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Review Article

An analysis of the essential understandings in elementary geometry and a comparison to the common core standards with teaching implications

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ABSTRACT

Geometry and spatial reasoning form the foundations of learning in mathematics. However, geometry is a subject often ignored by curriculum writers and teachers until high school, leading to students lacking in critical skills in geometric reasoning. As the United States moves into a new curriculum epoch, heralding the commencement of the national common core standards (CCS), one could question if CCS in geometry align with the essential understandings children need to be successful geometric thinkers. This paper begins with an examination of the essential understandings of geometric reasoning leading to an interpretation and critique of the elementary geometry CCS. Finally, the instructional implications are discussed, considering the common core progression through what we know about how children learn geometry.

Keywords: geometry, geometry education, mathematics education

INTRODUCTION

School geometry is comprised of an interconnected network of concepts, ways of reasoning, and axiomatic representational systems; these are used to mathematize spatial objects, relationships and transformations. Primarily, geometric reasoning consists of the creation and use of 'conceptual systems' adopted to investigate shape and space (Alghadari & Noor, 2021; Battista, 2001a, 2001b, 2002). For example, property-based conceptual systems can be used to analyze and define various types of triangles, by measuring the angles and lengths of the sides. A significant proportion of geometric thought involves spatial reasoning, and academics (vis., Battista, 2007; Clements, 1998; Lehrer et al., 1998) advocate for spatial reasoning to be considered alongside geometric conceptual systems in the study of geometry. Spatial reasoning refers to the set of cognitive processes involved in constructing and manipulating spatial objects, images, relationships, and transformations (Clements & Battista, 1992). These processes provide mental images for geometric reasoning and critical tools for geometric analysis (Battista, 2007). Therefore, throughout this paper, spatial reasoning will be discussed alongside geometry.

Geometry and spatial reasoning form the foundations of learning in mathematics and other academic subjects (Clements, 2004). Although geometry is important, it is often ignored by curriculum writers and teachers until high school (Clements, 2004; Lehrer et al., 1998). Due to this delayed progression, empirical evidence shows that a large majority of children in the United States have insufficient understanding of geometric concepts, and lack skills in geometric reasoning, and problem solving abilities (Beaton et al., 1996; Carpenter et al., 1980; Mullis et al., 1997, 1998). In addition, the instructional delay could result in crucial windows of opportunity being missed. For example, researchers (viz., Clements et al., 1999; Gagatsis & Patronis, 1990) found concurring evidence that children begin to form shape concepts in the preschool years, which can stabilize as early as age six.

While it is important to consider early instruction, curriculum developers and educators need to also be cognizant of constraints in the students' cognitive abilities due to biological maturation. In other words, children should not be asked to accomplish tasks that are beyond their cognitive abilities for that age. With these considerations, effective instruction should provide children with the appropriate opportunities to learn the 'essential understandings' in geometry, while progressing through these understandings in a developmental trajectory designed to supply the building blocks for further instruction. The term essential understanding is a neologism, defined by Karp et al. (2011) as the specific interconnected ideas of a larger mathematical concept. Other similar terms have been used in mathematics, for example Watt et al. (2002) referred to 'big ideas' as concepts that underlie understanding and mastering a strand of mathematics, and Wiggins and McTighe (2005) described 'enduring understandings' as the important understandings students need to retain to make meaning of the subject. These terms all refer to the critical transitions in students' development as they come to understand geometry.

Pierre van Hiele and Dina van Hiele-Geldof used empirical evidence to formulate the van Hiele model (van Hiele, 1984a, 1984b; van Hiele-Geldof, 1984), that describes how students' geometric reasoning develops. While researchers (e.g., Battista, 2007; Clements et al., 2001; Gutiérrez, 1992; Gutiérrez et al., 1991; Johnson-Gentile et al., 1994) have made revisions and elaborations to the model, research generally indicates that the van Hiele model is accurate (Battista, 2007; Burger & Shaughnessy, 1986; Clements & Battista, 1992; Fuys et al., 1988). Models such as these can be used by curriculum developers and teachers to ensure students gain the essential understandings needed in geometry. Unfortunately, Clements and Battista (1992) lament that the elementary geometry curriculum is impoverished, "the curricula consist of a hodgepodge of unrelated concepts with no systematic progression to higher levels of thought" (p. 422). As the United States moves into a new curriculum epoch, heralding the commencement of the national common core standards (CCS), one could question if CCS in geometry align with the essential understandings children need to be successful geometric thinkers and problems solvers.

This paper contains three sections. The first section begins with an explication of the essential understandings of geometric reasoning detailed within the van Hiele model. Then, several other research-based frameworks are described that contrast, or revise and elaborate on the van Hiele model, in order to create a refined picture of the essential understandings in geometry. These findings are then summarized into a list of four essential understandings with sub-components to highlight the finer essential understandings for each concept. The second section interprets and critiques the elementary geometry CCS, comparing the standards against the list of essential understandings in section one. Finally, the third section outlines instructional implications considering the common core progression through what we know about how children learn.

THE ESSENTIAL UNDERSTANDINGS

The van Hiele Levels of Geometric Thinking

Dutch educators, Pierre van Hiele and Dina van Hiele-Geldof, designed the van Hiele model (van Hiele, 1984a, 1984b; van Hiele-Geldof, 1984) to highlight students' development through five levels of geometric thought; beginning with a gestalt-like unanalyzed visuals, to a highly sophisticated complex level of thinking. While later articles (e.g., van Hiele, 1999) provide tacit references to the importance of age and biological maturation, the emphasis of the van Hiele model is placed on the purpose of effective instruction to facilitate progression throughout the levels.

The van Hiele's theorized that learning was a discontinuous process, with jumps in the learning curve that reveal the five discrete levels. The levels are sequential and hierarchical descriptions of how the student would demonstrate thinking at each level. In order to move through the levels, students need to become proficient in a significant portion of the lower level, before they can advance to a higher level (Hoffer, 1981). From observations of students' thinking, van Hiele (1984a) noticed that knowledge intrinsic at one level appears in an extrinsic way at the next. For example, while a child may be using particular properties to determine the name of a shape, the actual thinking at that level may not be cognizant of those features. In addition, language is different between the levels.

> Each level has its own linguistic symbols and its own system of relations connecting these symbols. A relation that is 'correct' at one level can reveal itself to be incorrect at another. Think, for example, of a relation between a square and a rectangle. Two people who reason at different levels cannot understand each other. Neither can manage to follow the thought process of the other (van Hiele, 1984b, p. 246).

van Hiele was keen to point out that rote learning can make a student appear knowledgeable about the concepts; but memorization of facts does not signify understanding, and students must not skip any levels. This belief concurs with the underpinning philosophy of essential understandings, that each critical understanding defines the way a student thinks about geometry within a developmental process, and if a component of that process is missed, the child will lack a crucial skill, which may hinder future learning.

The way in which the van Hiele levels are numbered has varied (Clements & Battista, 1992). For the purpose of this paper, the levels are listed as one through five. The terms visualization, analysis, abstraction, deduction, and rigor describe the cognitive levels that the students' progress through from level one to level five (de Villiers, 1987; Hoffer, 1981; Teppo, 1991).

Level 1: Visualization

Students at this initial level identify, name, compare, and operate on shapes, and other geometric configurations according to their appearance. Figures are seen as visual gestalts in that individual attributes, such as angle measurements, are not explicitly recognized; instead the figures are considered as a collection of a whole. Perception guides the students' reasoning, and visual prototypes are typically used to name a figure. For example, a student may say that a figure is a rectangle because it looks like a door (Clements, 1998).

Level 2: Analysis

Students at this level have progressed from gestalt perceptions, to analyzing figures according to their attributes, and are able to identify the relationships among the attributes to discover rules for how figures are named. For instance, a student may think of an equilateral triangle as a figure with three equal sides; therefore, the student has learned that the term "equilateral triangle" refers to a specific collection of properties.

Level 3: Abstraction

Students can provide abstract definitions and informal arguments. They can distinguish between the necessity and sufficiency of a set of properties for a concept, while also logically ordering those properties. It becomes clear, for example, why a square can also be a rectangle. Although the students are showing a method of logical organization, they do not know that it is a method by which geometric truths are established.

Level 4: Deduction

Students are able to formally reason and identify interrelationships within the axiomatic system through logically interpreting geometric statements, such as definitions, theorems, and axioms.

Level 5: Rigor

Students can reason by formally manipulating geometric statements such as definitions, theorems, and axioms, and through indirect proof and proof by contrapositives.

As movement through the levels relies on effective instruction, adults may still think at a level one or two if they did not receive instruction to support further progression in geometric thinking. That said, even with effective instruction, elementary students typically do not progress beyond the second or third level.

Piaget & Inhelder's Studies on Spatial Conceptions

While the van Hiele's emphasized instruction, Piaget and Inhelder (1967) studied children's development of space and geometry through the lens of genetic epistemology. Piaget's (1955b) early studies brought him to the conclusion that children construct *perceptual space* by infancy. Piaget and Inhelder (1967) furthered this research as they conducted a series of experiments in regard to children's conceptions of space in geometry; what Piaget and Inhelder (1967) termed *representational space*. Specifically, they studied children's haptic recognition, drawing, and spatial perceptions. This body of research led to the topological primacy thesis.

The topological primacy thesis refers to Piaget and Inhelder's (1967) claim that a young child's intrinsic geometry is initially topological, and then later projective and Euclidean (Darke, 1982). Similar to the van Hiele model, Piaget and Inhelder (1967) posited that there is a definite order in developmental progression that must be observed. Children are first able to learn about topological relations such as enclosure, connectedness, and continuity, then this is followed by the ability to learn projective (rectilinearly) and Euclidean (parallelism, angularity, and distance) relationships. Although, in regard to angularity, significant portions of Piaget's other studies identify children as young as six or seven developing a tacit knowledge of angle, developing to extrinsic knowledge around the age of nine (Olson, 1970). Lehrer et al. (1998) also found that children's knowledge of angles grows significantly during the elementary years.

Piaget and Inhelder's (1967) initial research concerned children's haptic recognition of shapes (that is, the perception of shapes by tactile stimulus, and the following visual identification). They found that preschool children could discriminate between open and closed features (topological), older children could identify the difference between straight or curved sides (rectilinearly) and identify shapes such as squares and diamonds (Euclidean). The haptic studies also led Piaget and Inhelder (1967) to the conclusion that representational space is not developed through a perceptual reading of the spatial environment, but by active manipulation. The act of touching resulted in tactile perception; when these actions are regulated by the child, relationships are built, providing an accurate representation of the shape (Clements & Battista, 1992; Piaget & Inhelder, 1967).

As Piaget and Inhelder (1967) studied children's perspective taking, they posit the difference between topological and projective or Euclidean perspectives, involve the relationship between the figures and the subject. Topological perspectives consider the figure in isolation, projective involves perspectives between the figure and the subject, and Euclidean, refers to perspectives between figures. Battista (2007), Clements (1998), and Piaget and Inhelder (1967) describe perspective taking as a critical developmental step in geometry. As students develop projective and Euclidean perspectives, they are able to move beyond their own perspective to the perspectives of others. For example, with the development of projective space, around the age of seven, students can construct straight lines by putting themselves as one of two points to be linked by a straight line. As students gain the perspective of Euclidean space, during middle childhood, concepts such as angularity and parallelism are developed. In later years, Clements (1998, 2004) theorized extensively on extending students' spatial perspective taking to include maps and navigation. Clements (1998) explicated the need for students to master environmental directions (e.g., up, down, left, and right) as well as global directions (north, south, east, west) to develop perspective and directional skills that will later lead to more complex coordinate frameworks.

Clements et al. (2004) and Piaget and Inhelder (1967) emphasized the importance for students to develop the ability to compose shapes, and that drawing is an act of representation that provides a window into students' geometric understandings. The evidence gathered from drawing experiments continued the topological primacy thesis. Piaget and Inhelder (1967) postulate that children under the age of four demonstrate topological features, with aimless scribbles, which are followed by squares and triangles that cannot be distinguished from circles. This drew them to the conclusion that the children do not have the cognitive ability to see the difference between curved and straight sides. Piaget and Inhelder (1967) provide further evidence to indicate that the results could not be attributed to the students' lack of fine motor skills. As children reach the age of four, they begin to provide a progressive differentiation of Euclidean shapes and are able to draw a square and a triangle. At seven years of age, students are effectively able to copy Euclidean shapes such as the rhombus.

Researchers, who replicated Piaget and Inhelder's (1967) study, have generally confirmed the findings (Laurendeau & Pinard, 1970; Lovell, 1959; Page, 1959; Peel, 1959). However, it appears that Piaget and Inhelder's (1967) thesis has had far less effect on classroom instruction than that of the van Hiele model (Pegg & Davey, 1998). Nonetheless, both works remain the most extensive early sources of information regarding children's perceptions of space in geometry, and trajectories of change (Lehrer et al., 1998, p. 137).

Revisions & Elaborations to the van Hiele Model

In recent years, Clements et al. (2004) followed a similar avenue of study to that of Piaget and Inhelder's (1967) use of drawings as representations of geometric understandings. Clements et al. (2004) used the research findings to extend the van Hiele model. The van Hiele theorized that students' knowledge of shapes begins with generic visual perceptions, leading to attribute recognition and hierarchical classifications. The work of Clements and Sarama (2007), Clements et al. (2001, 2004), Sarama and Clements (2004), and Sarama et al. (1996) found that composition and decomposition processes were also fundamental components in children's development; that students must gain the skills to recognize figures, but also the ability to manipulate shapes and their properties. Therefore, Clements et al. (2001) created a research-based learning trajectory to elucidate the competencies at each level. The trajectory spans ages four to eight years and consists of seven levels of thinking.

Pre-composers

Children manipulate shapes as individuals but are unable to combine them to compose a large shape.

Piece assemblers

Children at this level are similar to pre-composers, but they can concatenate shapes to form a picture. In free-form picture tasks, children can choose shapes to fulfill particular pictorial purposes. Simple frames can be filled using a trial and error technique (Mansfield & Scott, 1990; Sales, 1994), but children have limited ability to use flips or turns. The first two levels are similar to the van Hiele model in that children see gestalt-like shapes, not individual properties.

Picture makers

Children can concatenate shapes to form a picture in which several shapes play a single role, although students are using a trial and error approach, and do not anticipate creation of new geometric shapes. Names of shapes are chosen using a gestalt configuration, or from one attribute. Children at this level do not understand angle as a qualitative entity. Rotating and flipping are used typically through trial and error.

Shape composers

Children combine shapes to make new shapes or fill puzzles. Shapes are chosen using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes (Sarama et al., 1996). Imagery and systematicity begins to develop onwards from this level.

Substitution composers

Children deliberately form composite units of shapes (Clements et al., 1997), and recognize and use substitution relationships among these shapes.

Shape composite iterators

Children construct and operate on composite units intentionally. They can continue a pattern of shapes that lead to a "good covering", but without coordination of units (Clements et al., 1997).

Shape composers with superordinate units

Children can build and apply (iterate and otherwise operate on) units of units of units. The levels have been summarized from Clements et al. (2004, p. 276-278).

Within Clements et al.'s (2001) composition and decomposition trajectory, motions and transformations were included. While learning to conduct motions and transformations, students are developing additional skills that are necessary in order to gain a holistic operational view of geometric properties; students who have not been given opportunities to develop these skills will struggle in future geometrical tasks (Battista & Clements, 1988; Gutiérrez et al., 1991). Jamie and Gutiérrez (1989), Johnson-Gentile et al. (1994), and Lewellen (1992) also made additions to the van Hiele model to incorporate motions and transformations, describing these skills as part of the essential understandings students need to have. Slides appear to be the first form of motion students are able to master at first grade; this is followed by flips and turns (Perham, 1978). However, the level of difficulty can be dependent on direction of flips and turns (Schultz & Austin, 1983), and orientation clues provided (Rosser et al., 1984). Further studies indicate that second grade students are able to perform mental rotation of imagery (Perham, 1978; Rosser et al., 1988). While children often use rotational symmetry in work with pattern blocks (Sarama et al., 1996), many concepts of symmetry are not fully established until 12 years of age (Genkins, 1975).

Gutiérrez (1992) proffered a four-level visualization developmental trajectory to extend the van Hiele model. Visualization in his thesis refers to the way students perceive and move figures. In addition, Gutiérrez (1992) postulated that three dimensional shapes should also be included to the van Hiele model, these are referred to within each of the four levels.

Level 1. Students make comparisons of solids based on a holistic perception of the shapes of the solids or from some elements (e.g., faces, edges, and vertices), but pay no attention to properties such as angle sizes, parallelism, edge lengths etc. Students are not able to visualize solids, or the positions of solids that they cannot see, and students manipulate solids through a trial-and-error approach.

Level 2. Students use observation as the main basis for explanations. They compare solids based on a holistic perception of the solid, or properties of the solid. Students can visualize simple movements of solids between two concrete positions.

Level 3. Students make comparisons by mathematically analyzing their properties and visualize movements from non-visible positions.

Level 4. Students at this level have high visualization abilities. Students' reasoning is based on the mathematical structure of the solids or their properties, including those not seen, but formally deduced.

Summary of the Essential Understandings

Since the late 1950s, and the initial composition of the van Hiele model and the topological primacy thesis, there have been a number of empirical and theoretical additions that have created a more refined outline of the essential understandings students need in geometry. Similar to the initial theories, each new component typically includes some form of developmental trajectory that describes the basic, to the more advanced essential understandings in geometry and spatial reasoning. Each small step is a building block for future learning, tied to effective instruction and/or biological maturation.

The main components of the van Hiele model, the topological primacy thesis, and the further empirical and theoretical additions mentioned in this section, were compiled and four essential understandings emerged for the elementary years.

These four essential understandings each have multiple parts, but focus on a particular concept, or related concepts. The concept/s are shape attributes; spatial orientation (the ability for students to understand and operate on objects in space in relation to students' own positionality; Clements & Battista, 1992) and spatial perspectives; composition and decomposition of geometric figures; and motions, transformations, and reflections of geometric figures. These four essential understandings are listed below in full. Following each is a list of the names highlighting the main proponents, then a summary of the smaller building block understandings for each concept, listed in a quasi-developmental sequence.

1. *Recognize that two- and three-dimensional shapes have particular attributes that categorize the shape and can also be used to compare against other shapes.*

van Hiele (van Hiele, 1957/1984a, 1957/1984b; van Hiele-Geldof, 1957/1984), and Piaget and Inhelder (1967).

- Develop visual prototypes of shapes and corresponding names
- Analyze and name shapes according to attributes and relationships among those attributes.
- Compare shapes based on holistic perceptions.
- Compare shapes by analyzing their properties.
- Understand that shapes can look different within the same category (e.g., different types of triangles).
- Provide informal logical arguments to place shapes in particular categories.

2. *Recognize how to apply mapping and directional skills and understand that different spatial perspectives can be utilized.*

Clements (1998, 2004), Gutiérrez et al. (1991), and Piaget and Inhelder (1967).

- Use and create simple maps to provide a mental representation of the environment.
- Develop environmental (e.g., right, left, up, down) and global directions (left, right, east, west).
- Identify direct and indirect routes to a given location.
- Take the perspective of an abstract frame of reference (such as the perspective of a toy on a map, rather than self-perspective), and compare perspectives.
- Draw figures from various perspectives (such as a line from self to another point, then from two other points not related to self).
- Use a coordinate grid to interpret values

3. *Recognize how spatial visualization and knowledge of shape properties can be used to compose and decompose figures.*

Clements and Sarama (2007), Clements et al. (2001, 2004), Piaget and Inhelder (1967), Sarama and Clements (2004), and Sarama et al. (1996).

- Copy shapes from a visual cue (Shapes will increase in complexity).
- Copy shapes from memory (Shapes will increase in complexity).
- Use shapes to make a free-form picture (initially, picture may only be recognizable to the student).
- Use knowledge of the properties of shapes to fill a picture tray.
- Understand and use shapes to create other shapes.
- Rotate and flip to select and place shapes.

4. *Recognize that motions, transformations, and symmetry can be applied to figures, and know how to conduct such actions.*

Battista and Clements (1988), Clements et al. (2001), Gutiérrez et al. (1991), Jamie and Gutiérrez (1989), Johnson-Gentile et al. (1994), Lewellen (1992), and NCTM (2006).

- Name, model, draw, describe, and compare 2D and 3D figures
- Recognize and use slides on figures (with and without orientation clues).
- Recognize and use flips and turns on figures (with and without orientation clues).
- Use slides, flips, and turns in various directions.
- Perform mental slides, flips, and turns.
- Identify lines of symmetry.

A relationship can be inferred between van Hiele's theory, Piaget and Inhelder's (1967) theory of cognitive development, and the previously outlined four essential understandings in geometry. **Figure 1** depicts an individual engaging in a specific level of van Hiele's theory (Kamalodeen et. al., 2021) based on their grade level, age, and cognitive development level.

In recent years, academics and organizations have created similar collective lists. Watt et al. (2002) studied the growth and development of students' thinking about *big ideas* in K-5 geometry, and included other topics

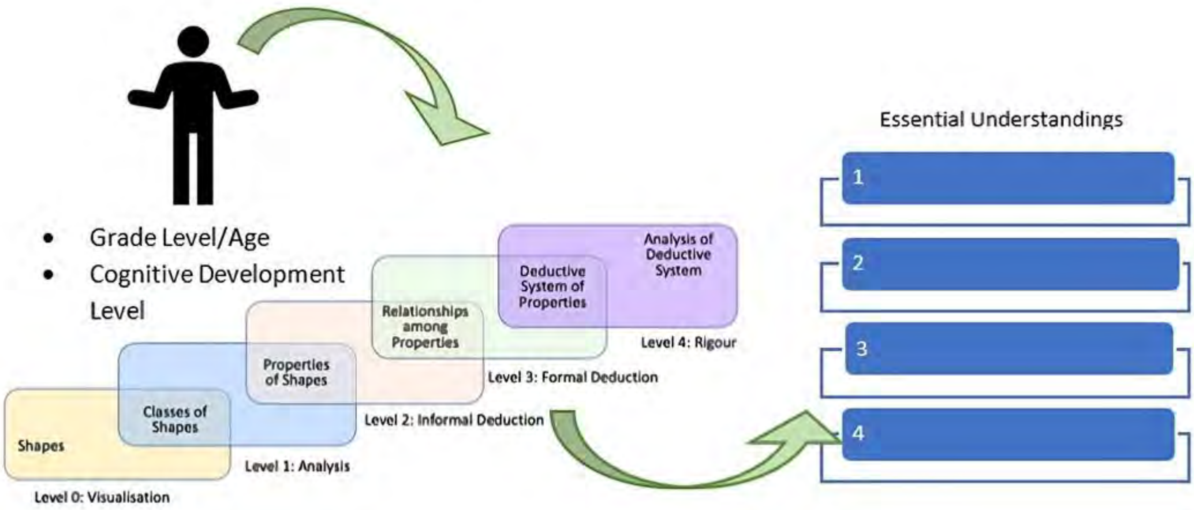

Figure 1. An individual engaging in a specific level of van Hiele's theory (adapted from van Hiele, 1999)

in mathematics, science, and the arts. They identified six areas: units, patterns and repetition; transformation of shapes; symmetries; composition and decomposition of shapes; similarity and scale; measurement and dimensionality; and three-dimensional/two-dimensional visualization. Interestingly, recognition of shape attributes and nomenclature were omitted, but it adds other mathematical concepts, such as patterns and repetition, typically within the operation and algebraic thinking strand, and measurement, from the measurement and data strand of mathematics. No additional geometric concepts were added beyond those identified.

National Council of Teachers of Mathematics (NCTM) highlighted four main areas of geometry; listing, properties of shapes, location and spatial relationships, transformations and symmetry, and visualization (NCTM, 2000). In addition, NCTM (2006) developed a set of curriculum focal points, described as *core structures* and *indispensable elements* for each grade level. Kindergarten, describing shapes and space; grade 1, composing and decomposing geometric shapes; grade 3, describing and analyzing properties of two-dimensional shapes; and grade 5, describing three-dimensional shapes and analyzing their properties. This list is similar to the research findings in this section; however, the initial four areas highlighted by NCTM (2000) are not clearly reflected in the indispensable elements for each grade level, with a lack of emphasis on transformations and symmetry.

Most recently, National Center and State Collaborative developed the learning progressions framework for K-12 mathematics (Hess, 2010, 2011). In these documents, the *enduring understandings* in geometry are described as "visualizations, spatial reasoning, and properties of two- and three-dimensional figures can be used to analyze, represent, and model geometric concepts and relationships" (Hess, 2011, p. 25). For each enduring understanding, there is a description of what skills students would be able to demonstrate, as well as progress indicators that describe specific skills for each elementary grade. Although there is a good similarity to the essential understandings developed in this section, the learning progressions framework omits a significant component, which is any form of motion transfer on two-dimensional shapes. Therefore, as the next section interprets and critiques CCS, this will be conducted based on the earlier list of essential understandings developed in this section, categorized as: shape attributes; spatial orientation and spatial perspectives; composition and decomposition of geometric figures; and movements, transformations, and reflections of geometric figures.

REVIEW OF COMMON CORE STANDARDS

Authors of CCS clearly describe their methodological priorities in the design of the standards. "By focusing on the identification of significant and recognizable clusters of concepts and connections in students' thinking that represent key steps forward, trajectories offer a stronger basis for describing the interim goals that students should meet …" (Daro et al., 2011, p. 12). In order to identify both the essential understandings and the developmental trajectory, the authors delineate the evidence based approach they used; studying empirical evidence through three lenses, cognitive development, instructional practice, and coherence of ideas (CCSSO/NGA, 2010a; Corcoran et al., 2009; Daro et al., 2011). However, Daro et al. (2011) pointed out that the lens of cognitive development played a substantial role in the formation of the elementary standards.

This section of the paper analyses the consistency of those research-based claims, as CCS in geometry are compared to the essential understandings explicated in the first portion of this paper.

1. *Recognize that two- and three-dimensional shapes have particular attributes that categorize the shape and can also be used to compare against other shapes.*

This understanding appears to underpin many of the other essential understandings. For example, in order to construct, deconstruct, and apply transitions, students must have a good understanding of shape attributes in order to apply these other skills. The elementary CCS incorporate this concept throughout K-5 standards, and it is also included within the overarching standard for each grade level. CCS trajectory seems well aligned to the research. Kindergarten children begin by naming shapes based on their gestalt appearances and begin to use non-formal language to analyze and compare shapes based on shape attributes. This trajectory follows the van Hiele model as the children are expected to have gained some familiarity with shapes during the prekindergarten years, to move from level one to level two during the kindergarten year.

The trajectory of the van Hiele model is also evident throughout grades one to five, as the students move within level two, and then onwards towards level three in grade five. First grade requires students to distinguish between defining attributes and non-defining attributes. At second grade they are expected to develop their knowledge of attributes to more complex shapes. Third and fifth-grade students are challenged to reach two significant milestones from the van Hiele model; they are required to develop the understanding that shared attributes define a large category in third grade, and to classify figures in a hierarchy in fifth grade. One author of CCS openly criticized this fifth-grade standard, describing it as “astoundingly trivial” (Milgram, 2010, p. 9). Milgram (2010) may have missed the point, that following CCS alignment to the van Hiele model, students would be moving to level three in which students logically organize shapes by their attributes into a logical hierarchy. This is a difficult concept for students as they may understand that there are many types of quadrilaterals, but they have to name shapes (e.g., squares) by working through a logical hierarchical process. The standards could have been worded a little clearer or provided further examples to avoid further misinterpretation.

CCS also included three dimensional shapes; a component added to the van Hiele model by Gutiérrez (1992) and Gutiérrez et al. (1991). CCS developmental trajectory is congruent with the thesis of Gutiérrez et al. (1991) and introduces three dimensional shapes in kindergarten and throughout van Hiele level two and three. It appears that shape attributes have been significantly addressed in CCS, and the developmental trajectory corresponds with the research. However, there are other components in CCS that may not be as clearly aligned. The fourth-grade standards require students to draw and to identify angles and parallel lines in two-dimensional shapes.

Piaget and Inhelder (1967) categorized angles and parallel lines as Euclidean concepts developing during middle childhood (middle to early high school). Clements (2004) described angles as a difficult concept for elementary students to grasp; but in accordance with other research findings (e.g., Lehrer et al., 1998; Olson, 1970), concluded students develop concepts of angle earlier than Piaget and Inhelder (1967) suggested, and the long learning process should begin in the elementary years. Battista (2007) and Clements (2004) highlighted students’ difficulties with parallelism, and Clements (2004) was unsure if curricula and teaching approaches would be able to facilitate lasting learning outcomes in the elementary grades.

2. *Recognize how to apply mapping and directional skills and understand that different spatial perspectives can be utilized.*

Researchers, such as Clements (1998, 2004), Gutiérrez et al. (1991), and Piaget and Inhelder (1967), clearly emphasized the importance of this particular essential understanding in students’ development. In addition, NCTM (2000, 2006) included location and spatial relationships as one of four core components in elementary geometry. CCS are somewhat aligned to the research with the inclusion of this concept and a similar trajectory; unfortunately, this is only consistent with the extreme poles of the elementary trajectory. The first standard in kindergarten requires the development of environmental directions, such as left, right, above, in front of, and next to, which was a component specifically described within Clements' (1998, 2004) framework. However, the next time this understanding is developed is in the fifth grade, as students have to graph and interpret values in a coordinate system. Many of the vital intermediate concepts are omitted from CCS.

Students are not predisposed to make and use coordinates for themselves (Clements, 2004). "Like so many processes in geometry, both the coordination process and the formation of frames of reference depend critically on creating appropriate mental models" (Battista, 2007, p. 891). Clements (1998, 2004) proffered the use of maps and navigation to develop mental models; to have students mathematize the directionality they have developed during preschool years. CCS begin with environmental directions, but the developmental sequence should continue with physical map building, using cutout shapes, models etc.; mental map building; global directions; recognition of features from various viewpoints; and understanding and use of map symbolization (Clements, 1998, 2004). In addition, mapping skills can be used to develop perspective, which is another essential understanding unrepresented within CCS. In order to facilitate development from topological, to the projective and Euclidean perspective described by Piaget and Inhelder (1967), the standards should have included the development of perspective taking from abstract frames of reference. For instance, matching views of the same structure from different perspectives (Downs & Liben, 1988) and providing directions to destinations from different starting locations (Clements, 1998, 2004).

3. *Recognize how spatial visualization and knowledge of shape properties can be used to compose and decompose figures.*

This particular essential understanding is covered substantially within CCS and is well aligned to the research. The developmental progression within CCS matches the composition and decomposition trajectory designed by Clements et al. (2001). Kindergarten standards match the *piece assembler* and *picture maker* level, grade one develops concepts described in the *shape composer* and *substitution composer* level. In grade two, the standards are concordant with the *shape composite iterators* and s*hape composers with superordinate units*. CCS also includes the composition and decomposition of shapes in grades three and four. Students work with various different types of quadrilaterals in third grade and various new figures in the fourth grade. Although the model of Clements et al. (2001) does not go above the age of eight, it is reasonable to assume that the essential understandings gained from composing and decomposing shapes in the lower grades would also be beneficial to those in the higher grades, as students are challenged with new shapes and figures. Continuation could also facilitate students' development towards, or within the Euclidean drawing stage, described by Piaget and Inhelder (1967). Concomitantly, CCS also correspond to the van Hiele model, in that it positions the model of Clements et al. (2001) at a place on the trajectory, where students will have acquired prerequisite essential understandings to support the composition and decomposition of figures.

In an interesting move, CCS requires students in grades one, two, and three to decompose shapes using fractions. This is somewhat similar to the decompose activities described in a study by Sarama and Clements (2006), as the students decomposed shapes and new shapes developed; but the decomposition in this case specifically involves fractions. Although fractions are not typically included in geometry curricula, fractions are a decomposition process. Underpinning conceptual understandings of shapes and shape attributes will have prepared students with the skills to tackle this new concept, before moving on to further studies within the fraction strand of CCS.

4. *Recognize that motions, transformations, and symmetry can be applied to plane figures, and know how to conduct such actions.*

NCTM (2000) and Watt et al. (2002) included transformations and symmetry within a list of the core essential understandings in elementary geometry. Jamie and Gutiérrez (1989), Johnson-Gentile et al. (1994), and Lewellen (1992), also added this to the van Hiele levels as a core component. Yet only symmetry is included within CCS, and transformations have been omitted. Symmetry is included within the fourth-grade standards, and students are required to recognize a line of symmetry, identify line-symmetric figures, and draw lines of symmetry. The authors of CCS posited that the fourth-grade symmetry component builds on the analysis of shape attributes, and composition and decomposition activities in grades one and two (CCSSO/NGA, 2010a). This is congruent with the argument for the introduction of symmetry in NCTM fourth grade curriculum focal points (NCTM, 2006), and corresponds to the empirical findings of Genkins' (1975), that concepts of symmetry are not established until the final elementary years. Therefore, it appears that symmetry is correctly included and placed within CCS, in concordance with the research. Unfortunately, the omission of transformations is not.

Shape transformations first appear in CCS in eighth grade and continue into high school. The authors of CCS claim that the K-12 progression of geometry standards leads toward an understanding of plane geometry from rigid transformations (CCSSO/NGA, 2010a). This has been highly criticized as an experimental move, unsubstantiated by research (Milgram, 2010; Milgram & Stotsky, 2010; Stotsky & Wurman, 2010). While the high school CCS cover complex transformations, research identifies transformation concept acquisition during the elementary years. Empirical evidence indicates that students master slides in first grade, and then flips and turns (Perham, 1978). Clements et al. (2001) showed a similar progression of transformations within the composer/decomposer framework, as students developed the ability to purposefully apply transformations as early as third grade. In addition, Battista (2008) theorized that transformations support students' developing understanding of shape attributes.

In order to match the findings of the research, students should formally begin learning transformations (e.g., translations and rotations) in the fourth or fifth grade. The early composition/decomposition activities, that are included in CCS, can support students' initial development of transformations, which could progress to a more formal understanding in the final elementary grades.

To summarize, the authors of CCS made the claim that the geometry standards are consistent with the research in regards to the essential understandings required by elementary students, and the developmental sequence of those understandings. From the interpretation and critique of CCS within this section, it would seem that the majority of the standards are consistent with the research. Although, there are essential components not fully addressed or missing, such as spatial orientation and perspectives, and shape transformations.

HOW STUDENTS LEARN GEOMETRY

While the argument has been made that the elementary CCS for geometry may not fully reflect the research, curriculum trajectories do play a vital role in helping all parties connect goals, curriculum components, teaching strategies and assessments (Clements et al., 2004). It enables teachers to learn about geometry, consider how students think about and learn geometry, and visualize potential developmental paths (Ball & Cohen, 1999). Nevertheless, as teachers use CCS, they must also consider how to develop effective pedagogies, activities, and tasks, to facilitate student acquisition of the essential understandings, and to support further progression. This section outlines instructional implications considering the Common Core progression through what we know about how children learn. To begin the overarching constructivist pedagogy is described, and then followed by tasks and activities identified within the research to support and enhance students' learning in geometry.

Constructivism and Assessment

In addition to the van Hiele model, which delineates the levels of geometric thinking, van Hiele (1984b), furthered the work of van Hiele-Geldof (1984), to develop a five-phase sequence of activities to assist progress through the levels. Instruction begins with the *inquiry phase*, as the students use materials to explore and discover mathematical structures. The second phase involves *direct orientation*; the characteristic shape structures appear gradually to the students during these activities. Next, during the *explication* phase, the teacher introduces relevant terminology. During *free orientation*, students are presented with tasks that can be solved in multiple ways. In the final phase, *integration*, students are given opportunities to consolidate their knowledge and ideas into a coherent network that can be applied to other situations. For the framework to be effectively used, the students may need to cycle through some of the phases multiple times during one particular topic (Mason, 1998). Pierre van Hiele's phases of instruction are similar to the constructivist approach, as the knowledge in the mind of the student is developed and altered by experiences and interaction with mathematical phenomena (Piaget, 1955a, 1971) and people (Vygotsky, 1978). The constructivist methodology is necessary to ensure that students gain mastery of the concepts, rather than memorized facts (Battista & Clements, 1995; Piaget, 1971; Soto-Johnson et al., 2009).

For growth to occur, it is crucial that the instruction matches the student's level of ability (Crowley, 1987). Piaget (1971) and van Hiele (1999) postulated that it is better to give no education at all, than give education at the wrong time. Teachers must remain cognizant that while CCS provide a developmental trajectory within the grades, students may not have reached the level of understanding to match their biological age or may even be working beyond that level. Furthermore, each standard has multiple parts of which students may have gained some components, and not others. Therefore, continuous assessment is required to determine what the student does, or does not understand, and any misconceptions he/she may have (Crowley, 1987).

Discussion, Reflection, & Language Development

Social interaction and reflective thinking are crucial components if students are to construct meaning and deep understanding of the concepts (Richardson, 1999; Van de Walle & Lovin, 2006). Vygotsky (1978), a keen advocate for social constructivism, believed that learning in isolation would not lead to cognitive development. He described social interaction as a way in which students can develop understanding, through internalization of the ideas they receive through conversations with others. As students become active in the conversations, they must also reflect on their own understanding of that particular concept (Chaplin et al., 2009; Clements, 1998). Vygotsky's (1978) zone of proximal development suggests that the geometric ideas developing within the student will interact with the information gained from the conversations, thus expanding the student's conceptual potential.

However, as students work in groups, the dynamic interrelationship of ideas or mathematical language can clash and be unheard, which will hinder, rather than aid development (Nyikos & Hashimoto, 1997; van Hiele, 1984b). Therefore, discussions need to be well planned and purposeful. Van de Walle and Lovin (2006) developed a set of suggestions for effective discussions, including, encourage questions, active participation, turn-taking, and reflective responses. In addition, Hiebert et al. (1997) and Richardson (1999) proffer that students should be given time to reflect on the ideas of others and understand that it is okay to make mistakes as the ideas can be discussed and corrected. Conversations are multifaceted as they allow students to clarify the meaning of geometric terms, such as sides, faces, and vertices (Clements, 1998); encourage shape descriptions and precise language; and provide opportunities for teachers and students to model the use of reasoning within hierarchical categorizations of shape (Clements, 2004).

Real-World Connections

Another way to support language development is through real-world experiences (Clements, 2004). There are a number of connections to the real-world within CCS. For example, the overarching standard for fifth grade requires students to work with "coordinate planes to solve real-world and mathematical problems" (CCSSO/NGA, 2010a, p. 34). Real-world contexts provide a solid base for building understandings (Clements & Sarama, 2009), and enhances students' ability to think and reason mathematically outside the classroom (Lehrer & Chazan, 1998). Researcher findings indicate that students using real-world contexts found geometric concepts easier to understand, more logical, interesting, and familiar to the students (Duatepe-Paksu, 2009). Real-world connections should be used to take advantage of these benefits. However, teachers should not try to use a theme (e.g., connections to the real-world), if the connection is not obvious, or if it does not enhance the students' understanding of that particular mathematical concept (Richardson, 2002).

Manipulatives

Manipulatives can provide a connection or representation of the real-world. Physical and virtual manipulatives are crucial in developing geometric and spatial thinking (Battista & Clements, 1996; Clements & McMillen, 1996; Piaget & Inhelder, 1967), and aid the student in making the connection from the tangible object to its abstraction (Kamina & Iyer, 2009). Physical manipulatives include objects such as blocks, geometric-solid models, and geoboards. Dynamic geometry environments, logo-based environments, and other similar computer programs provide virtual manipulatives, which are digital representations of physical objects that can be manipulated by mouse or touch screen.

Many geometry programs can provide additional features beyond their physical counterparts, including options to change the size and shape of the manipulatives; a drag feature, which modifies the shape corresponding with the geometrical properties; tools to allow the student to decompose the shape; and a record feature, to allow playback of actions carried out on the shape. Research findings indicate that use of manipulatives help students connect with real-world knowledge; increase memory and understanding (McNeil & Jarvin, 2007); assist students in creating definitions, and conjectures (Fuys et al., 1998), and support students in identifying shape attributes (van Hiele, 1984b). Nevertheless, while there are positive advantages to the use of virtual or physical manipulatives, studies conducted by Fennema (1972) and Resnick and Omanson (1987) found a lack of skill transfer and disconnect between the manipulatives to paper and pencil computations. From empirical evidence, such as this, researchers (viz., Martin et al., 2007; Sarama & Clements, 2009; Uttal et al., 2009) posit that instruction should be carefully organized to begin with manipulatives, ensuring that students are reflecting on their actions, and then move beyond the use of manipulatives while ensuring transfer.

Visual Prototypes

Concept learning and analysis play a prominent role in the development of geometric thinking, and an influential component of this process is the categorization of shapes (Battista, 2009). This process is crucial in the early elementary years as students develop prototypes (Clements, 1998, 2004), which are mental images, or examples of the visual appearance of particular shapes (Smith, 1995). Cultural influences, such as books, teaching supplies and toys engender prototype development (Clements, 1998). Therefore, teachers must remain cognizant of the visual images students are exposed to during play and instruction.

Students should not be continually exposed to shapes in rigid ways but experience many different visual examples of a particular shape; non-examples should also be displayed and discussed to draw attention to critical attributes (Clements, 1998, 2004). Also, teaching should not primarily focus on shape prototypes, as this can be detrimental to hierarchical thinking (Clements, 2004). Dynamic geometry environments, logo-based turtle geometry, and other similar computer programs allow students to view many different types of figures that go beyond the typical prototypes (Clements & Battista, 1992). In addition, as students have the opportunity to create and manipulate these computer-based representations, this process enables students to perceive the figures as geometric entities, and not just visual objects (Zbiek et al., 2007).

CONCLUSIONS

It is clear that geometry and spatial reasoning is important. It provides a means by which students can explore, interpret, and reflect on the physical environment. Since the early studies (Piaget & Inhelder, 1967; van Hiele, 1984a; van Hiele-Geldof, 1984), further research has led to numerous other theories and frameworks, each highlighting the essential understandings students need to gain within the elementary years. An aggregate list of the various frameworks led to the identification of four major essential understandings, *shape attributes*; *spatial orientation and spatial perspectives*; *composition and decomposition of geometric figures; and motions, transformations, and reflections of geometric figures*. The list also includes sub-components and a quasi-developmental sequence.

While research-based frameworks can be used by curriculum developers and teachers, geometry curricula have been described as having no systematic progression, full of unrelated concepts, and lacking in spatial reasoning (Clements & Battista, 1992). As CCS were compared to the essential understandings identified in section one, it appears that there has been a significant improvement to the geometry curriculum with CCS showing a high similarity to the research. The majority of the standards correlate to the essential understandings and are typically ordered in congruence with the identified trajectory. In addition, spatial visualization, which is the ability to understand and perform imagined movements of objects in two-dimensional and three-dimensional space (Clements & Battista, 1992; Gutiérrez, 1992), has been included within the skills students develop to compose and decompose shapes. However, CCS is still lacking another component of spatial visualization as shape movements and transformations have not been included in the elementary CCSs. Spatial orientation was also sadly lacking from CCS, which could have been incorporated into the standards with the inclusion of cartographic type activities suggested by Clements (1998, 2004).

The teaching of geometry in the elementary curriculum is led by CCS. Understanding these frameworks and the research is crucial for selecting and creating instructional tasks. Evidence supports a constructivist approach for the teaching and learning of geometry, incorporating discussion, reflection, real-world connections, and manipulatives. It is the role of the teachers to effectively match the activity to the understanding they are expecting the students to gain. The authors of CCS declare that the standards will be held to an "ongoing state-led developmental process that can support continuous improvement of the standards" (CCSSO/NGA, 2010b). While CCS have a strong connection to the research, it will be interesting to see if future revisions incorporate additional spatial reasoning components to further align the standards to the research findings.

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