

Visualizing Math: How Number Lines Can Empower Problem-Solving

Tiffany Berman
Department of Education, University of Cincinnati, USA

Casey Hord
Department of Special Education, University of Cincinnati, USA

Research has shown the importance of helping students, especially those with mild-to-moderate learning disabilities, to offload information during problem-solving. When students can get their thoughts onto paper, number line strategies can help them develop a firm foundation in mathematical problem-solving while understanding the relationships between mathematical operations. These strategies are helpful for the development of addition, subtraction, multiplication, division, and later, fractional mathematics. In this article, we describe the progression of number lines as a supportive strategy for elementary students and those with developmental delays in mathematics to improve mathematical understanding. This strategy is based on students being able to show their work and think about what they have written on paper or how they have used manipulatives.

Keywords: **mathematics, addition, subtraction, multiplication, division, number lines, math strategies, manipulatives**

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The development of a strong understanding of problem-solving strategies for addition, subtraction, multiplication, and division is crucial to building the foundation for student success in later mathematics (Huinker et al., 2020; Pape & Wang, 2003). Once students have established a solid base in early math strategies such as counting all and counting on, they are ready for the challenge of addition and subtraction (Van de Walle et al., 2017) and, eventually, multiplication and division. These transitions are not always easy for students and can be especially difficult if not taught in accessible and engaging ways. In this article, we will describe the use of number line strategies to engage and support students with mild-to-moderate learning disabilities and other students learning these concepts. Within the United States, learning disabilities are defined as "a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, that may

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manifest itself in the imperfect ability to listen, think, speak, read, write, spell, or to do mathematical calculations" (United States Department of Education, 2017).

Getting Thoughts on Paper

Often, students may struggle with new and challenging concepts. This can result in unnecessary stress for students who experience a lack of initial success when they try to do too much of the math "in their head" (Barrouillet et al., 2007; Hord et al., 2020). Teaching students to get their thoughts on paper (Risko & Dunn, 2010) can help them think through the mathematical process with less stress. They can benefit from writing down or building a model to represent their thoughts. This external representation allows students to connect concepts to what they can see rather than trying to remember many things all at once, all while thinking about the next step of the task (Hord & Marita, 2014; Hord & Xin, 2013). In our work, we have found that teachers can benefit from paying careful attention to helping students get things out of their heads and finding a strategy to represent concepts in front of them as they are worked through (Hord et al., 2020; Xin et al., 2008).

As educators, we know that many students struggle when math is taught in overly abstract or disconnected ways, especially when learning new concepts (Barrouillet et al., 2007; Keeler & Swanson, 2001). While students need to be able to engage with math in abstract contexts and with mathematical notation, we have found that a good "entry point" for many students is at the concrete or pictorial levels (Hinton & Flores, 2019; Hord & Xin, 2015). Students often benefit from progressing from concrete to semi-concrete or representational to abstract contexts (Van de Walle et al., 2017). This progression helps students build a firm foundation of math concepts, which can be built upon as they advance to the next level (Hord & Xin, 2013).

Additionally, research has shown that many learners, especially students with mild-to-moderate learning disabilities, are unique in the ways they build their knowledge (Lambert et al., 2022; Lewis, 2014), and can benefit from a combination of structure and freedom to build knowledge in ways that work for them (Hord et al., 2016). For instance, some students can gain knowledge and skills by being shown procedures, but others may learn better by doing tasks or activities. In these cases, teachers may find value in teaching strategies that allow students to play games, solve puzzles, or build models to "find their own" way. These strategies help students make meaningful connections that work for them based on their experiences inside and outside of school. Focusing on activities that help students connect with mathematics meaningfully makes it more likely that students will encode these procedures and knowledge into their long-term memory (Ormrod, 2020). In the next sections, we will illustrate some number line activities and strategies we have used to support and empower students in our classrooms and tutoring sessions.

Patterns in Our Examples

Readers will likely notice patterns in the strategies and activities used throughout the upcoming sections. Each of these strategies builds upon the previous strategies, increasing the connections students make as they advance through math concepts. It is important to remember that the goal is to help students get their thoughts down onto paper (or in a more representational way) rather than holding them in their working memory. This strategy reduces the cognitive load while allowing students to visualize the problem in parts as they work to understand the whole.

In early mathematical learning, students use counting strategies such as counting on and counting all and inventive strategies such as making 10, combinations of 10, doubles, and near doubles. While working with these strategies, students use concrete representations, known as manipulatives, such as fingers, snapping cubes, and base ten blocks, as well as pictorial representations, such as number lines, ten frames, and drawings of individual items. These same materials are helpful in multi-digit addition, subtraction, multiplication, and division. For this paper, we will focus on number line strategies.

The Use of Number Line Strategies

Number lines, a visual representation of numbers spread out at equal distances on a straight line that extends infinitely in either direction, are helpful throughout mathematical learning. The use of these strategies can help students build a powerful understanding of numbers that may lead to greater success in fractional mathematics, where research has indicated the benefits of number lines (Koc & Chambers, 2017; Lesner et al., 2024; Schumacher et al., 2018). Students often use number lines in the early grades when beginning to work on understanding numerical values and early counting skills. This familiarity lends itself to using number lines as a lower-stress mathematical tool for students struggling with the development of mathematical concepts. Lessening stress and anxiety when working on mathematics is vital to students' success and reducing the cognitive load (Ashcraft & Krause, 2007; Eysenck et al., 2007). With this in mind, we will move through a progression of addition, subtraction, multiplication, and division strategies in the following sections.

Strategies and Activities to Support Addition

Number lines are common when students are working on addition strategies in early elementary grades. However, this strategy continues to be useful throughout elementary and even into the middle grades and into high school. Number lines are helpful as a visual representation of multi-digit addition problems to allow students to break down complex problems into smaller chunks using knowledge of place values. Additionally, number lines can easily relate to meaningful events in students' lives, such as sports. Teachers can relate the movement on a number line to the bouncing of a ball connecting to a basketball, football, or even a ball on the playground. This approach allows teachers to show physical examples of how number lines can be used to support mathematical understanding and increase the connections students build. Traditionally, students were taught multi-digit addition using standard algorithms, as shown in Figure 1.

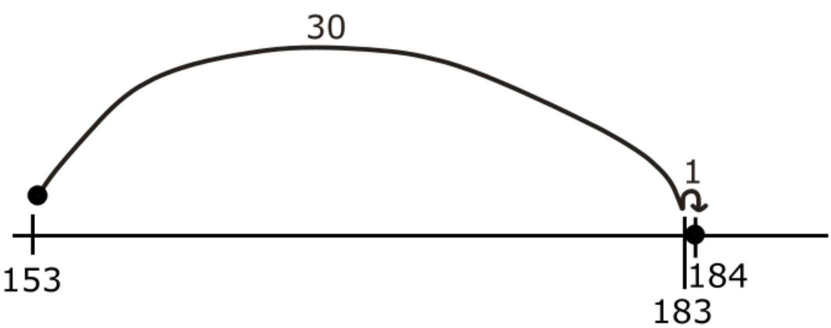
$$\begin{array}{r} 153 \\ +31 \\ \hline \end{array} \qquad 153 + 31 = \underline{\qquad}$$

Note. The standard algorithm for multi-digit addition is shown in two ways above: vertically and horizontally.

Figure 1. ***Addition Standard Algorithm***

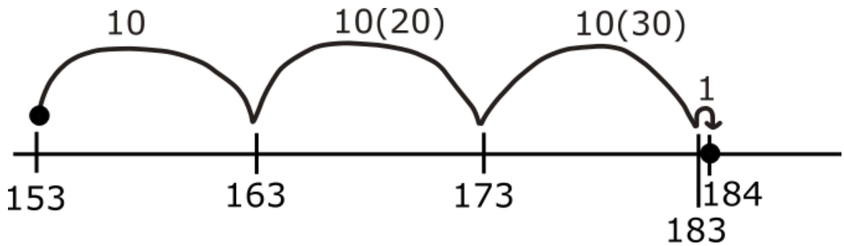
Over time, the focus has moved away from the standard algorithms toward strategies for helping students develop a stronger conceptual understanding of mathematical concepts and operations while reducing their cognitive load. In early grades, these inventive strategies include counting up, counting on, making ten, and near doubles. Number lines are often used at this time in combination with manipulatives, concrete objects, such as unifix cubes and ten frames with plastic counters.

The problem shown in Figure 1 above can be broken down into parts to reduce cognitive load and solved using number line approaches such as those shown in Figures 2 and 3.



Note. A number line addition problem with starting number 153 with the representation of a ball bouncing in a large bounce forward 30 to 183, then a small bounce forward one to 184.

Figure 2. ***Addition Number Line Strategy***



Note. A number line addition problem with starting number 153 and a ball bouncing in three medium bounces forward ten each time for 30 total to 183, then a small bounce forward one to 184.

Figure 3. ***Addition Number Line Strategy Breakdown***

Providing a physical manipulative that students can bounce on the number line may be helpful when beginning this strategy with students. The use of the manipulative in combination with teacher modeling provides students with a concrete representation to understand the movement of the number line better, increasing their conceptual understanding. This manipulative should be phased out as students progress in their understanding of using number lines for multi-digit addition.

Strategies and Activities to Support Subtraction

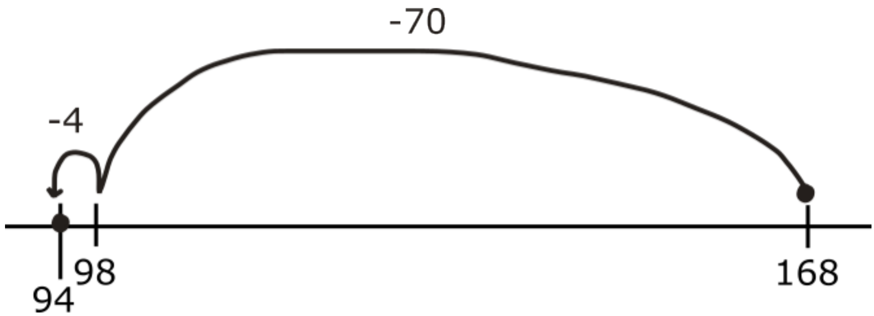
Once students have a firm grasp on using a number line for multi-digit addition and understand single-digit subtraction, the same approach can be applied, allowing students to build on prior knowledge of number lines. As with addition, the standard algorithm shown in Figure 4 is the traditional method of teaching multi-digit subtraction.

$$\begin{array}{r} 168 \\ -\ 74 \\ \hline \end{array} \qquad 168 - 74 = \underline{\qquad\quad}$$

Figure 4. ***Subtraction Standard Algorithm***

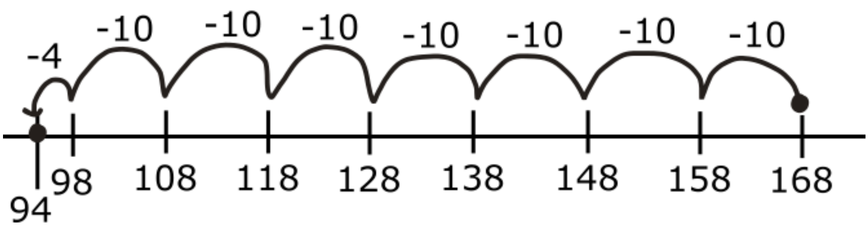
Note. The standard algorithm for multi-digit subtraction is shown in two ways above: vertically and horizontally.

As with addition, classroom teachers have begun moving away from using the standard algorithm for subtraction in favor of additional inventive strategies that will help students gain a conceptual understanding of the procedures of subtraction. In early mathematics, these include split strategies and jump strategies, as displayed in the number lines in Figures 5 and 6.



Note. A number line subtraction problem with starting number 168 and a ball bouncing in a large bounce backward 70 to 98, then a small bounce backward four to 94.

Figure 5. ***Subtraction Number Line Strategy***



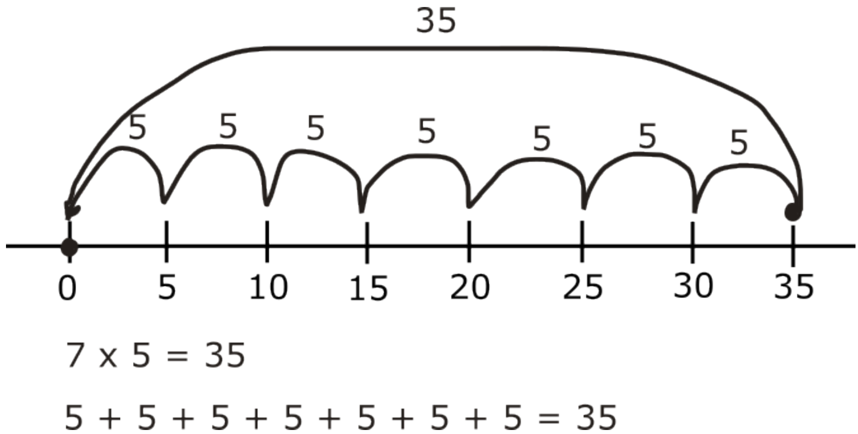
Note. A number line subtraction problem with starting number 168 and a ball bouncing in seven medium bounces backward ten each time for 70 total to 98, then a small bounce backward four to 94.

Figure 6. ***Subtraction Number Line Strategy Breakdown***

Similarly to the addition strategies, using physical manipulatives when introducing this strategy may be helpful for students. Some students may struggle to understand the differences in directionality between addition and subtraction problems. Many young children enjoy the use of small animals as manipulatives, such as the mini eraser animals often found in classrooms. Providing students with an animal that jumps backward on the number line may help improve their understanding of the directionality of subtraction. Similarly, a line of unifix cubes could be used, and with each backward jump, the student removes that number of cubes. This method allows students to check their work by counting the number of cubes removed at the end. Manipulative use should fade out as students develop more confidence using the number line method.

Strategies and Activities to Support Multiplication

Once students clearly understand number lines for multi-digit addition problems, they can explore using them for basic multiplication problems. As Figure 7 shows below, students can break down multiplication problems using a combination of the two number line methods shown in Figures 2 and 3 above.



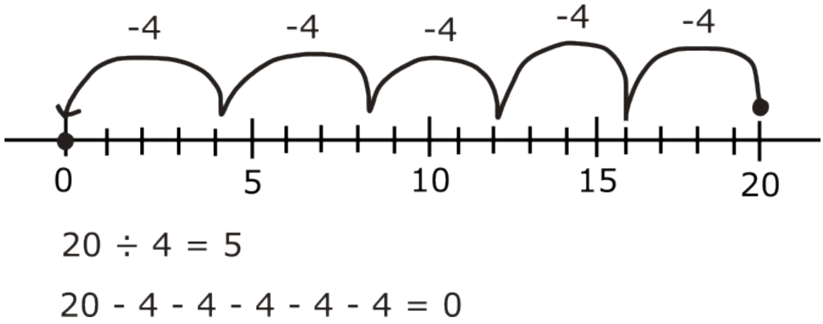
Note. A number line is used to solve the multiplication problem seven times five. Starting at the number zero students, the ball bounces forward seven times, moving five places each time. Students can then add each ball bounce using repeated addition to create a larger bounce forward and total it to 35.

Figure 7. ***Multiplication Number Line Strategy***

Using number lines to demonstrate that multiplication is repeated addition helps students understand multiplication in a manner that standard algorithms often do not while allowing them to break down the math problem on paper, reducing what needs to be held in their working memory. Small manipulatives such as Lego or dried beans can be used to further student understanding by creating groups of items with each bounce of the number line.

Strategies and Activities to Support Division

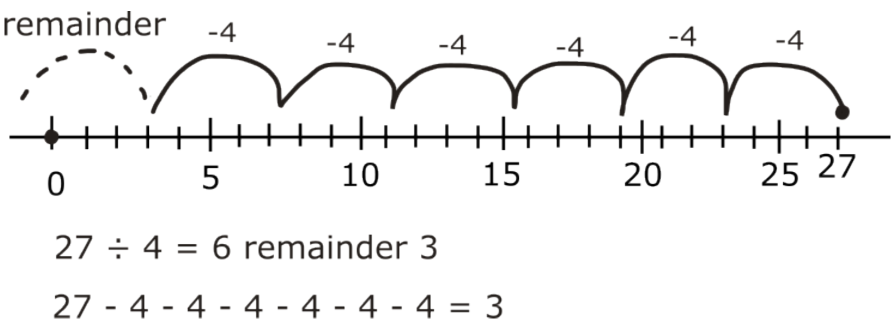
The use of number line strategies for division requires that students first have a firm grasp of using number lines for subtraction as the directionality is similar and an understanding of zero's place on the number line. For early division activities, students will bounce backward along the number line by a set amount until they reach zero. Students can then count the number of bounces to find the answer to their problem, as shown in Figure 8.



Note. A number line is used to solve the division problem 20 divided by four. Starting at the number 20, students bounce the ball backward by 4s until they reach zero, counting the number of bounces to see that 20 is divided by 5 times.

Figure 8. ***Division Number Line Strategy***

It will be important for teachers to note that students may not always get to zero in a division problem. Instead, the goal will be to get as close to zero as possible and to stop when the next bounce jumps over zero. This approach would allow teachers to begin illustrating the concept of remainders, as shown in Figure 9.



Note. A number line is used to solve the division problem 27 divided by four. Starting at number 27, students bounce the ball backward by 4s until they jump over zero, counting only the jumps before zero and how many more numbers they have until they reach zero. This number is the remainder.

Figure 9. ***Division with Remainders Number Line Strategy***

As with multiplication, students can be provided with Legos to break apart at each jump or a small manipulative such as dried beans to group with each jump. Students will likely develop a deeper understanding of the division process as repeated subtraction by combining manipulatives and number lines.

Teaching Summary

Given the struggle that students with learning disabilities often have when trying to keep their work organized, it is vital to teach them strategies that represent math concepts meaningfully. These strategies, such as the number line strategy, should help students get their thoughts on paper, visualize the parts of a problem, and reduce their cognitive load. Helping students learn how to get their thoughts on paper allows teachers to understand their thinking processes better and provides opportunities to recognize and correct errors. When teachers can visibly see where students are making errors, they can provide stronger support, helping students learn from their mistakes. It is important to remember that no strategy works for all students. We want to provide students with various strategies to help them find the ones they understand. The number line strategies described in this article can be used in various classroom contexts, including group work and one-on-one learning, at desks using small manipulatives, or in the gymnasium or playground using a taped number line and playground balls.

Connections between Research and Practice

Educational researchers and psychologists have explored various methods for students to offload information during mathematics to reduce their cognitive load and increase mathematical understanding. These include using diagrams, box methods, manipulatives, and gestures (Hinton & Flores, 2019; Hord et al., 2016; Xin et al., 2008). One strategy we have used with success in our classrooms and tutoring sessions involves using number lines. Number lines combined with physical manipulatives and meaningful scenarios are another strategy teachers can use to reduce students' cognitive load and improve their ability to work through multi-digit mathematical problems. As students are often familiar with number lines in the classroom, this strategy can be used successfully through the process of teaching addition, subtraction, multiplication, division, and, later, fractional math (Lesner et al., 2024). Given the unique needs of students in the classroom, especially students with learning disabilities, it is vital that teachers and students have access to strategies that may be usable throughout mathematical learning. These types of strategies allow students to develop and build upon stronger foundations, potentially increasing confidence in mathematics.

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Authors' Note

Correspondence concerning this article should be addressed toTiffany Berman, Department of Education, University of Cincinnati, 2150 Edwards 1, Cincinnati, OH 45221, USA, Email: bermantn@ucmail.uc.edu.