Assessing the understanding of the slope concept in high school students

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Abstract: This research reports the implementation of an evaluation instrument of the slope concept in high school students. The design of this study was based on four dimensions: Skills, Properties, Uses and Representations (SPUR model; Thompson & Kaur, 2011) and on three conceptualizations: constant ratio, behavior indicator and trigonometric conception. This work adopts a qualitative approach to analyze the students' productions and a quantitative approach when obtaining the percentages of the student's responses. The general objective of the work is to evaluate the effect produced by a group of tasks designed with the SPUR model on the slope concept in high school students. The results show that students have traditional conceptualizations of the slope as a constant ratio and trigonometric conception. However, these conceptualizations emphasize more procedural aspects than conceptual ones. This finding could partially explain why students can solve certain tasks of procedural nature but not conceptual tasks that usually require a multifaceted view of the slope.

Keywords: slope, SPUR model, conceptions, assessment

INTRODUCTION

The slope is considered to be a key concept since its understanding is an essential requirement for the study of concepts associated with linear functions or rate of change in calculus, linear regression in statistics, uniform linear movement in physics and other topics of high-level mathematics (Clement, 1989; Nemirovsky, 1992, 1997; Simon & Blume, 1994; Nagle & Moore-Russo, 2014; Casey & Nagle, 2016).

Different studies report the challenges that students face with this concept. In particular, they highlight the lack of connection and interpretation of the slope in different contexts (Thompson, 1994; Stump, 1999, 2001; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Lobato & Siebert, 2002; Moore-Russo, Conner, & Rugg, 2011).

This notion makes sense when considering the existing diversity of conceptualizations associated with the slope. For example, the pioneer works of Stump (1999; 2001) offer nine conceptualizations of the slope concept according to the context where it can be used. Furthermore, Moore-Russo et al. (2011) expand this categorization to eleven conceptualizations. These works constitute the basis for further investigations addressing the study of conceptualizations of students and teachers from all educational levels (Stump, 1999; Moore-Russo et al., 2011; Nagle, Moore-Russo, Viglietti, & Martin, 2013; Nagle & Moore-Russo, 2013).

In the case of students, the difficulties connecting the different conceptualizations of the slope or the emphasis on one or a few aspects of it stand out. For example, Zaslavsky, Sela and Leron (2002); Nagle and Moore-Russo, (2013); Nagle et al. (2013) and Deniz and Uygur-Kabael, (2017) show that students associate the slope with its geometrical and algebraical characteristics but are usually not able to connect these features in higher demand tasks.

Regarding the case of teachers (professional teachers or student teachers), other research has shown that they possess a more varied range of conceptualizations of the slope than students (Stump, 1999; Nagle et al., 2013; Nagle & Moore-Russo, 2013) but seem to omit it in their teaching, concerning the educational level where the teacher performs (Nagle & Moore-Russo, 2013).

For example, in secondary school teachers, there is an absence of conceptualizations of the trigonometric type (Azcárate, 1992; Stump, 1999); a dominant conceptualization of the slope interpreted as a geometrical proportion (Stump, 1999; Moore-Russo et al., 2013) and difficulties to work with the average rate of change (Coe, 2007).

Another body of work proposes didactical approaches that highlight the relevance of working with different conceptualizations of the slope in the classroom to work and foster the development of this concept (Deniz & Uygur-Kabael, 2017), or from work with teachers or student teachers (Moore-Russo et al. 2011; Diamond, 2020).

Other approaches emphasize the role of variation and covariation as essential to understanding the linear function (Thompson, 1994; Carlson et al., 2002). For example, DeJarnette et al. (2020) adopt a social semiotic perspective to analyze the interactions between students and teachers, highlighting that students interpret the slope more as a quotient and less as a parameter of the linear function.

Lastly, other studies analyze textbooks at the undergraduate level. These studies report inaccuracies in describing geometric and algebraic connections when defining the slope (Zaslavsky et al., 2002). Studies have found that the lack of connection between different slope conceptualizations could be due to the frequent use of problems with low cognitive demands (Arnold & Hicks, 2011). Another research indicates frequent use of problems with procedural features and fewer conceptual concepts (Tuluk, 2020).

Theoretical Debate

Nagle and Moore-Russo (2013) establish that research regarding the slope concept has focused on the study of conceptualizations in teachers and students in an isolated manner, and little has been explored regarding its articulation, which could enable a 'conception network' to improve the understanding of this concept. There is general agreement regarding the existence of a marked difference between students' and teachers' conceptualizations. In order to avoid this, teachers should promote a conception network according to slope usage to obtain an integrated view of the slope (Moore-Russo et al.,2011; Stanton & Moore-Russo, 2012).

Regarding this point, Nagle and Moore-Russo (2013) propose an analysis of slope conceptualizations through a frame of reference that considers the procedural and conceptual dimensions to cluster and generate a network of conceptualizations.

The previous accounts exhibit the need for instruments that evaluate the student's understanding level of the slope concept, which can reflect the diversity of conceptualizations and their connections. This specific need leads to wondering about the appropriate instruments to perform this evaluation.

According to Popham (2000, as cited in Heuvel-Panhuizen, Kolovou & Peltenburg, 2011), the evaluation can be understood as a process where teachers use students' responses (generated through stimuli or natural conditions) to make inferences about the knowledge, skills or the affective state of students. In this context, teachers and researchers show concern about the problems that the systematic or traditional evaluation may raise in contrast to more integrative perspectives whose focus is not only on evaluating procedural fluency (Thompson & Kaur, 2011). For example, interpretative skills for problem-solving, skills in contextualized tasks, problem-solving processes, etc.

Regarding the evaluation aspect, Thompson and Kaur (2011) promote the idea that if teaching reflects a multidimensional approach, the evaluation should also be in line with this perspective; thus, the evaluation instruments must consider these considerations.

With this context in mind, this research focused on answering the following question: What is the effect of a group of tasks on understanding the slope concept in high school students when evaluated through a SPUR model?

Based on the above, the reference framework used in this research is discussed (framework proposed by Nagle & Russo, 2013; and the SPUR model), the method used and the description of the activities that make up the evaluation instrument are reported. Finally, the results are exhibited and discussed based on the productions of the students.

THEORETICAL FRAMEWORK

In this research, we adopt the theoretical framework proposed by Nagle and Moore-Russo (2013), which considers research regarding procedural and conceptual understanding and its relation with visual and analytic interpretations of the slope for each of the eleven slope conceptualizations identified in Moore-Russo et al. (2011).

The authors propose five key slope components whose connection (established through tasks or problems that demand some knowledge of the slope) promotes a network of conceptualizations that offers a more integrated characterization of the understanding of the slope. These components are (1) constant ratio, (2) determining property, (3) behavior indicator, (4) trigonometry and (5) calculus.

Given our study sample, this research only considers components (1), (3) and (4). The features of each component according to the procedural or conceptual dimension are summarized in tables 1, 2 and 3.

Table 1. Description of the conceptualization of the slope as a constant ratio

	Examples of the slope as a constant ratio	
	Procedural Emphasis	**Conceptual Emphasis**
Visual approach	$R_{v,p}$: rise/run or vertical change/ horizontal change.	$R_{v,c}$: similarity of slope triangles yields a constant ratio of rise/run regardless of the position on the graph.
Analytic approach	$R_{a,p}$: change in y over change in x; $\frac{y_2-y_1}{x_2-x_1}$.	$R_{a,c}$: constant rate of change between two covarying quantities; and equivalence class of ratios and hence a function.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.130.

Table 2. Description of the conceptualization of the slope as a behavior indicator

	Examples of the slope as a behavior indicator	
	Procedural Emphasis	**Conceptual Emphasis**
Visual approach	$B_{v,p}$: increasing lines have a positive slope; decreasing lines have a	$B_{v,c}$: positive rise corresponds to a positive run for an increasing line,

	negative slope; horizontal lines have zero slope.	yielding a positive slope. For a decreasing line, a negative rise corresponds to a positive run, yielding a negative slope. A horizontal line has zero rise for any run, yielding a zero slope.
Analytic approach	$B_{a,p}$: value of m in the equation for a linear function (e.g.., in $y = mx + b$) indicates whether f is an increasing $(m > 0)$, decreasing $(m < 0)$ o constant $(m = 0)$ linear function [previously parametric coefficient].	$B_{a,c}$: application of the definition of increasing/decreasing/constant functions to explain positive/negative/zero slope respectively (e.g., f is increasing means that $f(x_1) < f(x_2)$ if $x_1 < x_2$, so $\frac{[f(x_2)-f(x_1)]}{x_2-x_1} > 0$.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.131.

Table 3. Description of the conceptualization of the slope as a trigonometric conception

	Examples of the slope as a trigonometric conception	
	Procedural emphasis	**Conceptual emphasis**
Visual approach	$T_{v,p}$: steepness of a line; slope as the angle of inclination of the line with a horizontal; as a line is rotated about a point, the slope changes.	$T_{v,c}$: the angle of inclination determines the rise/run; a steeper line has a greater rise per given run than a less steep line
Analytic approach	$T_{a,p}$: slope is calculated as $tan\,\theta$, where θ is the angle formed by the graph of the linear equation and an intersecting horizontal line.	$T_{a,c}$: the angle of inclination determines the ratio of $(\frac{y_2-y_1}{x_2-x_1})$, which is equivalent to $tan\,\theta$.

Source: "SLOPE: A Network of Connected Components", by Nagle and Moore-Russo, 2013, *North American Chapter of the International Group for the Psychology of Mathematics Education*, p.132.

SPUR Model

In this research, the SPUR model (Skills, Properties, Uses and Representations) proposed by Thompson and Kaur (2011) is used as a theoretical framework. This approach promotes a multidimensional evaluation that reflects the student's knowledge more broadly in four key aspects, the same of which constitute the acronym of the model (SPUR). The following section describes the characteristics of each dimension.

> ***Skills*** represent those procedures that students should master with fluency; they range from applications of standard algorithms to the selection and comparison of algorithms to the discovery or invention of algorithms, including procedures with technology.
>
> ***Properties*** are the principles underlying the mathematics, ranging from the naming of properties used to justify conclusions to derivations and proofs.
>
> ***Uses*** are the applications of the concepts to the real world or to other concepts in mathematics and range from routine "word problems" to the development and use of mathematical models.
>
> ***Representations*** are graphs, pictures, and other visual depictions of the concepts, including standard representations of concepts and relations to the discovery of new ways to represent concepts" (Thompson & Senk, 2008, p.2)

According to this, the authors emphasize that if our educational aim is to create students with a solid and flexible understanding of mathematics, then it becomes essential to evaluate more than their skills' knowledge, as is traditionally done. The authors warn that if teachers focus their evaluation on a single dimension, they might obtain a wrong view of their students' understanding. In contrast, if teachers expand their evaluation dimensions, they might gain a better view of the strengths and weaknesses of their students.

The SPUR approach has been employed on an international project regarding mathematical achievement (IMPA) in a longitudinal survey on elementary education in different countries. Their results have highlighted the students' low performance on the dimensions evaluated by the model (Thompson & Kaur, 2011), offering a more integrated picture of the evaluation.

METHOD

The design of this research is of a qualitative and interpretative nature. The study was conducted in three stages: 1) a literature review of the slope concept, which included terms related to the evaluation of this concept; 2) a selection of the theoretical framework and task design and 3) instrument application and analysis.

The study sample comprised 35 high school students from a private school in Mexico. The selection was made following the curriculum of this educational level and the average performance of students. Only three (out of a total of five) of the slope characterizations proposed by Nagle and Moore-Russo (2013) were selected.

The test was directly applied as an evaluation instrument of the slope concept. The participating students had previously studied the slope concept on several occasions throughout their academic training, which included courses such as algebra, geometry, trigonometry and analytic geometry.

The design of the evaluation instrument incorporated the SPUR approach so that each of the model's dimensions was reflected in the tasks according to the selected conceptualizations.

The Tasks

In **task 1,** students only require one slope conceptualization of the *constant ratio* type. The approach can be analytic or visual, but with a procedural emphasis for both cases. The situation posed requires thinking analytically about the slope as a change in the variable y over a change in the variable x. Visually it implies identifying, once the line has been drawn, the vertical change between the horizontal change by means of the drawing of a reference right triangle. Regarding the SPUR model, the task corresponds to the *Skills* dimension since it requires the application of a standard algorithm. Additionally, it covers the *Representation* dimension should the student correctly make the required graphical scheme.

Task 2 can be characterized by the cognitive processes that the student employs in the resolution process. The task involves a *constant ratio* conceptualization through a visual or analytic approach that can lead to a procedural or conceptual emphasis. The task comprises the *Skills* and *Properties* dimensions of the SPUR model.

For example, for part (a) in the visual approach, a student who correctly relates the change in the variable y to the change in the variable x by means of an appropriate representation showing that the similarity of two right triangles at any locations of the lines shown maintain a constant relationship of elevation and run, it will show a visual approach with a conceptual emphasis compared to those students who could limit their reasoning to a procedural emphasis, if it only shows the value of the slope from a single right triangle of reference.

To elucidate this variant, subsection (b) is introduced, which questions the student about which of the two lines shown has a greater angle of inclination? This would show the conceptual or procedural emphasis of the individual since the answer demands to relate the conception of *constant ratio* with the *trigonometric conceptualization* (which corresponds to the dimensions of properties, representations and skills of the SPUR model). More specifically, if a student associates that the angle of inclination with the horizontal axis is greater for the line with the least slope, then his emphasis is procedural. In contrast, if in her answer she argues that the angle of inclination determines the rise-run which implies that a steeper line has a higher elevation then her approach is visual but maintaining a conceptual emphasis.

On the contrary, an analytical approach with procedural emphasis is associated with responses based solely on the calculation of the $\tan\theta$ without emphasizing its relationship with the slope (this situation is related to the dimensions of *skills* and *representations* of the SPUR model). Within the analytical approach, a conceptual emphasis (*properties* dimension) can be characterized based on whether the student argues that the relationship $\frac{y_2-y_1}{x_2-x_1}$ is equivalent to $\tan\theta$.

Task 3 involves different approaches. On the one hand, it involves the slope conceptualization as a *behavior indicator* since this task requires the interpretation of the slope as a change indicator: increase, decrease or constant. The approach is initially visual; subsection (a) and subsection (b) demand analytic work. The emphasis can either be procedural or conceptual, according to the student's development. Finally, the task can be associated with the four dimensions of the SPUR model: *Skills*, *Properties*, *Uses* and *Representations*.

In this way, a student who exhibits the years in which there were greater losses and gains, but his reasoning is limited to considering only the calculation of the slopes or the inclination of these will show an analytical or visual procedural emphasis, respectively. On the other hand, if your arguments, in part (b), focus on the constant rate of change between the two given covariant quantities or if you propose a similarity relationship between any two right triangles on the respective lines, your reasoning will correspond to a conceptual emphasis.

In **Task 4,** the *Uses* dimension from the SPUR model was included since its focus relies on applications of the slope concept. In this case, a verbal problem is presented and involves the visual and analytic approach with a conceptual emphasis. This requires that the student understand that the situation involves determining the vertical change from the horizontal change and consequently associating it with the slope.

The aim of **task 5** is to make the student use the *trigonometric* conceptualization of the slope. The approach is initially analytic, but the student can rely on a visual approach to solve the task. The emphasis is conceptual and results in a procedural emphasis. The dimensions of the SPUR model associated with this task are *Properties, Skills and Representations*.

In this task, a student associates the angle formed with the horizontal axis and a given line with that property that describes the elevation-range ($m = \frac{y_2-y_1}{x_2-x_1}$) equivalent to $\tan\theta$ will show a trigonometric conceptualization with a conceptual emphasis. In another case, when the student does not exhibit reasoning that allows elucidating the transit between the equivalences shown, we will say that his reasoning is of a procedural type.

Task 6 is an adaptation of the task proposed by Zaslavsky et al. (2002). Subsection (a) corresponds to the *constant ratio* conceptualization. The emphasis could be visual or analytic according to the student's path for solving the task. Any of these paths involve the use of the *Skills* and *Representations* dimensions of the SPUR model. Additionally, the combination of *Representations* and *Properties* can be established if the arguments correspond to a conceptual emphasis.

For example, locating the coordinates of any two points on the line and calculating its slope is associated with an analytical approach with a procedural emphasis. In another case, if it is determined through the outline of a right triangle that exhibits the change in height-travel, then the approach will be visual with procedural emphasis.

Subsections (b) and (c) seek to determine if the students manifest the trigonometric conceptualization of the slope. Part (b) is used as scaffolding to try to get the student to notice the change of scale in each displayed representation so that the question in part (c) makes sense. According to the student's response, a visual or analytical approach could be associated, but, in any case, although the task derives from a procedural emphasis, a conceptual emphasis is initially required to be able to justify whether the slopes are the same or different. The latter can be associated with the dimensions of *properties* and *representations* of the SPUR model.

Finally, **task 7** involves the *Uses* dimension of the SPUR model since the suggested problem is of a verbal nature and involves data associated with temperature measurement scales. The slope conceptualizations needed for this task could be various: *constant ratio*, and *behavior indicator*, according to the activities' subsections. In any case, it is necessary to start from a conceptual emphasis to be able to carry out the appropriate approaches that culminate in the generation of a covariational model and understand its parameters to respond to subsections (b) and (c).

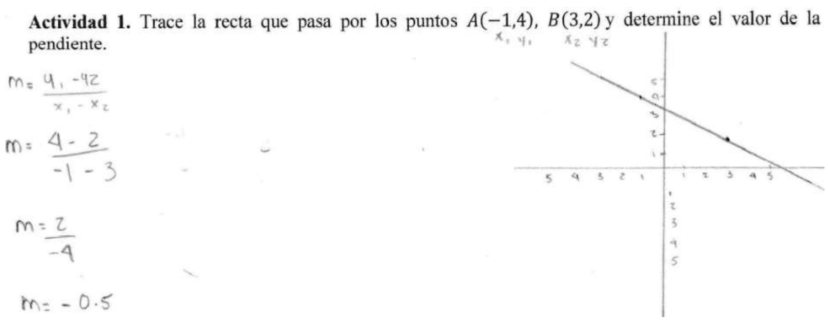
RESULTS

The theoretical framework of Nagle and Moore-Russo (2013) and the SPUR model were employed for the data analysis.

Below are examples of the conceptualizations of the slope of some students whose answers stood out for being more similar to the analysis framework.

The conceptualization of slope as a constant ratio

It was found that 67% of the total participating students show a conceptualization of slope as a constant ratio. From this percentage, 80% showed an analytic approach through a procedural emphasis associated with determining a change in variable y in relation to a change in variable x. This result is exemplified in the production of student E33, which inferred the corresponding slope value and presented the considered line's graphical representation but did not relate the two corresponding approaches (visual and analytic).

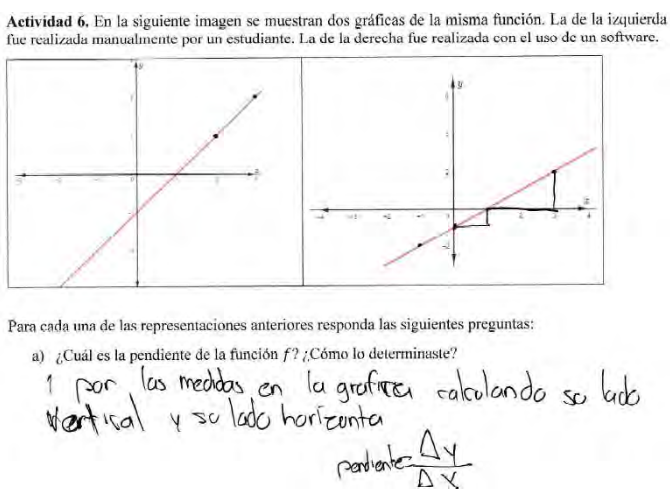


Task instruction: Draw the line that touches the coordinates A(-1,4), B(3,2) and find the slope value.

Figure 1. Response of participant E33 in Task 1.

This type of performance is reflected in most of the productions and is accentuated when observing the responses of Task 2, a task which demands the slope value by a graphical representation. The aim is to identify whether the student can solve this through a visual approach or needs to transform the task to an analytic approach.

On the other hand, student E11 (figure 2) is the only one showing a constant ratio conceptualization with a visual approach and conceptual emphasis, but only when making task 6 subsection (a).



Task 6. Instructions: The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.

Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?

Student's response: " 1^0 by measurements on the graph calculating its vertical side and its horizontal side".

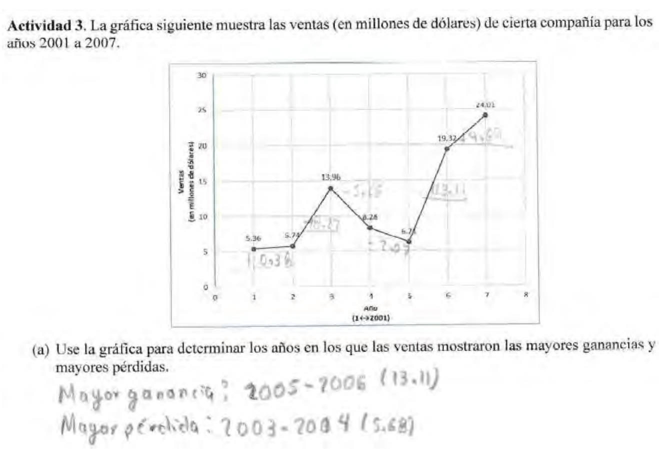
Figure 2. Response of student E11 in task 6(a)

As figure 2 shows, the student infers the slope value using a visual approach by drawing a pair of similar triangle rectangles that allow him to infer the slope value. His explanation even mentions vertical and horizontal change. Even though the student pointed out a pair of points in the left-sided graph, this relation was not displayed; in this case, this could be conditioned to the wording of the instruction when it mentions that it relates to the same function.

The conceptualization of the slope as a behavior indicator

The design of task 3 allowed the student to exhibit the conceptualization of the slope as a behavior indicator. According to our theoretical framework, this aspect relates the positive slope with the increment of the function, the negative slope with decrement and a zero slope with horizontal lines. According to the task, only 60% of the students showed part of this conceptualization, corresponding to a visual approach. Nevertheless, the remaining 40% of responses were possibly biased by the question's wording since it was stated in terms of higher and lower points of the graph and not in terms of the relation with the slope.

A standard answer made by the students can be seen in the case of student E7, which shows the correct answer (see figure 3). Also, the slope value can be observed for each pair of consecutive years.



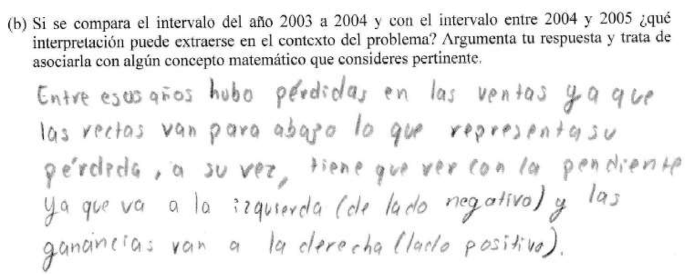
Task 3 Instructions: The following graph shows a company's sales (in million dollars) for 2001-2007.

(a) Use the graph to determine the years where the sales showed the highest earnings and the most significant losses.

Student's response: "Greater gain (2005-2006, slope value 13.11), greater loss (2003-2004, slope value 5.68)"

Figure 3. Response of student E7 in task 3(a).

These results show a conceptualization of the slope as a behavior indicator since its inference is not based on the highest or lowest points on the graph; instead, it is inferred from the slope value shown in the students' response. This claim can be reinforced with the student's E7 answer of subsection (b) of this task (see figure 4), where the answers are associated with the slope value.



(b) Si se compara el intervalo del año 2003 a 2004 y con el intervalo entre 2004 y 2005 ¿qué interpretación puede extraerse en el contexto del problema? Argumenta tu respuesta y trata de asociarla con algún concepto matemático que consideres pertinente.

Entre esos años hubo pérdidas en las ventas ya que las rectas van para abajo lo que representa su pérdida, a su vez, tiene que ver con la pendiente ya que va a la izquierda (de lado negativo) y las ganancias van a la derecha (lado positivo).

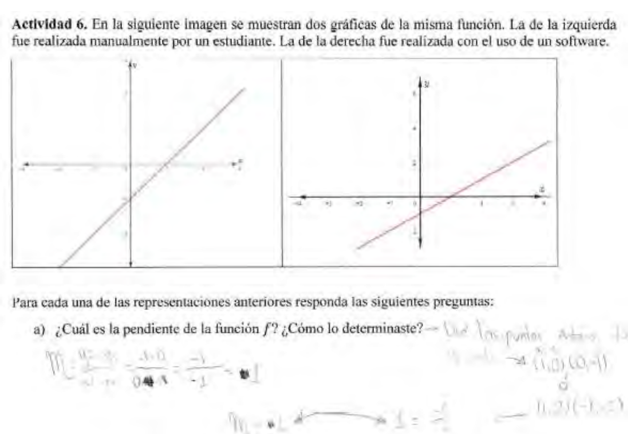
Task instructions: If you compare the 2003 - 2004 interval with the 2004 - 2005 interval, what interpretation can be drawn from the problem's context? Justify your answer and try to relate it to a mathematical concept of your consideration.

Student's response: "Between those years, there were losses in sales as the lines trend downwards, representing their decline. This is also related to the slope, as it goes to the left (negative side) for losses, and to the right (positive side) for profits."

Figure 4. The answer of student E7 in task 3(b).

The trigonometric conceptualization of the slope

This conceptualization was assessed in several stages of the evaluation: for example, in students' responses to subsection (b) of task 2 (22%), in task 5 (17%), or subsections (b) (28%) and (c) (32%) of task 6. In all of these tasks, the situation demands a trigonometric conceptualization associated with the slope, such as the angle's tangent of the line and the horizontal axis.



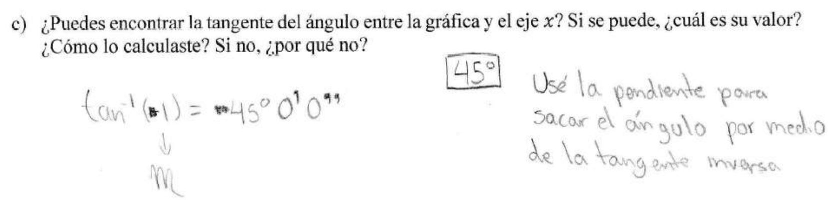
Task 6. Instructions: The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.
Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?

Student's response: "I used the integer points on the straight line".

Figure 5. Response of student E25 in task 6(a).

While, on average, 32% of the students showed some relations with this conceptualization, they could only present an analytic approach with a procedural emphasis rather than a conceptual one. A representative example of this type of response can be seen in student E25 (see figure 5). This student transformed the given task into an analytic approach by identifying a pair of points, in each case, to obtain the slope value. Nevertheless, when asking for the angle's inclination value in subsection (c), the student's response was restricted to an analytic one (see figure 6).

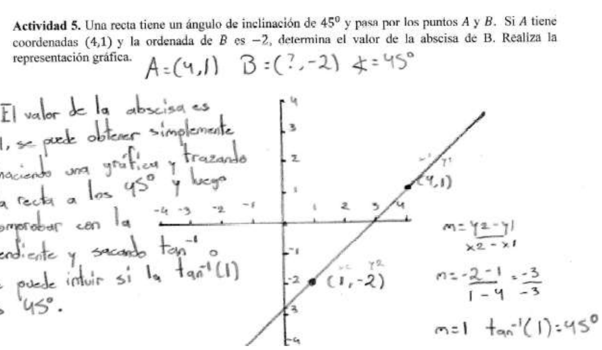


c) Can you find the angle's tangent between the graph and the x-axis? If it is possible to find it, which is the value? How did you calculate it? If it is not possible, state why.

Student's response: "I used the slope to calculate the angle using the inverse tangent".

Figure 6. Response of student E25 in task 6(c).

On the other hand, only 11% of the students showed the notion of trigonometric conceptualization from a visual approach with a conceptual emphasis. For example, in task 5 (see figure 7), E23 shows that the angle created by the line and the horizontal axis determines the sought relation and offers hints of the student's ability to infer the equivalence between the angle's tangent and the slope.



Task 5 Instructions: A line has an inclination angle of 45° and touches points A and B. If A has coordinates (4,1) and the ordinate of B is -2, infer the value of the abscissa of B. Draw the graphical representation.

Student's response: "The value of abscissa is 1, You can obtain it simply by graphing and drawing a line at a 45° angle and then you can verify it by checking the slope and calculating the tan to intuitively determine if the $\tan^{-1}$ (1) is 45°".

Figure 7. Response of student E23 in task 5

Considerations about the SPUR model's dimensions

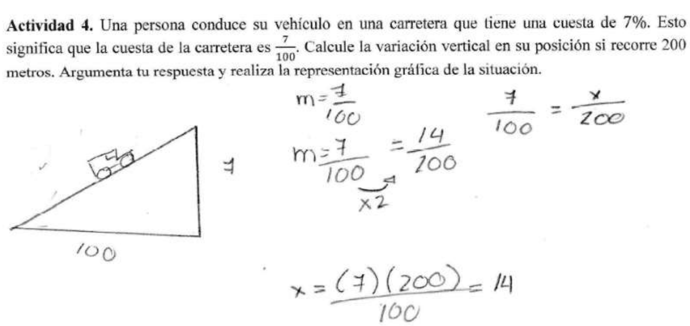
The SPUR model was included in this article as a theoretical framework that allowed us to observe a diversified evaluation of the slope conceptualizations. This multidimensional evaluation provided the conditions to show the connection level between the conceptualizations associated with the slope.

Another critical aspect is that these dimensions can suggest the type of activities that must be proposed in the classroom to encourage an articulated vision of the concept. For example, more than 90% of the participating students showed the *Skills* (S) and *Representations* (R) dimensions. However, in the case of tasks where more than one conceptualization was needed, like *Properties* (P) and *Uses* (U) dimensions, students had lower performance.

The evaluation of the *Uses* (U) dimension was categorized only for the constant ratio (tasks 4 and 7) and behavior indicator (task 3) conceptualizations. In the case of Tasks 4 and 7, their design did not focus on any visual or analytic approaches, and none of the procedural or conceptual emphasis was required. These tasks' resolution process requires an articulated vision of the concept in all its aspects, and its depth may be evaluated with the given responses.

The *Uses* (U) dimension of the SPUR model represents a vital referent to explore the level of integration of the different slope conceptualizations and the weaknesses or opportunity areas that can be developed with the students in the classroom.

For example, only 71% of the students showed a correct result in task 4. Nevertheless, their reasonings centered on a missing value task and not on visual or analytic reasoning of conceptual emphasis. It could be the case that the nature of the formulation of the problem may have caused this effect. The model's dimensions may contribute to the clarification of these aspects or also provide a broader picture of the event. A response example shown by most students can be seen in figure 8.



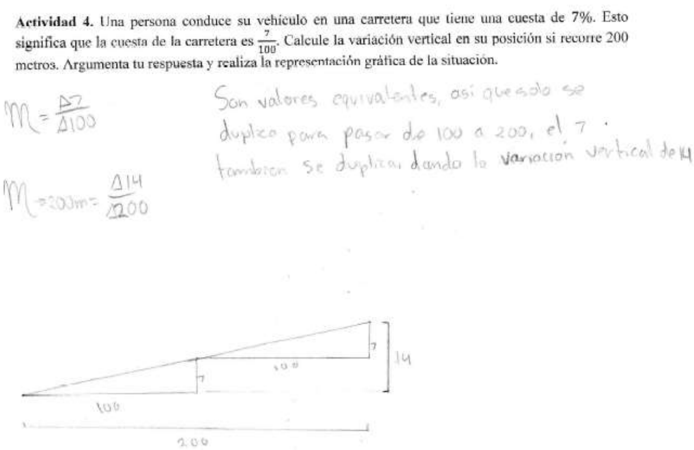
Task 4 Instructions: A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

Note. Obtention of the slope through the calculation of the missing value.

Figure 8. Response of student E15 in task 4

In the production of student E15, it is possible to observe that the student makes the corresponding calculation but does not show an articulation with the conceptual reasoning of the slope since the provided solution invokes an algebraic notion of a different nature that also allows the participant to express the correct answer.

Conversely, figure 9 illustrates a type of reasoning more related to the slope concept for the same task. In this case, student E25 alludes to the relation between the corresponding increments and shows a visual and conceptual approach in the answer.



Task 4 Instructions: A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

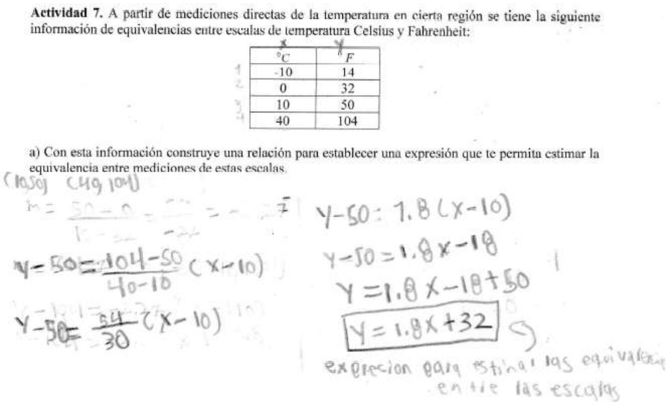
Student's response: "They are equivalent values, so it doubles to go from 100 to 200, the 7 also doubles resulting in a vertical variation of 14".

Figure 9. Response of student E25 in task 4

The previous contrast allows us to highlight the strength of the SPUR model by offering a more detailed view of the skills and deficiencies with the shown dimensions. To reinforce this idea, one could reflect that, in a traditional evaluation test, any student showing the right numerical answer would have a correct answer, regardless of how the answer was obtained. On the opposite, according to the examples of the two previous cases, only student E25 would show the *Properties* dimension (in terms of the SPUR model), which can be associated with an understanding level of conceptual nature.

Finally, results show that, in the case of task 7, only 17% of the students outlined some reasonings associated with the task, but none of them effectively finished the task. The task problem focused on the SPUR model's *Uses* (U) dimension. The student had to connect the slope with its rate of change conceptualization between two covarying quantities. This task implied an analytic approach and conceptual emphasis that requires a global understanding of the slope.

Although student E28 was able to build the expression that models the situation, it is impossible to confirm that he conceptualizes the slope as a rate of change (see figure 10) of two covarying quantities since his developments do not make this relation explicit.



Actividad 7. A partir de mediciones directas de la temperatura en cierta región se tiene la siguiente información de equivalencias entre escalas de temperatura Celsius y Fahrenheit:

°C	F
-10	14
0	32
10	50
40	104

a) Con esta información construye una relación para establecer una expresión que te permita estimar la equivalencia entre mediciones de estas escalas.

Task 7 Instructions: given the direct measurements of the temperature of a particular region, the following information on equivalences between Celsius and Fahrenheit temperature scales was collected.

a) Consider the previous information to build a relation that defines an expression that allows you to estimate the equivalence between the measurements of these scales.

Student's response: "Expression to estimate equivalents between scales ($y = 1.8x + 32$)".

Figure 10. Response of student E28 in task 7(a).

DISCUSSION

The results found in this study coincide with those reported by Thompson (1994); Stump, (1999, 2001); Carlson et al., (2002); Lobato & Siebert, (2002); Moore-Russo, et al., (2011) regarding the lack of connection between the different conceptualizations of the slope, which is evident in the low percentages obtained by students in tasks of conceptual and analytical emphasis, as well as in the *Properties* dimension of the SPUR model.

Another element to highlight is that the students showed poor performance in tasks associated with the trigonometric conception of the slope. According to Azcárate (1992); Stump, (1999) secondary school teachers showed a poor trigonometric conceptualization of the slope, which could be associated with the weak mastery of this conceptualization by the students in this study.

Finally, our work allows us to show how the Nagle and Moore-Russo (2013) framework in conjunction with the dimensions of the SPUR model can help promote connections between the various conceptualizations of slope.

CONCLUSIONS AND RECOMMENDATIONS

The evaluation instrument, designed according to the SPUR model dimensions, gave us a more global view of the student's understanding of the slope conceptualizations and their articulations.

The implementation of this instrument revealed that students possess strong conceptualizations of the slope as a constant ratio and as a trigonometric conception but are more marked by a procedural emphasis than conceptual and correspond to the *Skills* (S) dimension.

While students' works showed a frequent use of the *Representations* (R) dimension, this dimension was not effectively employed in solving the tasks that promoted the *Uses* dimension (U). In particular, the *Uses* dimension requires integration with the other dimensions (*Skills, Representations and Properties*). Concerning this matter, if there is no promotion of activities related to the *Uses* dimension (U) in the classroom, students will have fewer opportunities to acquire a more integrated vision between the different conceptualizations of the slope.

In agreement with this, it is possible to highlight that the isolated conceptualizations of the slope allow the resolution of specific tasks of procedural nature to some extent. However, this is not the case for tasks of conceptual nature since they generally require a multifaceted view of the slope. This notion is in line with the claims by Nagle and Moore-Russo (2013), which emphasize that, in order to achieve a global view of the concept, the creation of a network of conceptualizations of the slope is required.

The results of this project represent a proposal for evaluating the slope through the use of instruments that demand an integrated understanding of the slope with all its conceptualizations. At the same time, it can also inform the teacher about the possible changes needed to reach this purpose. Based on the results obtained in this study and inspired by the reflections expressed by Thompson and Kaur (2011), we propose some recommendations and suggestions that may be useful to design the instruction and evaluation of the mathematics class in general, and in particular, when it comes to the concept of slope.

1) Teachers could use the multidimensional approach SPUR to design activities that allow them to identify students who do not yet adequately handle mathematical content. Knowing more about students' conceptual understanding can promote instructional design that is more aligned with content assessment.
2) It is also advisable to use diagnostic instruments based on the SPUR model, which allows mathematics teachers to know in more depth the knowledge, skills, and representation capabilities of their students.
3) We consider that SPUR dimensions can suggest the type of activities that must be proposed in the classroom to encourage an articulated vision of the concept.

In another aspect, some limitations of this study include:

1) Improve the design of the activities so that they more deeply reflect the conceptual and analytical emphasis of the students. This is because activities such as 1, 2 and 6 may not have been the most appropriate to promote thinking with a conceptual and analytical emphasis and, on the contrary, encourage procedural strategies.
2) In addition to the previous point, it should be mentioned that the responses shown by students may be conditioned and limited by the type of activities. Although they were

designed with opportunities to showcase the conceptual emphasis and *Properties* dimension of the SPUR model, a better design is needed to prevent students from seeking quick answers that do not allow for conceptual reflection.

3) The designed test was applied as a direct evaluation of the concept of slope. Although this way allows us to observe a general state of students' knowledge. It is necessary to complement it with other types of more diverse approaches that include iterative follow-ups, clinical interviews, periods of exploration, replication, etc., which together can reveal in a more profound way the state of knowledge of the students. However, it is emphasized that the use of the SPUR model allowed, in part, to better explore students' conceptualizations of the slope.
4) The study only covers three of the five conceptualizations of the slope concept proposed by Moore-Russo et al. (2011), although it is noted that these were selected according to the student level of the study group.

Finally, it is highlighted that this work constitutes a first approach to introduce these results in the classroom and that the proposal of activities must broaden its characterization with the aim of systematizing the elaboration of evaluation instruments that reflect the level of construction of the network of conceptualizations of the slope.

REFERENCES

[1] Arnold, P., & Hicks, M. (2011). A stochastic approach to rainfall-induced slope failure. *Geotechnical Safety and Risk. ISGSR 2011*, 107-116.

[2] Azcárate, C. (1992). Estudio de los esquemas conceptuales y de los perfiles de unos alumnos de segundo de BUP en relación con el concepto de pendiente de una recta [Study of the conceptual schemas and profiles of some students of second of BUP in relation to the concept of slope of a line]. *Épsilon,* **24**, 9–22.

[3] Carlson, M., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education,* **33**, 352–378. https://doi.org/10.2307/4149958

[4] Casey, S. A., & Nagle, C. (2016). Students' use of slope conceptualizations when reasoning about the line of best fit. *Educational Studies in Mathematics,* **92**(2), 163-177. https://doi.org/10.1007/s10649-015-9679-y

[5] Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics,* **11**, 77-87.

[6] Coe, E. (2007). *Modeling teachers' ways of thinking about rate of change*. (Unpublished doctoral dissertation). Arizona State University, Phoenix, AZ.

[7] DeJarnette, A. F., McMahon, S., & Hord, C. (2020). Interpretations of slope through written and verbal interactions between a student and her tutors in Algebra 1. *Journal of Research in Mathematics Education,* **9**(2), 121-146. https://doi.org/10.17583/redimat.2020.4242

[8] Deniz, Ö., & Uygur-Kabael, T. (2017). Students' mathematization process of the concept of slope within the realistic mathematics education. *Hacettepe University Journal of Education* **32**(1), 123-142. https://dx.doi.org/10.16986/HUJE.2016018796

[9] Diamond, J. M. (2020). Mathematical Knowledge for Teaching Slope: Leveraging an Intrinsic Approach. *Investigations in Mathematics Learning,* **12**(3), 163-178. https://doi.org/10.1080/19477503.2020.1754091

[10] Heuvel-Panhuizen, M. V. D., Kolovou, A., & Peltenburg, M. (2011). Using ICT to improve assessment. In B. Kaur & K. Y. Wong (Eds.) *Assessment in The Mathematics Classroom: Yearbook 2011* (pp. 165-185). World Scientific. Association of Mathematics Educators. https://doi.org/10.1142/9789814360999_0008

[11] Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior,* **21**(1), 87-116. https://doi.org/10.1016/S0732-3123(02)00105-0

[12] Moore-Russo, D., Conner, A., & Rugg, K. I. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics,* **76**(1), 3-21. https://doi.org/10.1007/s10649-010-9277-y

[13] Nagle, C., & Moore-Russo, D. (2013). SLOPE: A Network of Connected Components. In Martinez, M. & Castro, A (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.127-135). Chicago, IL: University of Illinois at Chicago.

[14] Nagle, C., & Moore-Russo, D. (2014). Slope across the curriculum: Principles and standards for school mathematics and common core state standards. *The Mathematics Educator,* **23**(2).

[15] Nagle, C., Moore-Russo, D., Viglietti, J., & Martin, K. (2013). Calculus students' and instructors' conceptualizations of slope: a comparison across academic levels. *International Journal of Science and Mathematics Education,* **11**(6), 1491-1515. https://doi.org/10.1007/s10763-013-9411-2

[16] Nemirovsky, R. (1992). *Students' Tendency to Assume Resemblences between a Function and its Derivative*, Working paper 2-92, TERC Communications.

[17] Nemirovsky, R. (1997). On mathematical visualization and the place where we live. *Educational Studies in Mathematics,* **33**, 99–131. https://doi.org/10.1023/A:1002983213048

[18] Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *The Journal of Mathematical Behavior,* **13**(2), 183-197. https://doi.org/10.1016/0732-3123(94)90022-1

[19] Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A review of state standards. *School Science and Mathematics,* **112**(5), 270-277. https://doi.org/10.1111/j.1949-8594.2012.00135.x

[20] Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal,* **11**(2), 124–144. https ://doi.org/10.1007/BF032 17065

[21] Stump, S. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics,* **101**(2), 81–89. https://doi.org/10.1111/j.1949-8594.2001.tb18009.x

[22] Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education,* **1**, 21-44.

[23] Thompson, D. R., & Kaur, B. (2011). Using a Multidimensional Approach to Understanding to Assess Students' Mathematical Knowledge. In B. Kaur & K. Y. Wong (Eds.) *Assessment in The Mathematics Classroom: Yearbook 2011* (pp. 17-31). World Scientific. Association of Mathematics Educators. https://doi.org/10.1142/9789814360999_0002

[24] Tuluk, G. (2020). Knowledge of Slope Concept in Mathematics Textbooks in Undergraduate Education. *Journal of Curriculum and Teaching,* **9**(3), 161-171.

[25] Zaslavsky, O., Sela, H., & Leron, U. (2002). Being sloppy about slope: The effect of changing the scale. *Educational Studies in Mathematics,* **49**, 119–140. https://doi.org/10.1023/A:1016093305002

APPENDIX

Assessment instrument

Name: ____

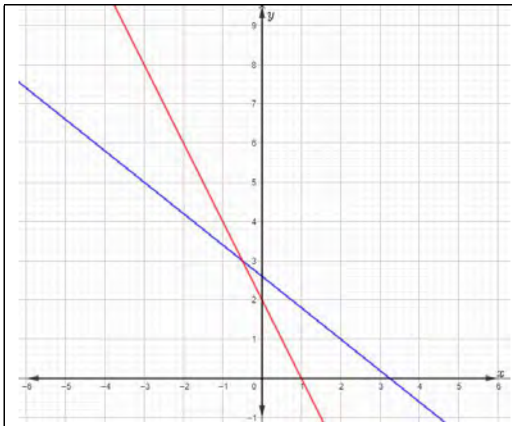
Task 1. Draw the line that touches the coordinates $A(-1,4)$, $B(3,2)$ and find the slope value.

Task 2. Use the following graph to determine

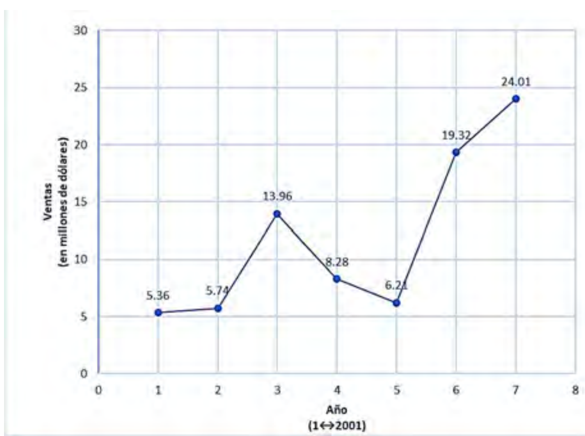
a) The slope value for each of the given lines

b) According to subsection (a), respond: which of the two lines has the largest inclination angle?

Justify your answer.

	Work space

Task 3. The following graph shows a company's sales (in million dollars) for 2001-2007.



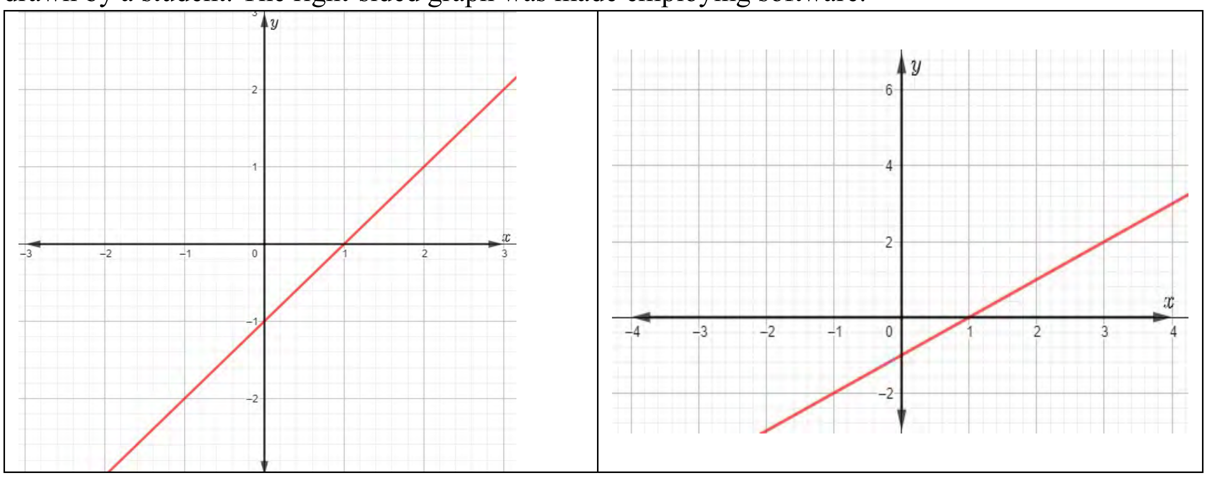
(a) Use the graph to determine the years where the sales showed the highest earnings and the most significant losses.

(b) If you compare the 2003 - 2004 interval with the 2004 - 2005 interval, what interpretation can be drawn from the problem's context? Justify your answer and try to relate it to a mathematical concept of your consideration.

Task 4. A person drives her vehicle on a highway that has a 7% slope. This means that the highway slope is $\frac{7}{100}$. Calculate the vertical variation in its position if she travels 200 meters. Justify your answer and draw a graphical representation of the situation.

Task 5. A line has an inclination angle of 45^0 and touches points A and B. If A has coordinates $(4,1)$ and the ordinate of B is -2, infer the value of the abscissa of B. Draw the graphical representation.

Task 6. The following image shows two graphs of the same function. The left-sided graph was manually drawn by a student. The right-sided graph was made employing software.



Answer the following questions for each of the previous representations:

a) Which is the slope of function *f*? How did you infer this?
b) Does the graph of *f* bisect the angle between the axes? How do you know?
c) Can you find the angle's tangent between the graph and the x-axis? If it is possible to find it, which is the value? How did you calculate it? If it is not possible, state why.

Task 7. Given the direct measurements of the temperature of a particular region, the following information on equivalences between Celsius and Fahrenheit temperature scales was collected.

^0C	^0F
-10	14
0	32
10	50
40	104

a) Consider the previous information to build a relation that defines an expression that allows you to estimate the equivalence between the measurements of these scales.
b) Determine the measurement at which the temperature is equal on both scales.
c) For which Fahrenheit scale measurement is twice the Celsius scale measurement.