-service primary school teachers involving elementary algebraic reasoning tasks in the context of designing a teaching unit. The activities related to algebraic reasoning proposed by the pre-service teachers suggest an arithmetic character mainly, followed by activities of a geometric and measurement nature. It also notes that pre-service teachers must fully prepare to include early algebra in the elementary school curriculum. Along the same lines, Aké (2013) evaluates the elementary algebraic reasoning of 40 pre-service primary school teachers based on a questionnaire with eight open-ended items focusing on arithmetic problems of multiplicative structure, pattern identification, properties of operations, and algebraic modeling. The results suggest that they are unfamiliar with the processes of developing algebraic ideas, considering the properties and relationships underlying elementary mathematical activities.

From the perspective of the didactic-mathematical knowledge model (Godino, 2009), for example, Godino et al. (2015b) evaluate elementary algebraic reasoning in 597 pre-service primary school teachers by administering a questionnaire with ten open-ended items that delve into algebraic content and didactic content involving structures, functions, and modeling. In their analysis of algebraic knowledge, they identify gaps in the teachers' knowledge of equations, relationships, and functions, while in their analysis of teaching knowledge, they note a deficit in epistemic knowledge and knowledge of instructional aspects. Similarly, Mejías (2019) evaluates the teaching-mathematical knowledge for teaching algebra by administering a questionnaire with eight open-ended items focusing, as in the previous study, on the notion of structure, functional thinking, and modeling, applied to 121 in-service primary school teachers, determining that this knowledge is insufficient. Their limitations are related to the scarce mathematical argumentation versus an algebraic justification or interpretation and deficiencies in the treatment of algebraic content.

From the perspective of the MKT model (Ball et al., 2008), the systematic review conducted by Pincheira and Alsina (2021b) shows that the studies carried out based on this knowledge model focus mainly on in-service teachers. Wilkie (2014), for example, analyzes the mathematical knowledge for teaching functional thinking of 105 in-service teachers through an open-ended questionnaire with six items on tasks involving functions, relationships, and variations that delve into four subdomains of the MKT model: specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of the curriculum. The analysis of the responses revealed that two-thirds of the teachers demonstrate knowledge of content in pattern generalization tasks. However, less than half demonstrated a reasonable pedagogical knowledge of content, especially when providing suitable examples for developing functional thinking.

Trivilin and Ribeiro (2015) analyze the specialized content knowledge and knowledge of the curriculum exhibited by ten in-service teachers involving the equal sign by administering an open-ended questionnaire, which revealed limitations in recognizing its meanings, such as the notion of operation when identifying the use of the equals sign as a symbol in mathematical operation sequence tasks, and the notion of equivalence when identifying tasks that refer to the use of the equals sign to relate two different representations of the same mathematical object. They also identified difficulties in determining the implications of teaching the different meanings of the equal sign in the curriculum. However, along these same lines, Barboza et al. (2020) and Barboza et al. (2021) report remarkable progress in the development of mathematical knowledge in a group of 6 in-service teachers through an intervention when moving between the operational, equivalence, and relational meaning of the equal sign.

Ferreira et al. (2017) identify the mathematical knowledge of 14 in-service teachers when discussing tasks with algebraic potential. The results show little familiarity with core questions of algebraic thought related to the generalization of arithmetic, such as number relationships, the properties of operations, and the meanings of the equal sign. Deficiencies are noted in identifying errors and recognizing the nature of an error.

As for studies that analyze the mathematical knowledge of pre-service primary school teachers, McAuliffe and Lubben (2013) analyze a teacher's performance when designing and teaching an early algebra lesson on patterns. These authors note the difficulty of helping students shift from focusing solely on the number pattern to simultaneously focusing on the function, a central transition in early algebra teaching.

Bernardo et al. (2017) apply an open-ended questionnaire to access the common content knowledge and specialized content knowledge that 60 pre-service teachers must interpret students' output in the context of an algebraic task. The results show the difficulties in assigning the semantic meaning involved in the students' solution to an equitable distribution task. Likewise, Zapatera and Callejo (2017) used two open-ended questionnaires to analyze the mathematical knowledge of 40 pre-service teachers in the context of a pattern generalization task. The first focuses on specific questions of the task, and the second involves the analysis of responses from three students, obtaining, as a result, a low level of specialized knowledge since they exhibit difficulties identifying the mathematical elements used by students and in abstracting observed regularities to interpret the characteristics of understanding generalization.

Oliveira et al. (2021) analyzed the aspects of functional thinking presented by 164 pre-service teachers at the beginning of their training program by using an open-ended questionnaire with three items. Their results reveal a lack of successful strategies to generalize functional relationships and difficulties in understanding and connecting the different representations of functions.

These studies show that in-service and pre-service teachers need to improve the mathematical knowledge required to tackle early algebraic tasks and promote the development of algebraic thinking in primary education. Likewise, the instruments used to measure this knowledge mainly involve questionnaires investigating different aspects of early algebra.

METHODS

In keeping with the goal of our research, we conducted an instrumental study (Montero & León, 2002) that consisted of designing and adapting tests, as well as studying their psychometric properties. For our study, we developed a questionnaire that considers four phases (Figure 2):

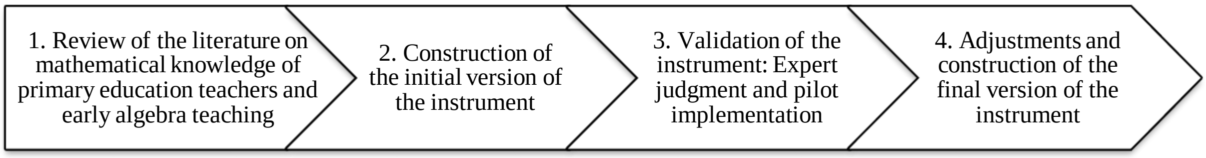


Figure 2. Phases of the Development of the Questionnaire

Construction of the Initial Version of the Questionnaire

The initial items of the questionnaire emerge from phase 1, where mathematical tasks proposed by Pincheira and Alsina (2021a) have been selected to address the characterization of early algebra for primary education. Then, based on the particular richness offered by each mathematical task selected, we delved into initial aspects of the domains and subdomains that comprise the MKT model (Ball et al., 2008) by using open-ended questions since they provide greater insight into the answers of the participants (Cohen et al., 2011). These questions aim to place pre-service teachers in teaching situations that allow us to analyze their mathematical knowledge for teaching early algebra.

Thus, the initial version of the questionnaire (phase 2) contains six items and a total of 22 open-ended questions, as shown in Figure 3. Likewise, a first section is included to collect general identifying data on the participants: gender, age, and previous studies.

Figure 3 shows the core algebraic features addressed by each item. Overall, the 6 items delve into the different subdomains of MKT: (a) CCK, focusing on solving the algebraic task and identifying the procedure as correct or incorrect; Horizon content knowledge, focusing on the link with more advanced algebraic contents of the curriculum; SCK, delving into the mathematical contents or properties that students must put into practice in order to give a solution to the algebraic task; KCS, paying attention to the description of the difficulties students face in solving the algebraic task; KCT, focusing on the teaching strategies proposed to help students solve the algebraic task; knowledge of content and curriculum, paying attention to the identification of the objective and school level of the algebraic task according to the proposed curriculum.

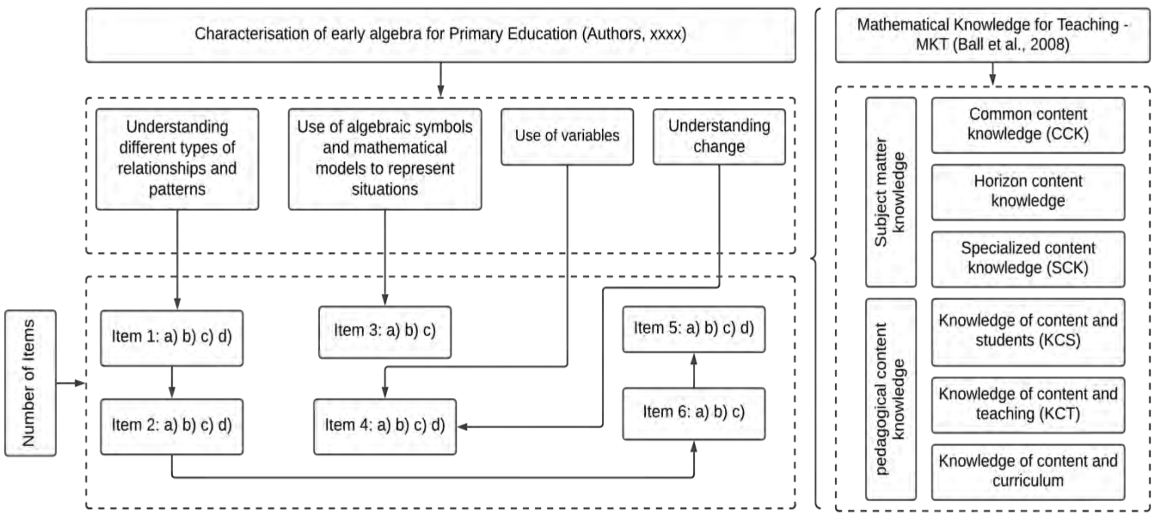


Figure 3. Structure of the Initial Questionnaire

Validation of the Instrument by Expert Judgment and Pilot Implementation

Once the initial version of the questionnaire was built, phase 3 underwent a process to validate the content, which involved expert judgment and pilot implementation to establish its reliability.

Expert Judgment

The expert judgment was carried out by a total of 12 researchers from various countries (two from Chile, one from Mexico, nine from Spain) with extensive experience in the field of Mathematics Education. The criteria for selecting the experts were: a) knowledge about the MKT model, b) knowledge of the study of early algebra, and c) expertise in designing instruments to assess the knowledge of pre-service and in-service teachers.

To carry out this validation, the experts were emailed the instrument and an assessment guideline to measure each item's degree of suitability based on the domains and subdomains of mathematical knowledge according to three categories: a) the degree of correspondence associated with whether it belongs to the MKT model or not (2: belongs, 1: does not belong); b) the formulation, associated with the language and clarity of each item (3: adequate, 2: needs improvement, 1: inadequate); and finally c) the relevance, which is related to the consistency of the item with respect to each subdomain of the MKT model (3: relevant, 2: with doubts, 1: not relevant). Likewise, the guideline contains a final section to express comments, proposals for improvement, or observations, both at the general level of the questionnaire and for each specific item.

Pilot Implementation

The validation of the expert judgment prompted us to modify the instrument before implementing the questionnaire with ten pre-service teachers. The sample was selected by considering a non-probabilistic sampling of an accidental or causal nature (Creswell, 2014) since the possibility of joining this group determined the selection criterion.

At the time of the pilot implementation, the ten pre-service teachers were in the third year of their Degree in Primary Education at a Spanish university: the criterion for inclusion was interested in participating in the study voluntarily after having informed them of the study's development. Therefore, the pilot implementation of the questionnaire was carried out during a lecture (90 minutes) that was given to the participants as part of the "Mathematics II" subject. They answered the questionnaire voluntarily after signing an informed consent.

Of the ten pre-service teachers, 7 were women and 3 were men, and their ages ranged between 20 and 22. 100% of the participants completed high school, 90% focusing on humanities and social sciences, and 10% on science and technology. The participants also took a previous subject called "Mathematics I," where they received instruction on numbering and calculation. The questionnaire's internal consistency and reliability (Cronbach's Alpha coefficient) were determined using the SPSS Statistics 27 data processing. We considered that for a scale to have internal consistency and be considered reliable, Cronbach's Alpha must be greater than 0.7 (Oviedo & Campo-Arias, 2005).

Finally, we analyzed the difficulty index of the items (DI), defined by Muñiz (2017) as the ratio of the number of subjects who answered the item correctly to those who provided an answer. The value of DI can fluctuate between 0 and 1, with 0 indicating the maximum difficulty and 1 the least difficulty, with the medium difficulty indices having the best ability to discriminate. All these steps gave rise to the final version of the instrument, called the MKT-Early Algebra Questionnaire (6-12-year-olds), which is shown in its entirety in Table 9.

RESULTS AND DISCUSSION

The data obtained from assessing the expert judgment and pilot implementation of the MKT-Early Algebra Questionnaire (6-12-year-olds) are described below.

Expert Judgment

The assessments provided by the 12 experts involving how well the questionnaire items correspond to the domains and subdomains of the MKT model allowed us to make a descriptive analysis of the scores (Table 1). For each item that makes up the questionnaire, these scores can range from a minimum of 3 to a maximum of 8 points.

Table 1. Descriptive Statistics for Each Item Based on the Expert Assessment (*N*= 12)

Items		Minimum	Maximum	Average	Standard deviation	Coefficient of variation
1	a)	6	8	7.41	0.996	8.812
	b)	7	8	7.91	0.228	2.882
	c)	7	8	7.91	0.228	2.882
	d)	7	8	7.75	0.452	5.832
2	a)	7	8	7.91	0.288	3.640
	b)	6	8	7.66	0.651	8.498
	c)	7	8	7.66	0.492	6.422
	d)	7	8	7.83	0.389	4.968
3	a)	7	8	7.83	0.389	4.968
	b)	6	8	7.50	0.674	8.986
	c)	7	8	7.83	0.389	4.968
4	a)	7	8	7.91	0.288	3.641
	b)	7	8	7.83	0.389	4.968
	c)	7	8	7.75	0.452	5.832
	d)	7	8	7.83	0.389	4.968
5	a)	7	8	7.75	0.621	8.012
	b)	7	8	7.75	0.452	5.832
	c)	7	8	7.83	0.389	4.968
	d)	7	8	7.91	0.288	3.640
6	a)	7	8	7.75	0.622	8.026
	b)	7	8	7.91	0.288	3.640
	c)	7	8	7.83	0.389	4.968

The mean, standard deviation, and coefficient of variation (standard deviation/arithmetic mean *100) were determined for each item to assess which ones to keep or eliminate. The criteria for eliminating an item were that it receives an average of less than 7 points or that it has high levels of discrepancy in the coefficient of variation, meaning a variation more significant than 25% (López & Sanz, 2021).

The statistical analysis in Table 1 shows that the instrument does not require eliminating any items. However, input from the experts by way of comments and observations indicates that the items should be reworded as follows:

- Item 1: In questions 1c) and 1d), the terms strategies and contents have been changed to teaching strategies and concepts, respectively. The wording of 1a), 1b) and 1d) was also improved.
- Item 2: Question 2a) has been modified since several evaluators indicated that the task already proposes general rules, and thus, the question should focus on which of these rules is correct. Finally, the wording of 2c) and 2d) was improved.
- Item 3: In 3a), the question has been posed in the third person, the wording of 3b) was improved, and 3c) now specifies that we are referring to the primary education curriculum.
- Item 4: The wording of 4b) was modified, and in 4c), the term strategy was changed to teaching strategy. In question 4d), explicit reference is made to the primary education curriculum, and a justification of the answer was added.
- Item 5: The wording of 5a) and 5b) was improved. Similarly, as in the previous item, the terms teaching strategy and primary education curriculum were added in 5c) and 5d), respectively.
- Item 6: The wording of 6a) and 6b) and the score of 6c) were improved.

Pilot Implementation of the Questionnaire

The pilot implementation of the questionnaire was conducted with a sample of 10 pre-service primary school teachers. At the start of the implementation, the instructions on how to respond to the instrument, the estimated time, and the purpose of its application were provided.

Then, field notes were taken from the comments and questions that the participants expressed during the implementation regarding the wording of the items and questions that made up the questionnaire. These notes made it possible to adapt the instrument and improve the understanding of the statements as follows: in item 1, the terminology of the context of the task was harmonized by referring only to euros, and in item 2, the figure and the table given were expanded, since they were not readable by the participants.

The time allotted to answer the questionnaire was deemed adequate since all the pre-service teachers finished the instrument in the assigned period (90 minutes). The results of the answers given by the pre-service primary school teachers were analyzed by constructing a rubric that established criteria based on the relevance of the answers.

The data were encoded and assigned scores based on the degree of correctness of the answers: 2 points for a correct answer, 1 for a partially correct answer, and 0 for an incorrect answer. As a result, the score on the questionnaire ranges from a maximum of 44 points to a minimum of 0 points. The actual results obtained ranged between 12 and 34 points, with an average score of 23.6 points, equivalent to 54% of the total score.

The instrument's degree of consistency and reliability was obtained by applying Cronbach's Alpha, yielding a value of 0.73, which is acceptable. This value is favorable and indicates that the instrument provides stable and consistent measurements with respect to the items that comprise the questionnaire. To calculate the DI of the instrument, the correct, partially correct and incorrect answers were classified, and the unanswered items were not considered. Table 2 shows a statistical summary of the data:

Table 2. Difficulty index of the items on the questionnaire

Item	1				2				3			4				5				6		
DI	a)	b)	c)	d)	a)	b)	c)	d)	a)	b)	c)	a)	b)	c)	d)	a)	b)	c)	d)	a)	b)	c)
(%)	70	44	40	0	70	0	40	60	80	50	38	90	44	44	67	44	29	14	29	38	88	25

The questionnaire has an average difficulty of 46%. On the one hand, items 1d) and 2b), associated with the knowledge of the mathematical horizon and the specialized content knowledge, respectively, exhibit the highest degree of difficulty. By contrast, the items that exhibit the lowest degree of difficulty are 1a), 2a), 3a) and 4a), all of which involve common content knowledge, as well as 6b), which is related to the knowledge of content and students.

The following describes our analysis of the key findings based on the answers received for each item on the questionnaire:

Analysis Item 1

This item (Figure 4) is taken and adapted from the study proposed by Barboza et al. (2020). The purpose of items 1a), 1b), 1c) and 1d) is to assess common content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of the mathematical horizon, respectively, and they focus on understanding different types of relationships linked to the concept of equality and the meaning of the equal sign as an expression of equivalence.

Item 1: A teacher was analyzing the answers of the students in her 4th grade class after giving them the following problem:

Arturo and Cecilia, who are siblings, got the same amount of money from their aunt. Arturo decided to save 20 euros in his piggy bank and save part of the money to take to school. Cecilia put 16 euros in her piggy bank and used the rest to buy some stickers.
Since both children got the same amount of money, we can write the equation:

$$20 + \_\_\_ = 16 + \_\_\_$$

Determine how much each child used for their expenses. Explain how you arrived at the result.

Carlos, Joaquín and Cristina:
"Arturo took €10 to school and Cecilia set aside €14 to buy her stickers.
We think that, if they got the same amount, then Cecilia spent 4 euros more than her brother and we think she took €10, because you can only take a maximum of 10 euros to school. So, we get 20 + 10 = 30 and 16 + 14 = 30"

Santiago, Raquel and Carolina:
"Arturo took €36 to school and Cecilia spent the same €36 on stickers, because they had the same amount. We arrived at this answer by adding the numbers that are given in the statement: 20+16"

Paula, Mateo and Mauricio:
"Arturo took 5 reals to school and his sister spent 9 reals on stickers. We think that, if they both got the same amount and he put €4 more in his piggy bank, then Cecilia had €5 plus €4 to spend on stickers. We get this answer by setting 20 + 5 = 16 + 9, because Arturo saved €4 more than his sister"

Questions:
a) Which answer(s) should the teacher accept as correct, and why?
b) What difficulties do the students in the class have in solving the problem?
c) What strategies would you use to help those students who have not been able to solve the task?
d) To which more advanced contents of the school curriculum do you relate the content involved in solving this problem?

Figure 4. Item 1 of the initial version of the questionnaire

Table 3 shows that most correct answers were given only for item 1a), which involves understanding the meaning of the equal sign. In the proposed task, 70% of the pre-service teachers analyzed each of the answers of a group of students, explaining why they were correct and demonstrating a good command of common content knowledge.

Table 3. Distribution by Percentage of Answers Given to Item 1 (n=10)

Item 1	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	70	10	20	0
b)	40	0	50	10
c)	40	60	0	0
d)	0	40	50	10

Regarding item 1b), which involves identifying the difficulties that students have in solving the task, and 1c), in relation to the teaching strategies used to help those students who were unable to solve the task, only 40% of the answers were correct. In relation to the difficulties, the correct answers include "understanding the equal sign as an equivalence" (pre-service teacher 3) and "understanding that the problem has several possible solutions" (pre-service teacher 6). Among the possible strategies, we note that "the use of manipulative material could facilitate algebraic understanding when solving the problem with a box of coins and analyzing the possible solutions" (pre-service teacher 9).

Consequently, we can infer that both knowledge of content and students and knowledge of content and teaching need to be improved. Finally, item 1d), on knowledge of the mathematical horizon, is the one that presents the highest degree of difficulty since none of the participants managed to relate the content involved in the task with other more advanced concepts of the school curriculum, such as, for example, first degree equations with an unknown.

Analysis Item 2

This item (Figure 5) is formulated from a task proposed by Demonty et al. (2018). The purpose of items 2a), 2b), 2c), and 2d) is to assess common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching, respectively, related to pattern understanding in functional relations.

Table 4 shows that more than 50% of the pre-service teachers could analyze the answers given by the students and explain why they were correct (item 2a). For example, they note that "students 1 and 2 gave a correct answer because they managed to identify the general rule that the number of chairs and the number of tables variables represent" (pre-service teacher 6). They also managed to determine teaching strategies to help students who were unable to solve the task (item 2d), including "using a table of values to represent the situation" (pre-service teacher 4) or "making predictions from manipulatives until they deduce the general rule" (pre-service teacher 1).

However, there needs to be more mastery of specialized content knowledge since none of the participants identified the mathematical content and properties that students must use to respond to the task of generalizing patterns (item 2b). Likewise, in item 2c), only 40% of the participants identified the difficulties faced by the students who answered the task incorrectly, which reveals a limitation regarding their knowledge of content and students.

Item 2: A teacher poses the following problem to her 6th graders:

Esteban's parents are throwing a birthday party for him. They get in touch with Mr. Gomez, the caterer, who only has a few small square tables. He suggests putting them side by side to form a long table where all the guests can sit, as shown below:

Determine a rule that can be used to find the number of chairs for any number of tables. Some examples of the rule that the students came up with are as follows:

Student	Visual representation	Rule suggested by the students
Student 1		Number of chairs = (number of tables x 2) + 2
Student 2		Number of chairs = (number of tables - 2) x 2 + 6
Student 3		Every time you add a table, you have 2 more chairs: Number of chairs = Number of tables + 2

Questions:
a) Write a rule to find the number of chairs for any number of tables you have. Justify your answer.
b) What mathematical content and/or properties should students use to answer the task correctly?
c) Describe the possible difficulties present in the incorrect answers, which have led the students to answer incorrectly.
d) What strategies would you use to help those students who have not been able to solve the task?

Figure 5. Item 2 of the Initial Version of the Questionnaire

The potential difficulties they identify include "not knowing how to write a general rule that represents the series given" (pre-service teacher 8).

Table 4. Distribution by the Percentage of Answers Given to Item 2 (*n*= 10)

Item 2	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	70	30	0	0
b)	0	40	50	10
c)	40	20	40	0
d)	60	20	20	0

Analysis Item 3

This item (Figure 6) is taken from the research by Bernardo et al. (2017). The purpose of items 3a), 3b), and 3c) is to assess common content knowledge, specialized content knowledge, and knowledge of the curriculum, respectively, associated with using algebraic symbols and mathematical models to represent mathematical situations.

Item 3: A teacher explains the following situation to his 5th graders:
Carlitos is a child who likes sweets. He has a box with 28 candies inside. Every day, he eats twice as many sweets as the day before. In three days, Carlitos has eaten all the sweets.
He then asks his students: How many candies did Carlitos eat each day?
Two students describe how they solved the problem.

Teresa: "The first day Carlitos eats some of the candies, and we don't know how many... [Teresa draws a square], the second day he eats twice as many as the first, so two servings [draws two squares] ... the third day twice as many as the second, so four servings [draws four squares]. Now the twenty-eight candies are divided among the seven servings I identified, and I know the value of each serving…"

Lucas: "I took the candies he had in the box and divided them by seven. The result is 4, which is how many candies he eats every day. The first day, then, he eats 4, the second day he eats 8, and the third day he eats 16."

Questions:
a) Solve the problem posed by the teacher. Justify your answer.
b) Explain whether you consider the students' productions to be mathematically correct or not. Justify the adequacy or inadequacy of the mathematical rationality shown by the students.
c) According to the school curriculum, what is the aim of the task proposed to the pupils?

Figure 6. Item 3 of the Initial Version of the Questionnaire

Table 5 shows that 80% of the pre-service teachers exhibited an excellent command of common content knowledge and solved the problem.

Table 5. Distribution by Percentage of Answers Given to Item 3 (*n*= 10)

Item 3	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	80	10	10	0
b)	50	10	40	0
c)	30	30	20	20

Regarding the specialized content knowledge (item 3b), 50% of the pre-service teachers adequately explained whether the contents of the student's answers were mathematically correct. Meanwhile, knowledge of the curriculum proved to be the most difficult, with only 30% of the pre-service teachers able to present the objective of the task in item 3c). Another 30% of the pre-service teachers gave a partially correct answer since they mentioned that the objective is linked to problem-solving but did not specify using equations. An example of this is evidenced in the answer of pre-service teacher 3, who proposed "working on mathematics through problem-solving."

Analysis Item 4

This item (Figure 7) is formulated from a study proposed by Tanisli and Kose (2013). The purpose of items 4a), 4b), 4c), and 4d) is to assess common content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of the curriculum, respectively, related to understanding change and use of variables.

As Table 6 shows, a high percentage of the pre-service teachers (90%) exhibited a good command of common content knowledge since they solved the problem by applying generalization as a modeling language, managing to determine that Pedro's height is $n + 4$, where n represents Clara's height. Likewise, more than half of the pre-service teachers (60%) correctly identified the target school level of the task, with pre-service teacher five noting: "The problem is relevant for the third cycle of primary school [10-to-12-year-olds] because of the use of variables". We can infer that the participants have some mastery of knowledge of the curriculum.

Item 4: Over the course of a class, the following situation is discussed:

"Pedro is 4 cm taller than Clara. If Clara is "n" cm tall, how tall is Pedro?"

Below is the discussion among three students:

Luis: Pedro is 4n tall
Pilar: No. Pedro is 104 cm tall.
Maria: I think Peter's height is x+4

Questions:
a) Determine Pedro's height. Explain your answer.
b) Describe the possible difficulties present in the incorrect answers, which led the students to answer the problem incorrectly.
c) What strategies would you use to help those students who have not been able to solve the problem?
d) According to the school curriculum, for which school level do you consider this problem relevant?

Figure 7. Item 4 of the Initial Version of the Questionnaire

However, only 40% of the participants managed to identify that the possible difficulties that led students to answer incorrectly (item 4b) are related to the interpretation of the variable and to determine strategies to help those students solve the task (item 4c), such as for example, guided questions of the type "What does 4 cm more mean? Or "What does 4 times more mean? How can we represent it?" (pre-service teacher 2). There needs to be more mastery of both knowledge of content and student, as well as knowledge of content and teaching.

Table 6. Distribution by Percentage of Answers Given to Item 4 (*n*= 10)

Item 4	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	90	0	10	0
b)	40	30	20	10
c)	40	20	30	10
d)	60	20	10	10

Analysis Item 5

This item (Figure 8) is taken from the study proposed by Ferreira et al. (2017). The purpose of items 5a), 5b), 5c), and 5d) is to assess specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of the curriculum, respectively, aimed at understanding different types of structural relationships associated with the generalization of arithmetic.

In general, this was the item with the highest degree of difficulty since only 40% of the pre-service teachers answered it correctly, as shown in Table 7. The item with the lowest number of correct answers is 5a), on the mathematical content and properties that students must use to correctly answer the task, revealing a poor mastery of specialized content knowledge. The pre-service teachers' answers noted the operations' properties: "commutative property, associative property, and neutral element of the sum" (pre-service teacher 6).

Item 5: A teacher asks his students to complete the following table, giving them the following instructions:

"Mark the following numerical expressions as true or false. Explain your answer"

	T	F	Explanation
24 + 37 = 37 + 24			
46 + 27 – 27 = 27			
◊ x 1 = ◊			
□ + 0 = □			

Some of the students' answers were as follows:

	T	F	Explanation
24 + 37 = 37 + 24	x		Because it's the same result, only the order changed

	T	F	Explanation
46 + 27 – 27 = 27		x	Because 46+27= makes 79 and 79 minus 27 makes 46

	T	F	Explanation
24 + 37 = 37 + 24		x	Calculations are not facts, the result never involves multiplication. So it's wrong!

	T	F	Explanation
46 + 27 – 27 = 27		x	It is incorrect because 27-27 gives 0, so there's 46 left over

	T	F	Explanation
◊ x 1 = ◊	x		Any number x 1 is equal to the number

	T	F	Explanation
□ + 0 = □	x		The square is zero, so the result is the square.

Questions:

a) What mathematical content and/or properties do students need to use in order to answer the task correctly?

b) Describe the possible difficulties, present in the incorrect answers, that have led pupils to answer incorrectly.

c) What strategies would you use as a teacher to guide those students who have given the wrong answer to the task?

d) For which school level do you consider this problem relevant, according to the current school curriculum?

Figure 8. Item 5 of the initial version of the questionnaire

Both the knowledge of content and students, as well as the knowledge of the curriculum, are limited since only two pre-service teachers (20%) correctly described that the potential difficulties that led students to answer incorrectly are related to the understanding of the properties of the operations and the meaning of the equal sign (item 5b).

Table 7. Distribution by the Percentage of Answers Given to Item 5 (*n*= 10)

Item 5	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	40	10	40	10
b)	20	20	30	30
c)	10	40	20	30
d)	20	50	0	30

They also mention that the relevant level of schooling in which to present the problem is the second cycle of primary education (8-to-10-year-olds) (item 5d). Likewise, the level of knowledge of content and teaching is insufficient, since in item 5c), only one pre-service teacher mentions ideal teaching strategies to guide students who gave a wrong answer to the task, noting "the use of manipulatives, like the scale, and guided questions" (pre-service teacher 3).

Analysis Item 6

This item (Figure 9) is formulated based on the study proposed by Barboza et al. (2020). The purpose of items 6a), 6b), and 6c) is to assess specialized content knowledge, knowledge of content and students, and knowledge of content and teaching, respectively, associated with understanding different types of relationships exploring meanings of the equal sign.

Item 6: A teacher writes on the board 3+2+2=5+2=7 and asks his 3rd graders to analyze whether the numerical expression is right or wrong.

Two students note the following:
"Carla explained that the expression is wrong, since not all the numbers were added together, so the final result would give 21, reading 3+2+2+5+2+7=21. Rodrigo said the expression was right and seven was the answer".

Questions:
a) What mathematical content and/or properties do students have to use in order to answer the task correctly?
b) Describe the possible difficulties that have led the pupil(s) to answer incorrectly.
c) What strategies would you use to help the pupil to realize and overcome her mistake? Justify your answer.

Figure 9. Item 6 of the Initial Version of the Questionnaire

Table 8 shows that item 6b) received the majority of correct answers since 70% of the pre-service teachers could describe the potential difficulties that led the students to respond incorrectly. Among the possible difficulties, they note, for example, that "students do not know the meaning of equality" (pre-service teacher 1) and "they do not consider the equivalence relationship and the"="element in operation" (pre-service teacher 4). This indicates good knowledge of content and students.

Table 8. Distribution by Percentage of Answers Given to Item 6 (*n* = 10)

Item 6	Correct answer	Partially correct answer	Incorrect answer	No answer
a)	30	40	10	20
b)	70	0	10	20
c)	20	20	40	20

Items 6a) and 6c) exhibit high difficulty since the correct answers are at most 30%. The pre-service teachers had difficulty determining the mathematical content and properties involved in solving the task (item 6a), as well as identifying teaching strategies to help students solve the task (item 6c). This indicates significant limitations involving specialized content knowledge and knowledge of content and teaching.

Adjustment and Construction of the Final Version of the Questionnaire

Based on the assessment of the expert judgment and the pilot implementation of the instrument, the questionnaire was improved in terms of the items formulated and the clarity of the questions that comprise it, resulting in the final version (phase 4). The final version of the questionnaire, which, as we have mentioned, is called MKT-Early Algebra (6-12), consists of six open-ended items and a total of 22 questions that evaluate the mathematical knowledge of pre-service primary education teachers to teach early algebra to students ages 6 to 12, presented in Table 9.

Table 9. Items on the Final Version of the MKT-Early Algebra Questionnaire (6-12)

Item 1

A teacher was analyzing the answers of the students in her 4th grade class after giving them the following problem:

Arturo and Cecilia, who are siblings, got the same amount of money from their aunt. Arturo decided to save 20 euros in his piggy bank and save part of the money to take to school. Cecilia put 16 euros in her piggy bank and used the rest to buy some stickers.
Since both children got the same amount of money, we can write the equation:

20 + ___ = 16 + ___

Determine how much each child used for their expenses. Explain how you arrived at the result.

Carlos, Joaquín and Cristina:
"Arturo took €10 to school and Cecilia set aside €14 to buy her stickers.
We think that, if they got the same amount, then Cecilia spent 4 euros more than her brother and we think she took €10, because you can only take a maximum of 10 euros to school. So we get 20 + 10 = 30 and 16 + 14 = 30"

Santiago, Raquel and Carolina:
"Arturo took €36 to school and Cecilia spent the same €36 on stickers, because they had the same amount. We arrived at this answer by adding the numbers that are given in the statement: 20+16"

Paula, Mateo and Mauricio:
"Arturo took 5 euros to school and his sister spent 9 euros on stickers. We think that, if they both got the same amount and he put €4 more in his piggy bank, then Cecilia had €5 plus €4 to spend on stickers. We get this answer by setting 20 + 5 = 16 + 9, because Arturo saved €4 more than his sister"

Questions:
a. What answer(s) should the teacher accept as correct? Why?
b. What difficulties do the students in the course exhibit when solving the problem?
c. What teaching strategies would you use to help those students who were unable to solve the task?
d. What advanced concepts from the school curriculum are relevant to the content addressed in the task?

Item 4

Over the course of a class, the following situation is discussed:

Pedro is 4 cm taller than Clara. If Clara is "n" cm tall, how tall is Pedro?

Below is the discussion among three students:

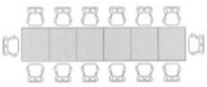
Luis: Pedro is 4n tall
Pilar: No. Pedro is 104 cm tall.
Maria: I think Peter's height is x+4

Questions:
a. Determine Pedro's height. Explain your answer.
b. Describe the potential difficulties that led the students to answer incorrectly.
c. What teaching strategies would you use to help those students who were unable to solve the problem correctly?
d. Judging by the primary education school curriculum, what grade do you think this problem is appropriate for? Explain your answer.

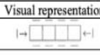
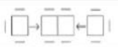
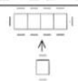
Item 2

A teacher poses the following problem to her 6th graders:

Esteban's parents are throwing a birthday party for him. They get in touch with Mr. Gomez, the caterer, who only has a few small square tables. He suggests putting them side by side to form a long table where all the guests can sit, as shown below:



Determine a rule that can be used to find the number of chairs for any number of tables. Some examples of the rule that the students came up with are as follows:

Student	Visual representation	Rule suggested by the students
Student 1		Number of chairs = (number of tables x 2) + 2
Student 2		Number of chairs = (number of tables - 2) x 2 + 6
Student 3		Every time you add a table, you have 2 more chairs: Number of chairs = Number of tables + 2

Questions:
a. What answer(s) should the teacher accept as correct? Why?
b. What mathematical content and/or properties should students use to answer the task correctly?
c. What difficulties might affect the students who responded incorrectly?
d. What teaching strategies would you use to help those students who were unable to solve the task?

Item 5

A teacher asks his students to complete the following table, giving them the following instructions:
"Mark the following numerical expressions as true or false. Explain your answer"

	T	F	Explanation
24 + 37 = 37 + 24			
46 + 27 − 27 = 27			
◊ x 1 = ◊			
□ + 0 = □			

Some of the students' answers were as follows:

	T	F	Explanation
24 + 37 = 37 + 24	x		Because it's the same result, only the order changed

	T	F	Explanation
46 + 27 − 27 = 27		x	Because 46+27= makes 79 and 79 minus 27 makes 46

	T	F	Explanation
24 + 37 = 37 + 24		x	Calculations are not facts, the result never involves multiplication. So it's wrong!

	T	F	Explanation
46 + 27 − 27 = 27		x	It is incorrect because 27-27 gives 0, so there's 46 left over

	T	F	Explanation
◊ x 1 = ◊	x		Any number x 1 is equal to the number

	T	F	Explanation
□ + 0 = □	x		The square is zero, so the result is the square.

Questions:
a. What mathematical content and/or properties should students use to answer the task correctly?
b. Describe the potential difficulties that led the students to answer incorrectly.
c. What teaching strategies would you use as a teacher to guide those students who answered the task incorrectly?
d. Judging by the Primary Education school curriculum, what grade do you think this problem is appropriate for? Explain your answer.

Item 3

A teacher explains the following situation to his 5th graders:

Carlitos is a child who likes sweets. He has a box with 28 candies inside. Every day, he eats twice as many sweets as the day before. In three days, Carlitos has eaten all the sweets.

He then asks his students: How many candies did Carlitos eat each day?
Two students describe how they solved the problem.

Teresa:
"The first day Carlitos eats some of the candies, and we don't know how many... [Teresa draws a square], the second day he eats twice as many as the first, so two servings [draws two squares] ... the third day twice as many as the second, so four servings [draws four squares]. Now the twenty-eight candies are divided among the seven servings I identified, and I know the value of each serving…"

Lucas:
"I took the candies he had in the box and divided them by seven. The result is 4, which is how many candies he eats every day. The first day, then, he eats 4, the second day he eats 8, and the third day he eats 16."

Questions:
a. Solve the problem presented by the teacher. Explain your answer.
b. Explain whether or not you consider the pupils' work product to be mathematically correct. Justify the appropriateness or insufficiency of the mathematical rationale shown by the students.
c. Considering the primary school curriculum, what might be the goal of the task proposed to the students?

Item 6

A teacher writes on the board 3+2+2=5+2=7 and asks his 3rd graders to analyze whether the numerical expression is right or wrong.

Two students note the following:
"Carla explained that the expression is wrong, since not all the numbers were added together, so the final result would give 21, reading 3+2+2+5+2+7=21.
Rodrigo said the expression was right and seven was the answer".

Questions:
a. What mathematical content and/or properties should students use to answer the task correctly?
b. Describe the potential difficulties that led the students to answer incorrectly.
c. What teaching strategies would you use to help the student realize and correct her mistake? Explain your answer.

CONCLUSIONS

This study presented the process of building and validating an instrument to assess primary school teachers' mathematical knowledge of early algebra during their initial training. This instrument is based on the MKT model proposed by Ball et al. (2008) and allows us to delve into the mathematical contents that characterize early algebra in primary education (Pincheira & Alsina, 2021a): a) understand different types of relationships and patterns; b) use algebraic symbols and mathematical models to represent situations; c) understand change; and d) use variables to determine a constant or unknown.

The validation process of the instrument considered the judgment of twelve experts and a pilot implementation with ten pre-service primary school teachers. This process established the reliability and internal consistency of the questionnaire's items and was used to adjust and refine them in terms of their clarity to achieve a better understanding of the problem statements. As a result, the final version of the questionnaire, called MKT-Early Algebra (6-12), contains six open-ended items comprising 22 questions.

The MKT-early algebra questionnaire (6-12) complements the questionnaires built and validated to evaluate the knowledge that teachers can draw on to present the teaching of algebra from other theoretical perspectives, as is the case of the studies developed by Aké (2013), Castro (2011), Godino et al. (2015b), and Mejías (2019). As concerns the MKT model, unlike other questionnaires that analyze mathematical knowledge for teaching early algebra in primary education from a specific subdomain (e.g., Bernardo et al., 2017; Oliveira et al., 2021; Trivilin & Ribeiro, 2015; Wilkie, 2014; Zapatera & Callejo, 2017), the MKT-Early Algebra questionnaire (6-12) presents a global perspective of the different subdomains that comprise the model. Likewise, while some studies have focused on a particular aspect of early algebra, such as functional thinking (e.g., Oliveira et al., 2021; Wilkie, 2014), the meaning of the equal sign (e.g., Trivilin & Ribeiro, 2015) and pattern generalization (e.g., Zapatera & Callejo, 2017), the questionnaire that we have constructed delves holistically into the different mathematical contents that make up the study of algebra in primary education, as mentioned earlier.

Regarding the answers given by the pre-service teachers in the pilot implementation of the questionnaire, they gave us an insight into their mathematical knowledge to teach early algebra. The results show teachers' limitations when faced with various teaching situations typical of the instruction that must be provided to impart this content block in primary education with respect to the domains and subdomains of mathematical knowledge.

These limitations reflect primarily the knowledge of the mathematical horizon, specialized content knowledge, and knowledge of content and teaching. These results, still incomplete, show similarities with other studies (e.g., Bernardo et al. 2017; Ferreira et al., 2017; Wilkie, 2014; Zapatera & Callejo, 2017) in relation to a) the inability to establish connections between the different meanings of the equality sign as an operator and as an expression of equivalence; b) misunderstanding the mathematical knowledge associated with pattern generalization; and c) difficulties determining implications for teaching numerical relationships and the properties of operations, linked to the generalization of arithmetic.

Blömeke and Delaney (2012) propose that teachers' knowledge is essential to students' mathematical achievement; however, primary school teachers have little experience teaching early algebra (Blanton & Kaput, 2011). From this perspective, we believe it is necessary to provide experiences during the initial and continuous training of primary education teachers that allow them to develop the mathematical knowledge required to adequately engage in teaching early algebra. Such experiences should incorporate reforms that address the teaching of early algebra in accordance with the demands of the school curriculum since the initial training of teachers becomes more reflective if it is explicitly directed toward school practice (Gellert, 2005).

In line with Hohensee (2017), we assume that teaching early algebra in primary education implies a restructuring of teaching practice that, as indicated above, directly challenges teacher education. Therefore, primary teacher education programs should incorporate specific elements of algebra teaching such as for example, the practices of algebraic thinking (Blanton et al., 2011): generalizing, representing, justifying, and reasoning. It is also necessary to promote professional tasks that bring teachers closer to classroom practices: selecting examples to explain algebraic content, anticipating the answers that students may give to a given algebraic task, or linking algebraic content with other mathematical content in the school curriculum, among others.

We conclude that the MKT-Early Algebra questionnaire (6-12) can be an effective diagnostic tool both to investigate the mathematical knowledge of early algebra called upon by pre-service primary education teachers and to reflect on teaching practices that promote algebraic thinking at this stage of schooling. On the other hand, as indicated, the research has comprehensively addressed the MKT model. However, we consider that subdomains still require a more detailed and exhaustive exploration, as is the case of the knowledge of the mathematical horizon, which constitutes a study limitation.

Finally, like the validation of other instruments that evaluate the algebraic knowledge of primary school teachers, the questionnaire focuses on a limited sample of pre-service teachers. Therefore, in future research, to obtain more meaningful results that can be generalized to other realities, the mathematical knowledge of the teaching staff should be analyzed by employing a larger sample size.

Acknowledgments

This research was supported by the National Agency for Research and Development of the Government of Chile (ANID) through a PhD scholarship abroad, Folio No. 72200447.

Declarations

Author Contribution	:	NP: Conceptualization, Writing - Original Draft, Formal analysis, and Methodology. ÁA: Conceptualization, Writing - Original Draft, Editing and Visualization, Review & Editing.
Funding Statement	:	This research was funded by the “M. Antonia Canals” Chair of Mathematics Education, the Department of Subject-Specific Didactics, and the Institute of Educational Research (IRE) of the University of Girona.
Conflict of Interest	:	The authors declare no conflict of interest.

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