The Thinking Process of Children in Algebra Problems: A Case Study in Junior High School Students

Wa Ode Dahiana [1,2*], Tatang Herman[1], Elah Nurlaelah[1]

[1]Universitas Pendidikan Indonesia, Bandung, Indonesia,
[2]Universitas Pattimura, Ambon, Indonesia
wdiana@upi.edu, tatangherman@upi.edu, elahnurlaelah@upi.edu

Abstract: *Mathematics is a collection of cognitive products that have unique characteristics from other scientific disciplines. As cognitive products, mathematics, and thought processes are two things that cannot be separated. Although in the literature there have been many approaches proposed to support the analysis of students' thinking processes, not many have used Harel's theory of thinking to reveal the characteristics of students' mathematical thinking. Therefore, this research aims to describe the thinking process of class IX students in solving algebra problems based on Harel's thinking characteristics. To achieve this goal, qualitative research was conducted with a case study design. The questions on the topic of algebra were adapted and developed from questions from the National Examination (UN) and the International Program for Student Assessment (PISA). Next, a test was given to 30 class IX students followed by interviews. The results of the data analysis concluded that in general students still use non-referential symbolic thinking and only a few have algebraic invariance thinking or proportional and deductive thinking. This shows that there are still many students who have difficulty understanding algebra. In increasingly complex problems, students' thinking processes become less flexible, and concepts are understood separately (procedural).*

Keywords: *Mathematical thinking process, algebraic thinking, algebraic problems, and problem-solving*

INTRODUCTION

Thinking is a mental activity carried out by humans in every domain of their lives, including in the fields of education and learning. Specifically for learning mathematics, all activities or activities in mathematics are mental activities. In this regard, Suryadi (2012) states that mathematics is a way and tool of thinking. The way of thinking developed in mathematics uses consistent and accurate reasoning rules so that mathematics can be used as a very effective thinking tool for looking at various problems, including those outside mathematics.

Mathematics is a researched subject in cognitive psychology and cognitive science has brought many changes. Such changes are reflected in policy-level agendas such as in the National

Council of Teachers of Mathematics standards (NCTM, 2000). These standards call for dramatic changes in schools, and how mathematics is learned, understood, and taught. To study mathematics, you must know its characteristics. Mathematics has abstract concepts and relationships, and this is what differentiates it from other sciences. Perception of mathematical concepts or relationships occurs in the mind and their operations are only possible using certain signs and symbols (Duval, 2017; Uzun & Arslan, 2009). Thus, mathematics and mathematical thinking are two things that cannot be separated in learning. In this regard, Harel (2008b) explains that mathematics consists of two complementary subsets. The first part is a collection, or structure, of structures consisting of certain axioms, definitions, theorems, proofs, problems, and solutions. The second part consists of all the ways of thinking that are characteristic of the mental acts whose products comprise the first set. Therefore, developing mathematical thinking processes should be the focus of attention for classroom educators and mathematics observers in general.

The thinking process is also called the problem-solving process because thinking is related to problems (Sari et al., 2019). Meanwhile, according to Mayer (in Purwanto, 2019), students' thinking processes include three main components, namely: (1) thinking is an invisible cognitive activity, but can be inferred based on visible behavior, (2) thinking is a process that involves some manipulation of knowledge in the cognitive system, and (3) thinking activities directed at solving problems. This description indicates that problem-solving activities absolutely need to be given to students in order to train their thinking processes. A trained thought process produces skills that can be applied in a variety of situations.

Algebra is a subject that deals with the expression of symbols and extensions of numbers beyond whole numbers to solve equations, analyze functional relationships, and determine the representational structure of systems, consisting of expressions and relationships. In such activities, algebra is referred to as a tool for modeling real-world phenomena. Therefore, Algebra is said to be not only a set of knowledge and techniques but also a process or way of thinking (Kieran, 2004; Lew, 2004).

As a way of thinking, algebra refers to a way of producing meaning through activities, modeling situations, and manipulating those models in certain ways (Nottingham & User, 1992). Algebraic thinking is also the activity of doing, thinking, and talking about mathematics from a general and relational perspective (Kaput & Blanton, 2005; Mason, 1996; Windsor, 2010). Through activities like this, it is hoped that someone (student) can gain meaning from the object or concept they are studying.

Driscoll et al. (2001), explain algebraic thinking as the capacity to represent quantitative situations so that the relationships between variables become more visible. Some other opinions also define algebraic thinking as "habits of mind" that enable students to identify and express mathematical structures and relationships, such as structures in arithmetic and symbolic expressions, relationships in numerical and geometric patterns, and numerical and geometric structures in tables, graphs, and lines. numbers (Carraher et al., 2006; Kaput & Blanton, 2005; Mulligan & Mitchelmore, 2009; Novita et al., 2018; Radford, 2000; Warren & Cooper, 2008).

Manly & Ginsburg (2010), describe algebraic thinking into four thinking activities which involve (1) looking for structures (patterns and regularities), (2) making generalizations, using symbols for the number of variables, (3) representing relationships systematically with tables, graphs, and equations, (4) logical reasons to address/solve new problems. In this regard, van Amerom, (2002), Kieran, (2004), & Seeley (2004) stated that to develop algebraic thinking, generalization approaches, modeling, problem-solving, and function approaches need to be applied in algebra learning.

The various definitions of algebra and algebraic thinking explained above, ultimately algebraic thinking is based on basic mathematical ideas and concepts that involve various cognitive strategies and in turn these ideas are used to help understand mathematical concepts and solve increasingly complex and diverse problems (Permatasari & Harta, 2018; Windsor, 2010; Windsor & Norton, 2011). Basic mathematical ideas and concepts that are not optimal in their development affect students' ability to solve problems. This was expressed by Jupri & Drijvers (2016) in line with the findings of their research that several students' difficulties in algebra have been identified, including difficulty understanding words, phrases, or sentences and difficulty compiling equations or creating mathematical models. These difficulties result in errors in interpreting and in the problem-solving process resulting in students' unproductive (undesirable) way of thinking, namely non-referential algebraic thinking.

The difficulties experienced by students as described above do not only occur in Indonesia, but they also occur in several other countries, such as Thailand. One of the weaknesses in mathematics education in Thailand is that students lack thinking and problem-solving skills even though the main aim of organizing mathematics activities is to encourage students to reflect on their thinking and use their mathematical abilities to solve problems (Chimmalee & Anupan, 2022; Natcha & Yeah, 2010). Iji, Abakpa, & Takor (in Ojo, 2022) stated that although many students are proficient in mathematical operations related to symbols, they are less proficient in solving algebraic problems.

In connection with symbols in mathematics, Manly & Ginsburg, (2010) explained the results of their research that students had difficulty understanding symbols (letters) because: (1) letters are first found in formulas to determine parameters such as area or volume, (2) letters can also represent certain unknown numbers, and (3) letters can represent general numbers that are not a specific value. Meanwhile, Harel (2008b) states that algebraic symbols are often understood by students separately or are not understood as coherent entities, which represent quantities and have quantitative relationships.

Based on the description above, it is very important to help improve students' way of thinking gradually so that they have a correct and scientific way of understanding and are useful in solving problems. In this case, teachers need to identify their students' way of thinking in order to then take corrective action, for example by accustoming students from an early age to interpreting and solving problems in more than one way. Apart from that, the principle of iterative reasoning can be used as a solution to help students have the desired way of thinking, namely algebraic invariance thinking (Harel, 2008b; Harel & Sowder, 2013). Therefore, the aim of this research is to analyze and describe students' thinking processes in solving algebra problems with different levels of difficulty. Based on these objectives, the problem in this research can be formulated: (1) What are the characteristics of students' thinking in solving algebraic problems, (2) What is the students' thinking process in more complex algebraic problems and alternative solutions that can be provided to develop their algebraic thinking.

How do you know the process or way of thinking of students in solving algebra (mathematics) problems? Explicitly, Harel (2008b) explains this by starting from a way of understanding a certain cognitive product of certain mental actions carried out by an individual. These cognitive products are the results of interpreting a concept or algebraic symbols, proving mathematical statements, and solving problems. In contrast, the way of thinking is the cognitive characteristic of the mental actions to create the product (Harel, 2013). So, how to think can be known from conclusions after observing the results of work or behavior (cognitive products) from someone's (student's) mental actions. For example, a teacher who observes a student's work related to problem-solving may conclude that the student's interpretation of mathematical symbols is characteristically inflexible, lacking quantitative references, or, alternatively, flexible and connected to other concepts, and so on. From these characteristics, the teacher can conclude that the student's way of thinking includes algebraic invariance thinking, non-referential symbolism, or something else. Likewise, teacher observations in relation to evidence can conclude that students' justifications for mathematical statements are based on empirical evidence, inductive, or based on deduction rules (Harel, 2008a). This way of knowing the thought process is used as one of the author's references in the data analysis process of this research.

The expected way of thinking that is the focus of the study in this paper is (1) The algebraic invariance way of thinking with indicators that students can explain the algebraic symbols used and interpret the operations used (applied) in solving problems. (2) Proportional reasoning with indicators, students can explain relationships and changes in quantity from the operations used. (3) Deductive reasoning with indicators, students can determine the general form or deductive generalization of real-world problems logically. The term thinking process in this article is based on the meaning that the phrase "way" has the connotation of a kind of process (which produces a product) so to think means to apply a way of thinking (Harel, 2008b). Therefore, practically to explain students' thinking processes in this paper both terms (process and method) are used.

METHODOLOGY

This research uses a holistic type case study design (Yin, 2014). A holistic type case study design was used to describe various field findings related to the research question, namely how the thinking process of class IX students in solving algebra problems is based on Harel's thinking characteristics. The research was carried out in class IX, one of the state schools in the city of Bandung, Indonesia. Participants in this research were 30 students (aged 15-16 years) for the 2021/2022 academic year.

Data collection techniques used semi-structured tests and interviews. The test instrument consists of three essay questions and a non-test, namely an interview guide. The test questions used are in the form of problem-solving on algebra material which is compiled and adapted from national exam (UN) questions and from the International Program for Student Assessment (PISA). Before use, the test questions were first validated by 2 lecturers and 3 mathematics subject teachers and then tested on students in different classes. The results of the trial were revised again, after which they were given to the students who were participants in this research. The test questions are arranged in the form of story questions and are based on the student's type of thinking process, namely algebraic invariance thinking, proportional reasoning, and deductive reasoning. In connection with algebraic thinking, the activities for the three questions can be described: (1) identification activities are generally found in the three questions, (2) using and interpreting symbols, composing equations or mathematical modeling is more dominant in question number 1, (3) making a representation of relationships in tabular form, graphics are dominant in question 2, and (4) determining patterns, compiling logical reasons, making generalizations, found in question number 3.

The test takes approximately 60 minutes, students work on the questions on the sheet that has been prepared. After carrying out the test, students were asked to rest while the researcher corrected and grouped the results of their work to determine the participants who would be interviewed. Next, 13 students were interviewed based on the grouping of their work results. First, there were 6 people from the group of students who answered almost all of the questions given correctly. Second, there were 5 people from the group who answered the questions almost 50 percent correctly. Third, there were 2 people from the group who answered almost all wrong. Interviews were carried out over 2 days. During the interview, students' written answers as well as test questions are presented and they are encouraged to explain the results of their work. As a guide in conducting interviews, general initial questions and follow-up questions have been prepared. Common interview questions include: explaining the meaning of the question according to your understanding. What is your strategy/way to solve this problem? Explain the solution steps you made. What is the meaning of the symbols you use? Follow-up questions include, for example:

Why did you take this step? What do you mean by this step? What is the next step? What does it mean? Interviews lasted between 25 and 45 minutes, depending on the student's response.

Students' success in working on problems, whether using algebraic invariance thinking, proportional reasoning, or qualitative deductive reasoning, is considered to have the expected way of thinking (Harel, 2008b). On the other hand, students are said to have an unexpected way of thinking (symbolic non-referential), namely manipulating symbols without understanding their meaning. Students who have a non-referential symbolic way of thinking and students who do not provide answers (blank answer sheets) are considered to have difficulty solving algebra problems. The data analysis technique used in this research is qualitative analysis techniques, namely data reduction, data display, and conclusions (Miles & Huberman, 1994). The data analysis steps in this research are as follows.

1. Data Reduction

At this stage, the researcher summarizes the results of valid test and interview data, simplifies them, selects the main points, and focuses on things that are relevant to the research objectives. The researcher's activities at this stage are: (a) All recordings of student speech during interviews are opened and transcribed; (b) Select interview notes by deleting unnecessary parts; (c) Re-examine the correctness of the results of the transcript by playing back the recorded interview until it is completely clear what the student expressed in the interview; (d) Typing and compiling transcript results to facilitate the analysis process.

2. Data Display

At this stage, the researcher presents data, which is the result of data reduction, namely data on students' thinking processes in solving algebraic problems.

RESULTS AND DISCUSSION

General Findings

The following table presents the findings of students' work in solving algebra problems and their applications. In general, as expected, question number 3 is the most difficult for most students of the other two questions, and question number 2, including literacy questions, is relatively more difficult than question number 1. Even question number 1 is relatively easier and can be solved by some (50%) students. The remaining part (50%) contributed to making mistakes. Based on the expected competency indicators for question number 1, namely making mathematical modeling (algebra) with thinking process indicators, namely being able to solve problems using algebraic invariance thinking, and proportional reasoning, it seems that some (50%) of the students can do it. Apart from that, the procedure for solving question number 1 is immediately known to students using the procedures they usually use, namely elimination and substitution. Solving with the way of thinking that is often used, namely proportional reasoning, can make it easier for them to find a solution.

For question number 2, 6 students (20%) could solve it correctly. Likewise, for type number 3, only 3 out of 30 students were able to complete it correctly (10%). The rest of the students made mistakes, some even did not write anything on the answer sheet provided. Different from question number 1, question number 2 requires literacy skills to be able to understand the question correctly before taking the solving steps. Likewise, question number 3, which is a non-routine question, requires generalization and deductive reasoning skills. Students find it very difficult to solve them because they are not used to dealing with similar questions.

Table 1. Results of Analysis of Student Answers (N=30)

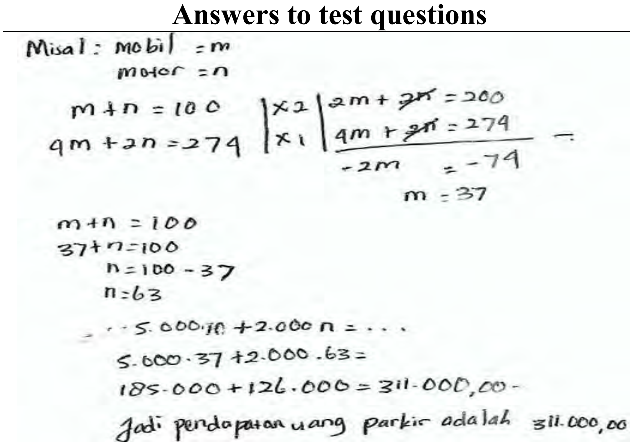
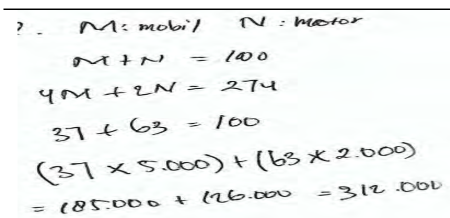
No.	Question Characteristics	Category Thinking Process			Correct Answer (%)
		Algebraic Invariance Thinking (%)	Proportional Reasoning (%)	Deductive Reasoning (%)	
1.	Demands students' ability to make examples, and construct mathematical equations or models. Use a solution method/strategy to determine the number of cars and motorbikes, and find the amount of income from a vehicle parking area.	15 (50)	15 (50)	-	15 (50)
2.	It is a matter of literacy, packaged in a verbal explanation as well as presented in a table in the form of information on the lengthening of the peanut tree each week. Students were asked to determine the height of the peanut tree on April 12, determine the time (date) when the peanut tree reached 50 cm high, determine the height of the peanut tree when it was harvested (3 months old), and make a graph of the growth of the peanut tree. using the calendar provided.	6(20)	6(20)	-	6(20)
3.	It is a matter of generalization that requires students to have the ability to think or	3(10)	-	3(10)	3(10)

inductive-deductive reasoning. Students are asked to determine the number of conifer trees and apple trees in the 8th row, find the general pattern or formula for the nth term from a number of rows of conifer and apple trees, and draw conclusions about the most planted trees.	

Results and Discussion of Problem Number 1

In Table 1, it appears that 50% of students can solve problem number 1. The following is a representation of student answers.

Table 2. Example of Representation of Student Answers to Question Number 1

Answers to test questions	Translate
 (a)	for example: Car = m Motorcycle = n m + n = 100 | x2 | 2m + ~~2n~~ = 200 4m + 2n = 274 | x1 | 4m + ~~2n~~ = 274 - -2m = -74 m = 37 m + n = 100 37 + n = 100 n = 100 – 37 n = 63 5000m + 2000n = … 5000.37 + 2000.63 = 185.000 + 126.000 = 311.000 So, parking revenue is 311.000,00 (a)
 (b)	M: Car N: Motorcycle M + N = 100 4M + 2N = 274 37 + 63 = 100 (37 x 5.000) + (63 x 2000) = 185.000 + 126.000 = 312.000 (b)

There were 14 students who answered correctly in part (a), while only 1 student answered correctly in part (b). The remaining 15 people gave wrong answers or provided solutions without understanding the meaning, both the meaning of the symbols used and the meaning and process of algebraic manipulation carried out. Judging from the thinking process, it is known that the 15 students who answered correctly knew the meaning of the symbols used and carried out the algebraic manipulation process correctly.

From the interview results, it was revealed that 14 students gave correct answers, they explained their answers based on the forms or solution strategies that had been given by the teacher. Only one person, namely student AS, has a different interpretation and solution method from the others (answer part (b)). Likewise, 15 people who gave wrong answers used the solution method given by the teacher but they did not understand the meaning of the algebraic symbols used nor did they understand the algebraic manipulation or the nature of the operations used. The results of interviews with RT and AS students related to answers (a) and (b) were asked (Q) how to answer problem question number 1, and the following explanation was given.

Interview with RT

Q : How do you answer the question in problem no.1?
RT : First make an example. For example, the car is m, and the motorcycle is n. Keep making the equations: m+n = 100 and 4m+2n = 274
Q : Next, what method is used to solve it?
RT : Using the elimination method. It is taken from the first and second equations. That's so that some can be eliminated, multiplied by 2 or 4. But here I'm multiplying by 2 so that they can be eliminated. [meaning RT elimination n]

RT students can explain the meaning of the symbols used, namely m and n as cars and motorbikes, and can construct equations or mathematical modeling of problem number 1, namely m+n = 100 and 4m+2n = 274. RT also understands the meaning of the operations used, for example, use the subtraction operation (-) to be able to eliminate n because if you use the addition operation (+) you cannot find any of the two values (m or n). Furthermore, RT can also determine the amount of income from parking using the substitution method after finding the values of m and n.

Interview With AS Students

AS : Cars are symbolized by M and motorbikes are symbolized by N. Then M+N= 100. 4M+2N= 274. Numbers 4 and 2 are the number of wheels. Then 37 + 63 = 100.
: Where do the values 37 and 63 come from?
AS : I can just guess, mother, the number of cars and motorbikes. The number of cars is 37 and the number of motorbikes is 63.

Try to explain how to guess it.
First I divide by 2 the number of motorbikes and cars (50 each). Then I try multiplying by 2 and multiplying by 4 until it produces 274. If it is less or more than 274 then you have to keep looking until you find the right result (4m + 2n = 274)

Q : Try to explain how to guess it.
AS : First I divide by 2 the number of motorbikes and cars (50 each). Then I try multiplying by 2 and multiplying by 4 until it produces 274. If it is less or more than 274 then you have to keep looking until you find the right result (4m + 2n = 274)
Q : OK, what are the next steps?
AS : Next, the number of cars and motorbikes multiplied by the price of each parking, namely 37 x 5000 = 185,000 and 63 x 2000 = 126,000. So, the total parking revenue is 185,000 + 126,000 = 312,000.
Q : Is the calculation correct? Try counting again!
AS : The result is 311,000. More than 1000 bu. [US realizes his mistake]

AS students can also explain the meaning of symbols and the steps for solving operations using a trial and error strategy or what they call guessing.

Both respondents (RT and AS) gave correct interpretations of the algebraic symbols used and in solving problems they were able to interpret the operations applied even though they had different ways or methods of solving them. From observations of the mathematical statements and behavior of RT and AS students, it was concluded that they have an algebraic invariance way of thinking and proportional thinking.

On the other hand, there were some students who had difficulty solving problem number 1. NP students, for example, wrote the equations: x + y = 100 and 2x + 4y = 274 when using the elimination method, the equations changed to:

$$x + y = 100x2 \rightarrow 2x + 2y = 200$$

$$2x + 4y = 257x1 \rightarrow 2x + 4y = 257$$

Next, y (car) = 28.5 is obtained

From the description of the NP students' answers above, it is known that students have difficulty interpreting the questions. students do not understand objects (e.g. vehicles m = 28.5. Is it possible that the number of vehicles is in decimal form, not integers? Students do not examine the meaning of symbols of objects.

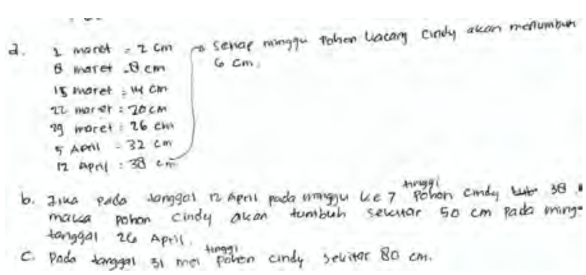
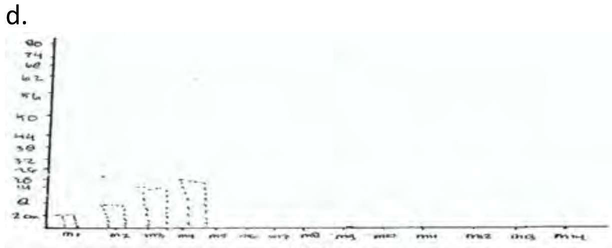
Another difficulty was also shown by MS students directly substituting values or prices for car and motorbike parking without understanding the objects and symbols used. MS does not interpret that 4m is a 4-wheeled car and 2n is a 2-wheeled motorcycle while 5000 and 2000 are the parking fees for each vehicle, not the number of vehicles, MS directly substitutes the value or price of car and motorcycle parking into the equation 4m + 2n = 100 substitutions parking fee: 4 (5000) + 2(2000) = 24,000.

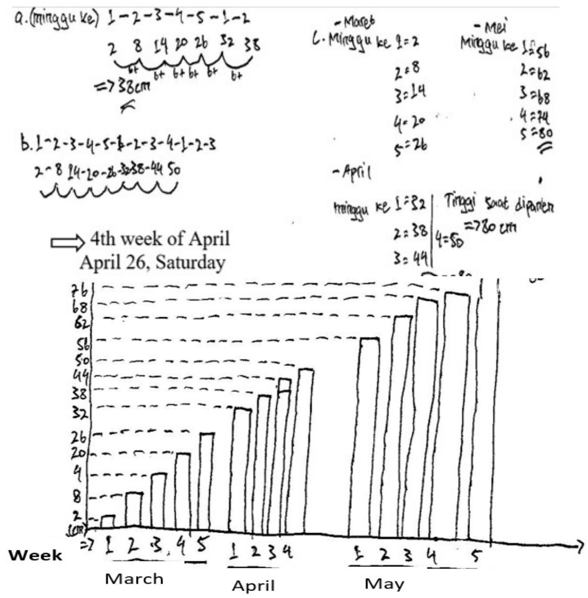
The difficulties experienced by students such as NP and MS are generally thought to be difficulties in understanding words, phrases, or sentences in word problems (such as problem number 1) which has become a difficulty for high school students around the world (Jupri & Drijvers, 2016). Difficulties like this result in errors in interpreting questions (concepts) and in the process of solving problems students cannot interpret the operations applied or manipulate without checking their meaning. Such a student's way of thinking is categorized as a non-referential symbolic way of thinking or an unwanted way of thinking.

Based on the description of the students' answers above, it appears that no one has used the substitution method to solve the system of linear equations or mathematical modeling that they have prepared. Even though this method is easier and more efficient than the elimination method. $x + y = 100$ as equation (1) can be changed to $y = 100 – x$ then substituted into equation (2): $4x + 2y = 274$ so that $4x + 2(100-x) = 274$, $2x = 74$, $x = 37$. The value of x is substituted into equation (1), obtaining $y = 63$. Next, to find the value or income of the parking lot, the function approach can be used: $f(x,y) = 5000x+2000y$, so $f(37,63) = 311,000$. The function approach is intended to develop their algebraic thinking abilities (van Amerom, 2002; Kieran, 2004; Seeley, 2004).

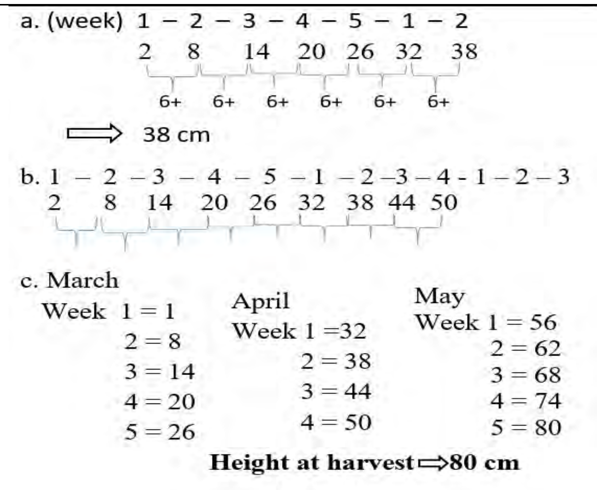
Results and Discussion of Problem Number 2

Table 3. Representative Example of Student Answers to Question Number 2

Answers to test questions	Translate
 d.  (a)	a.1 March = 2 cm 8 March = 8 cm 15 March =14 cm 22 March = 20 cm 29 March = 26 cm 5 April = 32 cm 12 April = 38 cm → Every week Cindy's beanstalk will grow 6 cm b. If on April 12 in the seventh week the height of the Cindy tree is 38 cm, then on April 26 the Cindy tree will grow to around 50 cm. c. On May 31, Cindy's tree was about 80 cm tall d. (a)



(b)



(b)

Table 3 is a representative example of students' correct answers to problem number 2. It is known in Table 1 that only 6 out of 30 people (20%) were able to give the correct answer. Question number 2 has a higher level of difficulty than question number 1 and is a literacy question. Students need to read the questions carefully and thoroughly and follow the instructions in order to provide the correct interpretation and solution to the problem. From the results of the answers and interviews, it was discovered that the majority of students did not read the questions well and did not follow the instructions on the questions, so they gave wrong interpretations. The instructions in question are instructions for using the calendar in the problem to carry out the solution process. Completing the answer in part (a), students use procedural skills, follow the instructions, and continue the table in the question to determine the change in height (length) of the nut tree every week. The following is the explanation of ZP students in the interview regarding the answer to part (a).

Q : How did you finish number 2?
ZP : By looking at the question in the question, how tall is the peanut tree on Saturday, April 12? On what date does the height of green beans reach 50 cm? Mung bean height at harvest at the age of 3 months from the first measurement. So, in answer to the first question, we write down the weekly increase in the length of the beanstalk by 6 cm.
Q : Does that mean that on April 12 the height of the peanut tree reached 38 cm? [while pointing at the settlement table made by ZP]
ZP : Yes, ma'am.
Q : When solving this, did you use a calendar or something?

ZP : I'm using tables. I examined the increase in length every week and it turned out to be 6 cm. Then I looked around for a height of 50 cm and it turned out to be on the 26th. [Respondent using Table means continuing the existing table in the question and referring to the calendar]

The student's answer to part (b) by FR, is a unique (different) solution from all students who took the test. The following is a fragment of an interview regarding the strategy used by FR students in solving question number 2.

Q : What is your strategy to solve question number 2?
FR : First, you have to be based on the questions, the benchmark questions are every week on Saturday. The first and second weeks are looking for the difference, for example in the first week 2cm then in the 2nd week it increases to 8cm obtained from 2 + 6 then the height increase is 6 cm

Even though the two solutions by ZP and FR obtain the same answer, using the same difference value, namely 6, they both have different basic understandings. ZP in its calculations uses procedural knowledge by creating a tabular form to find the answer asked in the question. ZP describes changes in the height of nut trees every week by referring to the initial height, namely 2 cm on March 1, 8 cm on March 8, and so on increasing by 6 every 7 days. The answer to the question, "How tall will the beanstalk be on April 12?" ZP immediately looks in the table in the column for the date "April 12" and the resulting "height", as well as to determine the date that shows the height of the nut tree is 50cm. Meanwhile, FR in the process of solving the answer uses intuitive abilities, namely trying, guessing, guessing, then finding the answer. As FR explained, look for the difference, "for example, if in the first week, it was 2 cm, in the second week it was 8 cm and this is obtained from 2 + 6 then the increase in height is 6 cm". The explanation of the answer is in the form of a pattern, namely the height of the nut tree increases by 6 (6+) every week.

From the results of observations on student worksheets and interviews, it is known that ZP and FR students provide interpretations that have a quantitative relationship, for example predicting the height of the next week's nuts by referring to the initial height of the nut tree (2 cm). FR students provide flexible interpretations, connected to other concepts, for example, when asked about solving problem number 2 in another way, FR answered that he could use the concept of arithmetic series. ZP and FR students also solved the problem by interpreting the operations applied, namely always increasing by 6 every week (week), doing a counting jump (prediction) when determining the height of the beanstalk after three months and in what week the beanstalk reached a certain height (50cm). Based on the characteristics of this way of understanding, it is generally concluded that ZP and FR students have an algebraic invariance way of thinking.

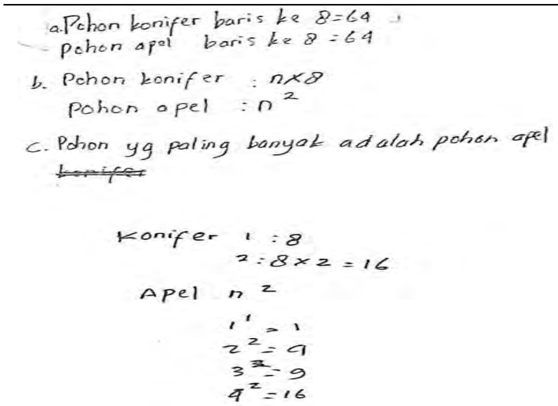
Some students encountered difficulty in solving question number 2, for example, student NS, namely on the question, "determine the height of the beanstalk at 3 months", NS gave the answer: 2 cm x 90 days x 6 cm = 1880 cm. He explained that 2 cm is the initial height, 90 days is 3 months of age and 6 cm is the increase in height every week, still 6 cm. This shows that NS's interpretation of problem number 2 has no quantitative relationship and problem-solving without interpreting operations or manipulating without checking the meaning. There is also another difficulty, namely the question "What date will the nut tree reach 50 cm in height?" PN students gave their answers, namely April 28 (Monday). Based on the question information, measurements are carried out every Saturday. These are some of the interpretation and problem-solving errors made by the majority (80%) of students.

Even though both students (ZP and FR) have the expected way of thinking (algebraic invariance) they both still use procedural abilities. Problem number 2 above can be solved using the concept of arithmetic sequences, to teach students that mathematics has more than one way of understanding (solution), namely using the formula for the nth number of rows: Un = a + (n-1)b with a = 2, b = 6, where Un is the amount of time (nth week). In this way, the answer to the question of how tall the tree will be on April 12 can be determined. This can be seen on the calendar, namely n = 7, so U_7 is 38. For the question, how tall will the tree be in three months? In this case n = 14, so U_{14}= 80. The next question is What date does the tree reach 50 cm in height? This section will be easier to complete with a functional approach. The basis for this understanding can be built from the formula for the number of nth terms: Un = a + (n-1)b, where Un is taken as the variable y which indicates the height of the tree, a and b are constants whose values are 2 and 6, and n can be taken as the variable x which indicates time. In this way, a function equation y = 2 + 6(x-1) can be formed, which is then simplified to y = 2(3x-2). When it reaches a height of 50cm (y = 50), then x = 9. In this case, the 9th week can be seen on the calendar, namely April 26. Through the function obtained, it will be easier to draw a graph of the growth of the beanstalk as requested in the problem.

Results and Discussion of Problem Number 3

Table 4. Example of Representation of Student Answers to Problem Number 3

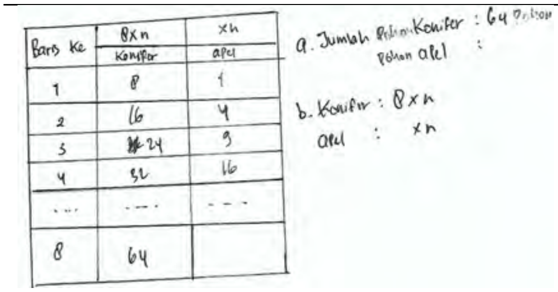
Answers to test questions	Translate



(a)

a. 8th row conifers = 64
 8th row apple tree = 64
b. Conifer Trees: n *x* 8
 Apple Tree: n^2
c. The most planted trees are apples
 Conifer 1 : 8
 2 : 8 *x* 2 = 16
 Apple n^2
 $1^2 = 1$
 $2^2 = 4$
 $3^2 = 9$
 $4^2 = 16$

(a)



(b)

Baris ke	8*x*n	*x*n
	Konifer	Apel
1	8	1
2	16	4
3	24	9
4	32	16
…	…	…
8	64	

a. Number of trees
 conifers: 64 trees.
 The Apple tree:

b. Conifers: 8*x*n
 Apple: *x*n

(b)

Table 4 is an example of a representation of students' correct answers and it is known based on Table 1 that only 3 out of 30 people (10%) can give the correct answer. Problem number 3 is a pattern generalization problem and has a higher level of difficulty than questions number 2 and 1. Based on the examples of answers to part (a), RT students and answers (b) by AS students provide interpretations of the images (patterns) and determine the number of each tree in each row using reasoning or inductive thinking. Next, students make deductive generalizations, namely for apple trees nxn or n^2 and for conifer trees 8n.

The following is the explanation of RT and AS students in the interview.

Interview with RT

Q : Tell me about the picture! [Points at question]
RT : The inside of the picture is an apple tree, and the outside is a conifer.
Q : : How do I find the answer to that question?
RT : To the question, "how many apple trees and conifers are in row 8?" I first make the formula, so it's easy to find the answer.
Q : What's the formula?

RT : Conifers = n x 8 (n is for the nth tribe or row, while 8 is the number of conifers in the first row)
Q : Before you conclude it is n x 8, what is the thought process to arrive at n x 8?
RT : Well Ma'am. Since the first equation has 8 conifers, 16 in the second, 24 in the third, and 32 in the fourth, that means nx 8.
Q : Okay for the apple tree how do I find it?
RT : Here I use the formula for apples=n^2
Q : How did you find the formula for n^2?
RT : Because first row 1 x1 =1, second row 2 x2=4, third row 3 x3=9 and fourth row 4x4=16
Q : OK, back to n, n can be any number or what does n mean?
RT : n can be any number because it is unknown.

RT students describe their understanding of problem number 3 and its solution clearly. RT can understand symbols and can also interpret the operations used. For example, he explains the symbol n as the nth term or line, making the correct guess by using reasoning or inductive thinking. For example, explain that in the first row of the Apple tree pattern: 1x1, second row: 2x2, third row: 3x3, and so on until the question the eighth row is 8x8 = 64. In the same way, for Conifer trees the first row is 8, the second row is 16, third row is 24. Next, RT can determine a formula to determine the number of apple trees in the nth row or term, namely nxn or n^2, and for conifer trees 8n. Forms n^2 and 8n are the result of a deductive process or way of thinking. The conclusion that apple trees are more numerous than conifer trees was obtained by RT after applying and comparing the n^2 and 8n formulas.

Q : Why are there so many apple trees?
RT : Well, ma'am, the conifer trees in the ninth row are 72, while the apple trees in the ninth row are 81.

Interview with AS

Q : What do you understand from question 3?
AS : Number 3 is that there are plants that have been planted in a rectangular area that continues to fold every few rows.
Q : Try to explain in the first line!
AS : For the first row there are 8 conifer trees and 1 apple tree. The second row of apple trees, it multiplies by 4 to 4 apple trees. For conifers 8 x 2 = 16 conifers. In the third row of conifers 8 x 3 = 24.
As for Apple, I don't know Mrs.

AS students appear to be trying to make a temporary conjecture that the number of trees in a row is obtained from the number of trees in the first row multiplied by that row. This is true for conifer trees but not for apple trees. At first, the US also made the conclusion that the number of coniferous and apple trees was the same. This conclusion was obtained because the US used the formula for determining the number of apple and conifer trees only in the eighth row. However, after being asked during the interview discussion, which one had more results between 8xn and the next? AS revised his answer, that there were more apple trees. As a result of the interview discussion with AS regarding question number 3, information was obtained that students need to be given indirect assistance in the form of encouragement to reflect on their thinking when they encounter difficulties in solving problems. This can help students develop their way of thinking about problems (Natcha Kamol & Yeah Ban Har, 2010).

From the description above, both statements on student worksheets and in interviews show that RT and AS students' interpretations of problem number 3 characteristically have a quantitative relationship, understand symbols, interpret the operations they apply, and can determine the general form (nth row). Based on this way of understanding, it is concluded that RT and AS students have an algebraic invariance way of thinking as well as a deductive way of thinking.

Based on Table 1, it is known that the majority of students (90%) made mistakes in solving problem number 3, including those who could not provide an answer (the answer sheet was blank). There are several types of errors made by students, for example, NA students try to give answers using the concept of arithmetic sequences but errors occur in understanding the concept. NA students write the formula for the nth term: Un = (a+1n)b $\rightarrow U_8=(8+1x8)8 \rightarrow U_8=128$. NA provides an explanation or statement that U_8: 8th term, a=8 because the first row of conifers totals 8, likewise for the difference (b)=8 because the first row totals 8. The students' interpretation and problem-solving do not have a quantitative relationship, do not understand symbols, not interpret operations applied and manipulated without checking their meaning. To the question, which tree has the most? 90% of students answered conifer trees with the reason that conifers protect apple trees so there should be more of them. Based on the characteristics of this way of understanding, it is concluded that students have a non-referential symbolic way of thinking.

Question number 3 is a pattern generalization question that is expected to develop students' algebraic thinking abilities. The thing that needs to be emphasized in this problem is the shape of the pattern and also the difference (difference) resulting from the existing number pattern or image. Based on known information or data, there are two types of trees, namely apples and conifers. If you look closely, the apple tree forms a perfectly square pattern, the number of sides is the same, so it can easily be determined that the first row is 1x1, the second row is 2x2, the third row is 3x3, and. From the shape of the pattern, it can be determined that the nth row is nxn. The difference (difference) produced varies and continues to increase, forming a series of odd numbers: 3, 5, 7, ... Meanwhile, conifer trees form a square image but only on the outside, forming a pattern with the number of first rows: 8, second row: 16, third row; 24, and so on with the resulting difference being fixed namely 8. From the shape of the resulting pattern it can be determined that the first row is 8x1, the second row is 8x2,... and so on. So, it is concluded that the number of conifer trees will increase by multiplying 8 by the corresponding row. Thus the formula for the nth term can be determined as 8xn where n is the number of rows. This description concludes that the most numerous trees if land continues to be expanded are apple trees. This is because the difference between apple trees continues to increase while conifers remain constant, and is based on the nth-term formula, where nxn is more than 8xn.

The results of the analysis of students' answers from both written answers and interviews, it is known that the majority of students have a characteristic way of thinking that is very far from the expected thinking, namely algebraic, proportional, and deductive invariance. Students perform manipulations without the ability to investigate the meaning of symbol relationships and any transformations involved in them. In other words, symbols are not understood as representations of a coherent mathematical reality. This shows that classroom learning does not pay attention to students' thinking processes in solving problems (Harel, 2008b).

CONCLUSIONS

In general, students have a non-referential symbolic way of thinking. This is based on the thinking characteristics of students, namely not understanding the symbols and operations involved, not being flexible or concepts being understood separately, and not connected to other concepts. This way of thinking illustrates that students' understanding is procedural in nature. There are also findings that the more complex the algebra problem given, the more students experience difficulty in solving it. In problem 1, which is relatively easier than problems 2 and 3, some students (50%) can solve it and have a way of thinking about algebraic invariance and proportional reasoning. In problem 2, which is relatively more difficult than problem 1, only 20% of students answered correctly and had an algebraic invariance way of thinking and proportional reasoning, while the rest (80%) of students had a non-referential symbolic way of thinking. Problem 3, which is a problem of pattern generalization that requires deductive thinking and algebraic invariance, only 10% of students answered correctly, and the rest (90%) of students had a non-referential symbolic way of thinking.

These findings can lead to further investigations, for example how to present material or an algebraic (mathematics) concept so that it can develop the desired way of thinking, namely algebraic invariance thinking, proportional reasoning, and deductive reasoning. To form the desired thinking habits, namely by providing opportunities for students to understand mathematical objects or problems in different ways or more than one way of understanding. This can be done since students are still in elementary school. In addition, generalization, modeling, functional, and problem solving approaches can be used to develop algebraic thinking.

References

[1] Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*(2), 87–115. https://doi.org/10.2307/30034843

[2] Chimmalee, B., & Anupan, A. (2022). Effect of Model-Eliciting Activities using Cloud Technology on the Mathematical Problem-Solving Ability of Undergraduate Students. *International Journal of Instruction*, *15*(2), 981–996. https://doi.org/10.29333/iji.2022.15254a

[3] Driscoll, M., Zawojeski, J., Humez, A., Nikula, J., Goldsmith, L., & Hammerman, J. (2001). *The Fostering Algebraic Thinking Toolkit A Guide for Staff Development*. *3*.

[4] Duval, R. (2017). Understanding the mathematical way of thinking - The registers of semiotic representations. In *Understanding the Mathematical Way of Thinking - The Registers of Semiotic Representations*. https://doi.org/10.1007/978-3-319-56910-9

[5] Harel, G. (2008a). DNR perspective on mathematics curriculum and instruction, Part I: Focus on proving. *ZDM - International Journal on Mathematics Education*, *40*(3), 487–500. https://doi.org/10.1007/s11858-008-0104-1

[6] Harel, G. (2008b). What is Mathematics? A PedagogicalAnswer to a PhilosophicalQuestion. In B. Gold & R. A. Simons (Eds.), *Proofandother Dilemmas Mathematics andPhilosophy* (p. 346). The Mathematical Association of America.

[7] Harel, G. (2013). DNR-Based Curricula: The Case of Complex Numbers. *Journal of Humanistic Mathematics*, *3*(2), 2–61. https://doi.org/10.5642/jhummath.201302.03

[8] Harel, G., & Sowder, L. (2013). Advanced mathematical-thinking at any age: Its nature and its development. *Advanced Mathematical Thinking: A Special Issue of Mathematical Thinking and Learning*, *December 2013*, 27–50. https://doi.org/10.4324/9781315045955

[9] Jupri, A., & Drijvers, P. (2016). Student difficulties in mathematizing word problems in Algebra. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(9), 2481–2502. https://doi.org/10.12973/eurasia.2016.1299a

[10] Kaput, J. J., & Blanton, M. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, *36*(5), 412.

[11] Kieran, C. (2004). Algebraic thinking in the early grades: What is it. *The Mathematics*

Educator, *8*(1), 139–151.

[12] Lew, H.-C. (2004). Developing Algebraic Thinking in Early Grades: Case Study of Korean Elementary School Mathematics. *The Mathematics Educator*, *8*(1), 88–106.

[13] Manly, M., & Ginsburg, L. (2010). *Algebraic Thinking in Adult Education*. *September*, 20. internal-pdf://0169661638/algebra_paper_2010V.pdf

[14] Mason, J. (1996). *Expressing Generality And Roots Of Algebra* (N. B. et al. (eds.) (ed.)). Kluwer Academic Publishers.

[15] Miles, Matthew B & Huberman, A. M. (1994). *Qualitative Data Analysis* (Second Edi). Sage Publications, Inc. https://books.google.co.id/books?hl=id&lr=&id=U4lU_-wJ5QEC&oi=fnd&pg=PA10&dq=miles+and+huberman+1994+qualitative+data+analysis+pdf&ots=kFZB5IRYVS&sig=uFmd7-FQIbCihaBJObVbrkFNC94&redir_esc=y#v=onepage&q&f=false

[16] Natcha Kamol, & Yeah Ban Har. (2010). Shaping the future of mathematics education: Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia. In *Upper primary school students' algebraic thinking* (Issue July).

[17] NCTM. (2000). *Using the NCTM 2000 principles and standards with the learning from assements materials*. http://www.wested.org/Ifa/NCTM%0A2000.PDF. %0A

[18] Nottingham, T., & User, N. E. (1992). *Lins , R . C . (1992) A framework for understanding what algebraic thinking is . PhD thesis , University of*.

[19] Novita, D., Cahyaningtyas, & Toto. (2018). Analysis of Student Algebra Thinking Process. *Jurnal Pendidikan Matematika Dan Sains*, *4*(1), 50–60.

[20] Ojo, S. G. (2022). Effects of Animated Instructional Packages on Achievement and Interest of Junior Secondary School Student in Algebra. *Mathematics Teaching-Research Journal*, *14*(1), 99–113.

[21] Permatasari, D., & Harta, I. (2018). Kemampuan Berpikir Aljabar Siswa Sekolah Pendidikan Dasar Kelas V Dan Kelas Vii: Cross-Sectional Study. *Jurnal Pendidikan Dan Kebudayaan*, *3*(1), 99. https://doi.org/10.24832/jpnk.v3i1.726

[22] Purwanto, W. R. (2019). Proses Berpikir Siswa dalam Memecahkan Masalah Matematika Ditinjau dari Perspektif Gender. *Prosiding Seminar Nasional Pascasarjana UNNES*, 895–900.

[23] Radford, L. (2000). *Signs and Meanings in Students' Emergent Algebraic Thinking: A Semiotic Analysis*. 237–268.

[24] Sari, I., Marwan, M., & Hajidin, H. (2019). Students' Thinking Process in Solving Mathematical Problems in Build Flat Side Spaces of Material Reviewed from Adversity Quotient. *Malikussaleh Journal of Mathematics Learning (MJML)*, *2*(2), 61–67. https://doi.org/10.29103/mjml.v2i2.1468

[25] Seeley, C. L. (2004). *President ' s Message A Journey in Algebraic Thinking*. *September*, 2004.

[26] Suryadi, D. (2012). *Membangun Budaya Baru dalam Berpikir Matematika*. Rizqi Press.

[27] Uzun, S. C., & Arslan, S. (2009). Semiotic representations skills of prospective elementary teachers related to mathematical concepts. *Procedia - Social and Behavioral Sciences*, *1*(1), 741–745. https://doi.org/10.1016/j.sbspro.2009.01.130

[28] van Amerom, B. (2002). Reinvention of early algebra. In *Developmental research on the transition from arithmetic to algebra*.

[29] Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, *67*(2), 171–185. https://doi.org/10.1007/s10649-007-9092-2

[30] Windsor, W. (2010). Algebraic Thinking : A Problem Solving Approach. *Proceedings of the 33rd Annual Conference of the Mathematics Education Research Group of Australasia*, *33*, 665–672. https://research-repository.griffith.edu.au/bitstream/handle/10072/36557/67823_1.pdf?sequence=1&isAllowed=y

[31] Windsor, W., & Norton, S. (2011). Developing Algebraic Thinking : Using A Problem Solving Approach in A Primary School Context. *Mathematics : Traditions And [New]Practices*, *March*, 813–820.

[32] Yin, R. K. (2014). Design and Methods, Third Edition, Applied Social Research Methods Series, Vol 5. In *Sage Publications* (pp. 1–181).