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Research Article

Special tutorials to support pre-service mathematics teachers learning differential equations and mathematical modelling

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ABSTRACT

Special tutorials both online and off-line were experimented in order to provide extra support for the senior pre-service mathematics teachers at an Australian regional university to improve their learning experience and achieve the best possible learning outcomes in an advanced mathematics course focusing on solving ordinary differential equations and applying mathematical modelling. Two types of special tutorials were offered to the students, the progressive tutorials on solving the same problem with different methods according to the learning progression and student's instant requests, and the targeted tutorials to address the common problems shared by many students in attempting questions in the formal assessments. The experiments on these special tutorials indicated that the targeted tutorials were immensely useful for the students to either expand the scientific knowledge related to a real-world scenario described by words so as to begin problem solving with correct setting-ups or streamline multiple mathematical processes together to solve a complicated real-world problem described in words. This approach motivated most students to achieve their best possible learning outcomes. The progressive tutorials were effective in addressing student's curiosity of solving the same problem by multiple techniques and hence improving student's mathematical thinking and problem-solving skills in general. This exploratory study also found that there were common problems with a lack of general science knowledge and retention of the previously learnt mathematical techniques among most students. There also existed a portion of students who showed no interest in engaging with learning regardless of how much extra learning support provided to them.

Keywords: progressive tutorial, targeted tutorial, ordinary differential equations, mathematical modelling, pre-service mathematics teachers, regional students and universities

INTRODUCTION

In most undergraduate programs for pre-service mathematics teachers in Australia, the topic of ordinary differential equations (ODEs) is the most advanced subject in the mathematics curriculum in which the main focus is the first-order ODEs required in designing and conducting scientific modelling for the talented secondary school students engaged in the specialist mathematics course. For example, in the specialist mathematics curriculum recommended by the Queensland curriculum and assessment authority (QCAA, 2019), the secondary school students engaged in the specialist mathematics are required to learn how to solve some first-order ODEs by separation of variables. To effectively guide secondary school students to solve first-order ODEs, the mathematics teacher would be best prepared to be able to apply multiple techniques, including separation of variables, to solve first-order ODEs, which must be the goals of the tertiary study and training for the pre-service mathematics teachers.

Solving ODEs with multiple techniques requires the student mathematics teachers to be efficacious in calculus, preferably in both elementary calculus and multivariable calculus as the prerequisite recommended in many popular textbooks (Greenberg, 1998; James, 1996; Kreyszig, 2011). However, the pre-service mathematics teachers and most engineering students are only required to complete elementary calculus in their mathematics curriculum in most Australian institutions. Hence, the coverage of ODEs in these popular textbooks is inappropriate for the pre-service mathematics teachers in Australian universities.

Compared to the engineering students in their second or third year who are more scientifically oriented in learning and able to find abundant information on almost every topic of mathematics from various online sources, the feedback from many pre-service mathematics teachers who are more socially oriented in learning shows that they prefer to learn mathematics by a more structured and directed procedure with a preset learning plan with a prescribed textbook. As the topic of ODEs for the pre-service mathematics teachers covers only a small portion of a formal course in ODEs for science and engineering students (Guo & Wang, 2019), a well-suited textbook for science and engineering students would be 'too heavy' for the pre-service mathematics teachers. Hence, the successful pedagogy in teaching ODEs for engineering students (Guo, 2021b; Guo et al., 2021) must incorporate additional measures to make learning ODEs more effective for the pre-service mathematics teachers. The additional measures that have been experimented include special tutorials either targeting specific areas of a challenging question in the official assignments, where the students may encounter extreme difficulties to begin with or to understand the scenario described in words (denoted as the targeted tutorial in this case study), or widening student's vision in problem solving by connecting solving a specific problem with multiple alternative approaches whilst learning is progressing from one technique to another technique (denoted as the progressive tutorial in this case study). Note that some of these special tutorials can be delivered during the formal online live class especially requested by students whereas other tutorials can be recorded offline after receiving student's request outside of the official online class.

This case study reports the rationales of, practices in, and student's learning outcomes from employing the special tutorials to provide extra learning support for the pre-service mathematics teachers to better handle learning ODEs and applying ODEs for solving and modelling scientific problems at Central Queensland University (CQU) of Australia. Since this is a technical report focusing on an advanced topic, the comparative case study (Christenson et al., 2020) combined with simple statistical analysis is the most appropriate research method for this work. As the special tutorials were initiated by student's requests, informal feedback through chats during the online classes, rather than from formal qualitative data collection, was embedded during the presentation wherever appropriate.

We present the background information on how this new pedagogical addition was initiated through teaching students solving ODEs in recent offerings. Next, examples of employing the special tutorials to support learning and applying ODEs for the pre-service mathematics teachers are presented. We then discuss students' learning outcomes with the extra support by means of the special tutorials, and then concludes this case study. Note that the linked videos in this case study are recaptured recordings for public access accompanying this case study, which are different from the original videos shared with the students on the internal course website.

BACKGROUND

Australia has a large territory with relatively a small population. Australian regions are classified into five categories: major cities, inner regional, outer regional, remote, and very remote areas (Regional Education Expert Advisory Group [REEAG], 2019). Except major cities, the other four regions are commonly referred to as the regional, rural, and remote (RRR) areas, which cover more than 95% of Australian territory. There is a lower rate of participation of school leavers in tertiary education in RRR areas. To retain as many students in tertiary programs as possible, regional universities encourage educators to provide as much support as they can to assist students' learning, in addition to formal lectures and tutorials, particularly in STEM programs.

The mathematics specialty in Bachelor of Education at CQU is a program to train future mathematics teachers for secondary schools, particularly RRR schools. CQU is a regional university with multiple campuses in RRR areas located in local centers along the north-eastern coast of Australia. The mathematics specialty

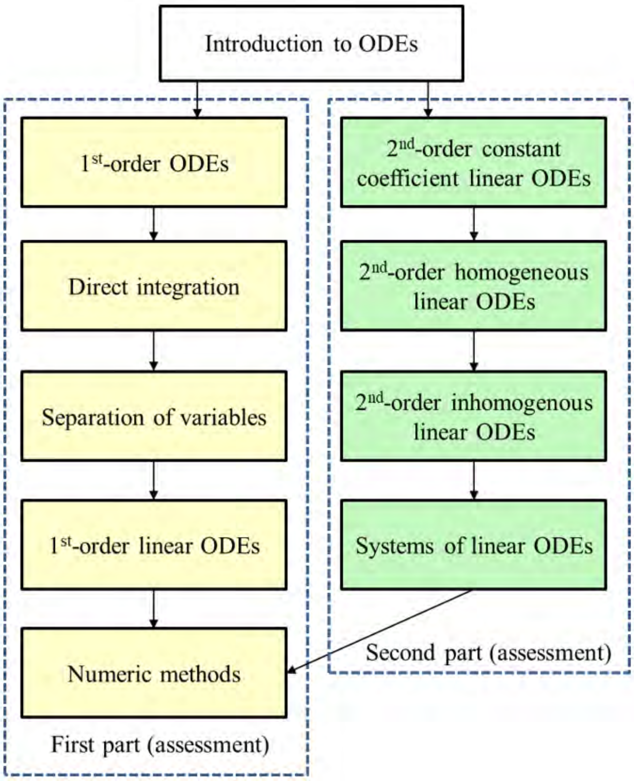

Figure 1. Optimized teaching & learning sequence for advanced mathematics course (Source: Author)

consists of one statistics course and five mathematics courses across multiple academic levels over three years of full-time study. The five mathematics courses have been gradually realigned to the Queensland senior mathematics syllabus (QCAA, 2019) since 2019 with one foundation mathematics course, two intermediate mathematics courses, and two advanced mathematics courses. The final advanced mathematics course in the third year covers ODEs and mathematical modelling, which will be required in the specialist mathematics course for the senior mathematics syllabus.

Due to the combined effects of a low rate of tertiary participation of students in RRR areas, a large proportion of part-time students, and a higher attrition rate due to financial constraints and/or concurrent work and/or family commitments, there are usually about 40 students enrolled in the first-year mathematics course. The number is then reduced to around 25 in the second-year mathematics courses, and further reduced to around 10 students in the third-year mathematics courses. Hence, helping every student going through the final mathematics course in their third-year study is also the motivation for the teacher to do whatever could be done.

For achieving optimized teaching and learning outcomes, this advanced mathematics course has been structured in a seamless sequence of weekly activities outlined in **Figure 1**. Note that advanced techniques for solving ODEs, such as Laplace transform, Fourier series and transformation, and eigenvalues and eigenvectors, are not covered in the school senior mathematics syllabus.

In terms of teaching, learning and assessments, the course is divided roughly into two parts: solving first-order ODEs including numeric methods in the first part and solving second-order ODEs including systems of linear ODEs in the second part. Mathematical modelling is embedded into both parts wherever there is an appropriate opportunity to include modelling in problem solving. Given the fact there would be around 10 students enrolled in the third-year mathematics course and many of these students would be studying part-time with other work commitments, online live classes were scheduled weekly so that any student could join or leave the online class in their convenience. An online class was normally running for up to three hours with lectures and tutorials being seamlessly articulated together as short segments on individual focused topics. The live class was recorded, and the edited segments (or instructional videos) were then shared with all students on the course website.

Table 1. Learning sequence & methods related to special tutorials (Guo & Wang, 2019)

Topic	Extension	Method	Extra tutorial
Direct integration for $y' = f(x)$ or $x' = g(y)$		Integration by - Substitutions - By parts - Complete differentials - Partial fractions	Tutorial 2: By complete differentials Tutorial 4: Application
Separation of variables for $y' = f(x)g(y)$	$y' = f(\frac{y}{x})$	By substitution $u = \frac{y}{x} \longrightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$	Tutorial 1: By $u = \frac{y}{x},\ v = \frac{x}{y}$
1st-order linear ODE $y' + P(x)y = Q(x)$		Integrating factor $\begin{cases} \mu(x) = e^{\int -P(x)dx} \\ y = \mu(x)\left[c + \int \frac{Q(x)}{\mu(x)} dx\right] \end{cases}$	
	Bernoulli equation $y' + P(x)y = Q(x)y^n$	$\begin{cases} z = y^{1-n} \quad n \neq 0,1 \\ z' + (1-n)P(x)z = (1-n)Q(x) \end{cases}$	Tutorial 3: By Bernoulli equation
2nd-order homogenous constant coefficient linear ODE $y'' + ay' + by = 0$	Newton's second law $F = ma$	By Characteristic equation $r^2 + ar + b = 0 \to r_1, r_2$ $y = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x} \leftarrow r_1 \neq r_2 \\ c_1 e^{rx} + c_2 x e^{rx} \leftarrow r_1 = r_2 = r \\ e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \leftarrow r_{1,2} = \alpha \pm i\beta \end{cases}$	Tutorial 5: Setting up 2nd-order homogenous ODE through $F = ma$

After delivering the designated contents in a live class, the online students at the time could ask any question or share any difficulty in learning, including difficulties in dealing with problems in the formal assessment. Wherever appropriate, an extra online tutorial would be conducted immediately to address the issue raised by the online student. The recordings of such extra tutorials would be shared on the course website with all students soon after. Students could also ask any question or share any difficulty in learning outside of the online live class.

Extra special tutorials addressing the raised issues would be recorded offline and then shared on the course website with all students once edited. These extra special tutorials have proven to be immensely helpful in supporting students' learning and motivating them to achieve their best learning outcomes, particularly in challenging mathematics topics such as ODEs

PROGRESSIVE TUTORIALS

First Progressive Tutorial–Separation of Variables by Substitutions

After introducing the fundamentals of ODEs, the first set of techniques to solve first-order ODEs was taught and demonstrated to the students with various solved examples. The sequence of this set of techniques was from direct integration to separation of variables, and then extended to substitution by *u*=*y*/*x* for some special types of ODEs (**Table 1**). In the weekly exercises recommended to the students, the most challenging question was to solve the following ODE:

$$2xydx + (y^2 - 3x^2)dy = 0, \quad y(0) = 1$$

The reference solution provided to the students was based on substituting *x*=*yv*, or *v*=*x*/*y*, rather than the standard substitution of *y*=*xu* or *u*=*y*/*x* demonstrated in the textbook (Guo & Wang, 2019). The step-by-step solution for this question using *x*=*yv* is shown below (also in tutorial 0 at: https://youtu.be/VlwO3WW3eiU):

$$2xydx+(y^2-3x^2)dy=0 \longrightarrow 2xydx=-(y^2-3x^2)dy$$
$$\frac{dx}{dy}=\frac{3x^2-y^2}{2xy}=\frac{3x^2}{2xy}-\frac{y^2}{2xy}=\frac{3}{2}\frac{x}{y}-\frac{1}{2}\frac{y}{x}$$

Let $v=\frac{x}{y}$; then $x=yv$ and $\frac{y}{x}=\frac{1}{v}$..

Hence,

$$\frac{dx}{dy}=\frac{3}{2}\frac{x}{y}-\frac{1}{2}\frac{y}{x}=\frac{3v}{2}-\frac{1}{2v}=\frac{3v^2-1}{2v}=f(v) \longrightarrow \frac{dv}{f(v)-v}=\frac{dy}{y} \longrightarrow \frac{dv}{\frac{3v^2-1}{2v}-v}=\frac{dy}{y}$$
$$\frac{dv}{\frac{3v^2-1-2v^2}{2v}}=\frac{dy}{y} \longrightarrow \frac{2vdv}{v^2-1}=\frac{dy}{y} \longrightarrow \frac{d(v^2-1)}{v^2-1}=\frac{dy}{y} \longrightarrow \int\frac{d(v^2-1)}{v^2-1}=\int\frac{dy}{y}$$
$$\ln(v^2-1)=\ln y+c_1 \longrightarrow v^2-1=e^{\ln y+c_1}=e^{c_1}e^{\ln y}=cy \longleftarrow v=\frac{x}{y}$$

Thus,

$$(\frac{x}{y})^2-1=cy \longrightarrow (\frac{x}{y})^2=cy+1 \longrightarrow x^2=y^2(cy+1) \longrightarrow x^2=cy^3+y^2$$
$$y(0)=1 \longrightarrow x^2=cy^3+y^2 \longrightarrow 0=c+1 \longrightarrow c=-1$$
$$x^2=-y^3+y^2 \longrightarrow y^3=y^2-x^2$$

Students could understand the mathematical process of solving this problem by *x=yv* in the reference solution, but after having attempted this question by themselves, in the next live online class, one student was curious about why *x=yv* was preferred over *y=xu* in the solution provided. To answer this question, at the end of the live class, an extra tutorial was immediately conducted for the students remained online, and the recording was then shared on the course website with all students to clarify the similar curiosity (tutorial 1 at: https://youtu.be/lfOnA36Nzv8). This video contained the alternative solution using the 'usual' substitution *y=xu*, and its process is shared below for interested readers.

$$2xydx+(y^2-3x^2)dy=0 \longrightarrow 2xydx=-(y^2-3x^2)dy \longrightarrow (y^2-3x^2)dy=-2xydx$$
$$\frac{y^2-3x^2}{xy}\frac{dy}{dx}=-2 \longrightarrow (\frac{y}{x}-\frac{3x}{y})\frac{dy}{dx}=-2$$

Let $u=\frac{y}{x}$; then $y=ux$ and $\frac{x}{y}=\frac{1}{u}$.

Hence,

$$(u-\frac{3}{u})\frac{dy}{dx}=-2 \longrightarrow (\frac{u^2-3}{u})\frac{dy}{dx}=-2 \longrightarrow \frac{dy}{dx}=\frac{-2u}{u^2-3}=f(u) \longrightarrow \frac{du}{f(u)-u}=\frac{dx}{x}$$
$$\frac{du}{\frac{-2u}{u^2-3}-u}=\frac{dx}{x} \longrightarrow \frac{du}{\frac{2u+u(u^2-3)}{u^2-3}}=-\frac{dx}{x} \longrightarrow \frac{u^2-3}{u(u^2-1)}du=-\frac{dx}{x}$$
$$\int\frac{u^2-3}{u(u^2-1)}du=\int-\frac{dx}{x} \longrightarrow \int\frac{u^2-3}{u(u^2-1)}du=\ln\frac{1}{x}+c_1$$
$$\frac{u^2-3}{u(u^2-1)}=\frac{A}{u}+\frac{Bu+C}{u^2-1}=\frac{A(u^2-1)+u(Bu+C)}{u(u^2-1)}=\frac{Au^2-A+Bu^2+Cu}{u(u^2-1)}=\frac{(A+B)u^2+Cu-A}{u(u^2-1)}$$
$$\begin{cases}A+B=1\\C=0\\-A=-3\end{cases} \xrightarrow{A=3} \begin{cases}A=3\\B=1-A=-2\\C=0\end{cases} \longrightarrow \frac{u^2-3}{u(u^2-1)}=\frac{3}{u}-\frac{2u}{u^2-1}$$

Hence,

$$\int \frac{u^2-3}{u(u^2-1)}du = \int (\frac{3}{u} - \frac{2u}{u^2-1})du = 3\ln u - \int \frac{d(u^2-1)}{u^2-1} = \ln u^3 - \ln(u^2-1) = \ln \frac{u^3}{u^2-1}$$

$$\ln \frac{u^3}{u^2-1} = -\ln x + c_1 \longrightarrow \frac{u^3}{u^2-1} = e^{c_1} e^{\ln \frac{1}{x}} \longrightarrow \frac{u^3}{u^2-1} = \frac{c}{x} \longrightarrow u^3 = \frac{c}{x}(u^2-1) \longleftarrow u = \frac{y}{x}$$

$$\frac{y^3}{x^3} = \frac{c}{x}(\frac{y^2}{x^2}-1) \longrightarrow \frac{y^3}{x^3} = \frac{c}{x^3}(y^2-x^2) \longrightarrow y^3 = c(y^2-x^2) \longleftarrow x=0, y=1$$

$$1 = c(1-0) \longrightarrow c = 1 \longrightarrow y^3 = y^2 - x^2$$

This is the same result as the original reference solution provided to students, but its process is more tedious than using *x=vy* and requires using partial fractions to assist in integration, which was why *x=vy* was preferred in the original solution.

Second Progressive Tutorial–Direct Integration by Complete Differentials

After watching the processes of solving the same exercise by different substitutions demonstrated in the previously shared tutorial 1, another student asked privately whether this ODE could be solved by direct integration. To address this asynchronous request, a new special tutorial was recorded offline and shared on the course website with all students (link to tutorial 2 at: https://youtu.be/-6LG6XxIi0M). This video contained the alternative solution by means of integration by complete differentials, a method belonging to direct integration. This solution is shared below for interested readers.

$$2xydx + (y^2 - 3x^2)dy = 0 \longrightarrow 2xydx + y^2dy - 3x^2dy = 0$$

$$2xydx - 3x^2dy = -y^2dy \xrightarrow{\times y^2} 2xy^3dx - 3x^2y^2dy = -y^4dy$$

$$y^3d(x^2) - x^2d(y^3) = -y^4dy \xrightarrow{/y^6} \frac{y^3d(x^2) - x^2d(y^3)}{y^6} = -\frac{y^4}{y^6}dy$$

By quotient rule for differentials (or derivatives),

$$\frac{y^3d(x^2) - x^2d(y^3)}{y^6} = \frac{y^3d(x^2) - x^2d(y^3)}{(y^3)^2} = d(\frac{x^2}{y^3})$$

Hence,

$$d(\frac{x^2}{y^3}) = -\frac{dy}{y^2} \longrightarrow \int d(\frac{x^2}{y^3}) = \int d(\frac{1}{y}) \longrightarrow \frac{x^2}{y^3} = \frac{1}{y} + c \xrightarrow{\times y^3} x^2 = y^2 + cy^3$$

$$cy^3 = x^2 - y^2 \longleftarrow x = 0, y = 1$$

$$c = -1 \longrightarrow -y^3 = x^2 - y^2 \longrightarrow y^3 = y^2 - x^2$$

This is the same as the solutions obtained by using substitutions through separation of variables solved previously. This seems the most direct approach, but it requires higher-level of integration skills.

Third Progressive Tutorial–By Bernoulli Equations

After learning the first sequence of techniques from direct integration to separation of variables and its extension of substitution by *u=y/x* to solve first-order ODEs, the course then progressed to solving first-order linear ODEs by integrating factor and its extension to solving Bernoulli equations. One student reflected that in general she followed the standard mathematical procedures well in applying integrating factor to solve first-order linear ODEs and using substitution $z = y^{1-n}$ to solve Bernoulli equations. However, she thought that Bernoulli equations might be less useful as it would be rare to encounter a Bernoulli equation if not deliberately set up. This feeling about Bernoulli equations may be true to some extent as Bernoulli equations are treated as a special case by a special substitution $z = y^{1-n}$. In practice, some first-order ODEs can indeed be transferred to Bernoulli equations for solutions. To help students further understand the usefulness of Bernoulli equations in solving ODEs, a new special tutorial for solving the same exercise through Bernoulli equations was recorded offline and shared on the course website with all students (link to tutorial 3 at: https://youtu.be/T1CsIGeCZi8). The process of solving the question by Bernoulli equations is shared below.

$$2xydx+(y^2-3x^2)dy=0 \longrightarrow 2xydx=-(y^2-3x^2)dy$$

$$\frac{dx}{dy}=\frac{3x^2-y^2}{2xy} \longrightarrow \frac{dx}{dy}=\frac{3x^2}{2xy}-\frac{y^2}{2xy} \longrightarrow \frac{dx}{dy}=\frac{3}{2y}x-\frac{1}{2}yx^{-1}$$

$$\frac{dx}{dy}-\frac{3}{2y}x=-\frac{1}{2}yx^{-1} \longleftarrow \text{Bernoulli equation } (n=-1)$$

Let $z=x^{1-n}=x^2;\ P(y)=-\frac{3}{2y}, Q(y)=-\frac{1}{2}y, 1-n=2$

$$\frac{dz}{dy}+(1-n)P(y)z=(1-n)Q(y) \longrightarrow \frac{dz}{dy}-\frac{3}{y}z=-y \longleftarrow \text{1st-order linear ODE}$$

$$p(y)=-\frac{3}{y}, q(y)=-y \longrightarrow \mu(y)=e^{\int -p(y)dy}=e^{\int \frac{3}{y}dy}=e^{3\ln y}=y^3$$

$$z=\mu(y)\left[c+\int\frac{q(y)}{\mu(y)}dy\right]=y^3\left[c+\int\frac{-y}{y^3}dy\right]=y^3\left[c-\int\frac{1}{y^2}dy\right]=y^3\left[c+\frac{1}{y}\right]=cy^3+y^2$$

$$z=x^2 \longrightarrow x^2=cy^3+y^2 \longleftarrow x=0, y=1$$

$$0=c+1 \longrightarrow c=-1 \longrightarrow x^2=-y^3+y^2 \longrightarrow y^3=y^2-x^2$$

This is the same result as the solution obtained by the three different methods demonstrated previously. Note that the transformation to a Bernoulli equation for this question can only be made through *dx*/*dy*. The usual way by means of *dy*/*dx* cannot transfer the ODE to a Bernoulli equation in this case. Hence, this solution also demonstrated a need for students to become adaptive to different conditions by applying the same technique.

TARGETED TUTORIALS

One of the teaching strategies the author has adopted in teaching mathematics is *learning by challenging*. This pedagogical strategy is realized in the assignments for students by including at least one word problem in an assignment. The word problem challenges students to solve the problem by integrating the relevant knowledge and techniques obtained thus far, including those obtained in previous mathematics courses and the general science knowledge learnt in secondary schools. Of course, necessary hints and assistance would be provided to students according to their progression in attempting the problem. If multiple students request assistance in the similar knowledge area as a shared point of difficulty, an extra video will be prepared and shared with the students to target the area, where many students feel difficult. This usually takes one of the two forms: providing a similar solved example for students to understand the correct procedure, or explaining further the background information and related facts about the word problem so that students can start attempting the problem correctly. Two examples of such targeted tutorials are presented in this section.

First Targeted Tutorial

The following word problem was assigned to all students in the first assessment (problem 1). It aimed at testing students' ability to capture a real-word scenario described in words as a simple differential equation, then solve the differential equation with the known conditions correctly, and finally demonstrate basic modelling skills for this scenario. The required mathematical procedure and solutions are detailed in **Appendix A** for reference.

Problem 1. Word problem assigned to students in assignment 1

If a stone were thrown on to the flat surface of a frozen lake with an initial speed of 10 m/s, the speed of the stone would be gradually reduced. Assume its change in speed was proportional to the product of the square of the current speed with its travelling time. The speed of the stone was observed to be 5 m/s at the 5th second mark.

a. Find how fast the stone would travel on the surface with time, i.e., *v*(*t*).

b. Determine how long (i.e., time) the stone would have travelled when its speed was 25% of the initial speed.

c. Plot the result as a *t*–*v* curve for the first 20 seconds with an interval of 0.5 seconds.

Many students attempted to solve this question but felt not confident in the entire procedure and solutions they came up with as this was the first time they had to streamline multiple processes for different purposes together cohesively. Upon observing student's unconfident sentiment on their attempts, the author crafted a similar word question shown as problem 2 below and recorded the procedure of solving this word problem off-line. The recording was then shared on the course website with all the students (link to tutorial 4 at: https://youtu.be/jKKRevdgmUM). This procedure is shared below with interested readers.

Problem 2. A problem recorded in targeted tutorial to assist in solving problem 1

A motorbike travelling with a speed of 10 m/s took a quick brake to avoid a potential accident. If the change in speed is proportional to the product of the cube of the speed with the time from the moment the brake was pressed, and the speed was reduced to 2 m/s in 1 second,

a. How fast the motorbike would travel since the brake was pressed?

b. How long would the speed of the motorbike be reduced to 1 m/s?

c. How far would the motorbike have traveled in 4 seconds from the moment the brake was pressed?

How fast the motorbike would travel since the brake was pressed?

$$\frac{dv}{dt} = -kv^3t \longrightarrow \frac{dv}{v^3} = -ktdt \longrightarrow \int \frac{dv}{v^3} = \int -ktdt$$

$$-\frac{1}{2v^2} = -\frac{1}{2}kt^2 - d \xrightarrow{\times(-2)} \frac{1}{v^2} = kt^2 - 2d \xrightarrow{c=-2d} v^2 = \frac{1}{kt^2 + c} \longrightarrow v = \frac{1}{\sqrt{kt^2 + c}}$$

$$t = 0, v = 10m/s \longrightarrow 10 = \frac{1}{\sqrt{c}} \longrightarrow \sqrt{c} = \frac{1}{10} \longrightarrow c = \frac{1}{100}$$

$$v = \frac{1}{\sqrt{kt^2 + c}} = \frac{1}{\sqrt{kt^2 + \frac{1}{100}}} = \frac{1}{\sqrt{\frac{100kt^2 + 1}{100}}} = \sqrt{\frac{100}{100kt^2 + 1}} = \frac{10}{\sqrt{100kt^2 + 1}}$$

$$t = 1s, v = 2m/s \longrightarrow 2 = \frac{10}{\sqrt{100k + 1}} \longrightarrow \sqrt{100k + 1} = 5 \longrightarrow 100k + 1 = 25 \longrightarrow k = \frac{24}{100}$$

$$v = \frac{10}{\sqrt{100 \times \frac{24}{100}t^2 + 1}} = \frac{10}{\sqrt{24t^2 + 1}}$$

How long would the speed of the motorbike be reduced to 1 m/s?

$$v = 1\ m/s \longrightarrow \frac{10}{\sqrt{24t^2 + 1}} = 1 \longrightarrow \sqrt{24t^2 + 1} = 10 \longrightarrow 24t^2 + 1 = 100$$

$$24t^2 = 99 \longrightarrow t = \sqrt{\frac{99}{24}} \approx 2.03\ s$$

How far would the motorbike have traveled in 4 seconds from the moment the brake was pressed?

$$v = \frac{10}{\sqrt{24t^2 + 1}} = \frac{ds}{dt} \longrightarrow ds = \frac{10}{\sqrt{24t^2 + 1}}dt$$

$$s = \int_0^4 \frac{10}{\sqrt{24t^2 + 1}}dt = \frac{10}{\sqrt{24}}\int_0^4 \frac{1}{\sqrt{t^2 + \frac{1}{24}}}dt = \frac{10}{\sqrt{24}}\int_0^4 \frac{1}{\sqrt{t^2 + (\frac{1}{\sqrt{24}})^2}}dt \longleftarrow a = \frac{1}{\sqrt{24}}$$

$$= \frac{10}{\sqrt{24}}\ln\left[t + \sqrt{t^2 + \frac{1}{24}}\right]_0^4 = \frac{10}{\sqrt{24}}\left[\ln\left(4 + \sqrt{16 + \frac{1}{24}}\right) - \ln\frac{1}{\sqrt{24}}\right] \approx 7.5\ m$$

For more detail, refer to **Appendix B**.

Table 2. Summary of solving first-order ODEs & problem 1 in assignment 1 by students

Method/problem	Correct	Incorrect	No attempt	Average rate of correction
Separation of variables 1	5	3	0	83%
Separation of variables 2	8	0	0	
Separation of variables 3	7	1	0	
Integrating factor 1	5	2	1	56%
Integrating factor 2	4	3	1	
Bernoulli equation	5	2	1	63%
Problem 1	6	1	1	75%

Second Targeted Tutorial

The following word problem was assigned to all students in the second assessment (problem 3). It aimed at testing students' ability to capture a real-word scenario described in words as a simple differential equation, then solve the differential equation with the known conditions correctly, and finally demonstrate basic modelling skills for this scenario. The required mathematical procedure and solutions are detailed in **Appendix C** for reference.

Compared to problem 1, most students seemed unsure how to start with the second question. *Two students thought this would need a system of two first-order ODEs as the resistance had two components*. By the time the formal delivery of a live online class was completed, *the two students asked the lecturer (the author) to comment whether their idea of using a system of two first-order ODEs was in the right direction*. In the meantime, *one other student in his email was concerned with how to apply the Newton's second law to this situation*. All students' questions were pointing to their weak knowledge in basic physics, specifically the Newton's second law. This is understandable as most of these education students did not have a strong background in science due to the shallow coverage in science subjects in their secondary schools and a lack of connections between science and mathematics courses in the current tertiary curriculum.

Problem 3. Word problem assigned to students in assignment 2

When the power of a boat travelling in a lake was cut off, its speed was 8 m/s. The resistance force against the boat was proportional to

(a) the speed of the boat by a factor of 400 kg/s and

(b) the distance the boat moved since the power was cut off by a factor of 50 kg/s^2.

Suppose the boat had a mass of 500 kg.

a. Determine the distance the boat travelled as a function of time since the power was off.
b. Find the distance the boat travelled till stopped since the power was off.
c. Plot the speed and distance curves against time with an interval of 0.1 seconds.

Upon realizing student's problem with basic physics knowledge, the lecturer immediately opened and shared problem 3 with the two students online and explained the Newton's second law in general and how to analyze the physical scenario presented in problem 3. Based on the situational analysis for the problem, the lecturer then demonstrated how to apply Newton's second law to this situation so that students could begin their attempt with a correct strategy. Unlike problem 1, where students were unsure whether their attempts and solutions were correct against the described scenario, solving a simple second-order ODE by means of the method of characteristic equations should be manageable by the students for problem 3. Hence, no further elaboration on solving problem 3 was provided in the recording that was shared with all the students on the course website (link to tutorial 5 at: https://youtu.be/EOppFoHxu44).

DISCUSSION & CONCLUSIONS

In the first assignment assessing student's efficacy in solving the 1st-order ODEs with different methods, students seemed able to use separation of variables to solve the three explicitly given ODEs satisfactorily in general (**Table 2**), with an average rate of correction of 83% for the three questions. One student seemed only able to use separation of variables and hence attempted these three questions only in the entire assignment,

Table 3. Summary of solving second-order ODEs & problem 3 in assignment 2 by students

Problem	Correct	Incorrect	No attempt	Average rate of correction
Homogeneous ODE	8	0	0	100%
Inhomogeneous ODE 1	8	0	0	96%
Inhomogeneous ODE 2	8	0	0	
Inhomogeneous ODE 3	7	1	0	
System of linear ODEs	6	2	0	75%
Problem 3	5	2	1	63%

with two corrections and one incorrection. The other three incorrections were due to errors in conducting integrations during the process, which has been reported as one of the common mistakes students made in applying integration by parts (Guo, 2021a, 2021b; Guo et al., 2021).

The next best was solving the word problem to which a similar case was demonstrated in tutorial 4. Six out of eight students correctly solved this word question following the similar process demonstrated in tutorial 4. Except the student who only attempted the three questions by separation of variables, one other student consistently used methods outside of the coverage in this course and obtained incorrect solutions to most questions in this assignment including the word problem. The course log showed that this student did not access any of the special tutorials on the course website. Compared to other explicitly given ODEs, this word problem would be the most challenging question in this assignment. However, all those students who watched tutorial 4 had managed to solve this challenging problem correctly, which indicates that the targeted tutorial indeed provided the students with effective learning support that enabled them to achieve their best possible outcomes.

Solving the first-order linear ODEs by means of integrating factor and Bernoulli equations were more problematic for the students. With the help of the worked examples, plus the extra progressive tutorials for both the first-order linear ODEs and Bernoulli equations, many students could choose a correct procedure in which they were most confident in approaching individual questions, except the two students who either did not attempt the rest of questions or used a method outside of this course wrongly. The major cause for the students to get incorrect solutions for these questions was again the mistakes in conducting integrations, particularly integration by parts. Hence, retaining skills and knowledge gained in previous mathematics courses is vital for students' smooth progression in advanced mathematics courses, regardless of how much extra effort the educators can make to the pedagogy of the course.

For the second assignment, there was almost no problem for all the students to solve the four second-order constant-coefficient linear ODEs explicitly expressed (**Table 3**). Six out of the eight students also solved the system of two linear ODEs correctly whereas the two students who could not solve problem 1 correctly could not follow the correct procedure to solve this system of linear ODEs. With the help of the targeted tutorial 5, five out of the eight students solved the word problem 3 and modelled the scenario correctly. The two students who did not watch tutorial 5 were either attempted the question in a completely wrong way or gave it up all together. One other student who watched tutorial 5 solved part of the first sub-question correctly but were wrong in the rest of question due to inefficient understanding of the physical relationships and conditions involved in this scenario.

It is evident that once being guided on how to apply Newton's second law to a scenario described by words, most students had the required mathematical skills to solve the problem correctly. On the other hand, without basic scientific knowledge, the learnt mathematical skills would always be regarded as just logical or symbolic games or tools only useful in an abstract world. In fact, modern mathematics was mainly shaped by the demands of scientific exploration and inventions. Therefore, there have been an increasing number of calls to interconnect STEM subjects in all levels of education in the past two decades, particularly in the recent ten years (e.g., Carrejo & Marshall, 2007; Ghani, et al., 2021; Kertil & Gurel, 2016; Rizza, 2021; Surif et al., 2012; Tan et al., 2022; Tursucu et al., 2020). The artificial separation of mathematics from science, particularly physics in most education curricula for secondary students in the world, should be seriously reviewed, restructured, and rationalized in future curriculum reform (Al-Mutawah et al., 2022; Incikabi & Serin, 2017; Jones et al., 2003; Teo et al., 2021).

Given the extremely challenging nature of ODEs and mathematical modelling, the teacher's best effort on supporting student's learning must be echoed by student's active engagement with learning and courage to deal with the challenges during the course. No matter how much extra effort an educator can make to the pedagogy and curriculum, not all students would pay attention to or appreciate the teacher's effort. Although there were only eight students enrolled in this course, two of them (or 25%) were not actively engaged with the teaching and learning practices of this course, similar to the findings of existing studies that some students enrolled in a course would not follow any instruction or would do nothing during the course (Guo et al., 2021; Henry et al., 2019; Massingham & Herrington, 2006). Hence, aiming to achieving a correction rate of 100% from all students would be an unrealistic task on most occasions, particularly in advanced mathematics courses.

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APPENDIX A: SOLUTIONS TO PROBLEM 1

a. Find how fast the stone would travel on the surface, i.e., *v*(*t*)?

$$\frac{dv}{dt} = -kv^2 t \longrightarrow \frac{dv}{v^2} = -ktdt \longrightarrow -\frac{1}{v} = -\frac{1}{2}kt^2 + d \longrightarrow \frac{1}{v} = \frac{1}{2}kt^2 - d = \frac{kt^2 - 2d}{2} = \frac{kt^2 + c}{2}$$

$$v = \frac{2}{kt^2 + c} \longrightarrow v(0) = 10 \longrightarrow \frac{2}{c} = 10 \longrightarrow c = \frac{2}{10} = \frac{1}{5} \longrightarrow v = \frac{2}{kt^2 + \frac{1}{5}} = \frac{10}{5kt^2 + 1}$$

$$v(5) = 5 \longrightarrow \frac{10}{5k(5)^2 + 1} = 5 \longrightarrow 5^3 k + 1 = \frac{10}{5} = 2 \longrightarrow 5^3 k = 1 \longrightarrow k = \frac{1}{5^3}$$

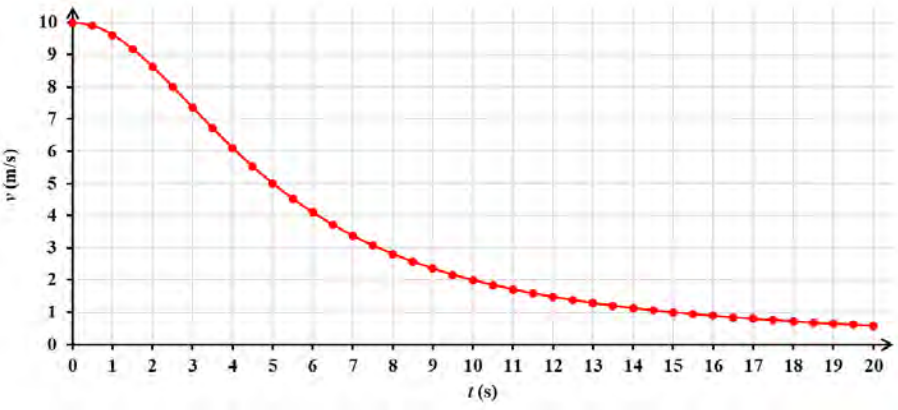
$$\therefore v = \frac{10}{5(\frac{1}{5^3})t^2 + 1} = \frac{10}{\frac{t^2}{25} + 1} = \frac{250}{t^2 + 25}(m/s)$$

b. Determine how long (i.e., time) the stone would have travelled when its speed was 25% of the initial speed?

$$\frac{250}{t^2 + 25} = 10 \times 0.25 = 2.5 \longrightarrow t^2 + 25 = \frac{250}{2.5} = 100 \longrightarrow t^2 = 100 - 25 = 75$$

$$t = \sqrt{75} \approx 8.66(s)$$

c. Plot your result as a *t*–*v* curve for the first 20 seconds with an interval of 0.5 seconds.



(Source: Author)

APPENDIX B: SOLUTION TO THE INDEFINITE INTEGRAL IN PROBLEM 1

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh u du}{\sqrt{a^2(\sinh^2 u+1)}} \longleftarrow x = a\sinh u,\ dx = a\cosh u du,\ 1+\sinh^2 u = \cosh^2 u$$

$$= \int \frac{a\cosh u du}{a\cosh u} = \int du = u + c = \ln\left[\frac{x}{a}+\sqrt{1+\left(\frac{x}{a}\right)^2}\right] + c_1 \longleftarrow \sinh u = \frac{x}{a},\ u = \sinh^{-1}\frac{x}{a} = \ln\left[\frac{x}{a}+\sqrt{1+\left(\frac{x}{a}\right)^2}\right]$$

$$= \ln\left(\frac{x}{a}+\sqrt{\frac{x^2+a^2}{a^2}}\right) + c_1 = \ln\left(\frac{x}{a}+\frac{\sqrt{x^2+a^2}}{a}\right) + c_1 = \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c_1 = \ln\left(x+\sqrt{x^2+a^2}\right) - \ln a + c_1$$

$$= \ln\left(x+\sqrt{x^2+a^2}\right) + c \longleftarrow c = -\ln a + c_1$$

APPENDIX C: SOLUTIONS TO PROBLEM 3

a. When the power was off, the only external force exerted on the boat was the resistance,

$$R = 400v + 50x = 400\frac{dx}{dt} + 50x$$

where *v* is the speed of the boat at time *t* and *x* is the distance the boat travelled since the power was off. By Newton's second law,

$$ma = -R \longrightarrow ma = -400\frac{dx}{dt} - 50x \longrightarrow 500a = -400\frac{dx}{dt} - 50x \longrightarrow a = -0.8\frac{dx}{dt} - 0.1x$$

$$a + 0.8\frac{dx}{dt} + 0.1x = 0 \longrightarrow \frac{d^2x}{dt^2} + 0.8\frac{dx}{dt} + 0.1x = 0$$

where *m*=500 kg is the mass of the boat and *a* is the acceleration of the boat. This is a second-order homogeneous ODE.

$$\frac{d^2x}{dt^2} + 0.8\frac{dx}{dt} + 0.1x = 0 \longrightarrow r^2 + 0.8r + 0.1 = 0$$

$$r = \frac{-0.8 \pm \sqrt{0.8^2 - 4\times 0.1}}{2} = \frac{-0.8 \pm \sqrt{0.24}}{2} = -0.4 \pm \sqrt{0.06} \longrightarrow r_1 = -0.1551, r_2 = -0.6449$$

$$x = c_1e^{r_1t} + c_2e^{r_2t} = c_1e^{-0.1551t} + c_2e^{-0.6449t} \longrightarrow v = \frac{dx}{dt} = -0.1551c_1e^{-0.1551t} - 0.6449c_2e^{-0.6449t}$$

$$\begin{cases} x(0) = 0 \\ v(0) = 8 \end{cases} \longrightarrow \begin{cases} c_1 + c_2 = 0 \\ -0.1551c_1 - 0.6449c_2 = 8 \end{cases} \xrightarrow{c_2 = -c_1} -0.1551c_1 + 0.6449c_1 = 8 \longrightarrow 0.4898c_1 = 8$$

$$c_1 = \frac{8}{0.4898} = 16.3332 \longrightarrow c_2 = -c_1 = -16.3332$$

$$x = 16.3332e^{-0.1551t} - 16.3332e^{-0.6449t} = 16.3332(e^{-0.1551t} - e^{-0.6449t})$$

$$v = \frac{dx}{dt} = -2.5333e^{-0.1551t} + 10.5332e^{-0.6449t}$$

b. Find the distance the boat travelled till stopped since the power was off. When the boat stopped, the speed must be zero.

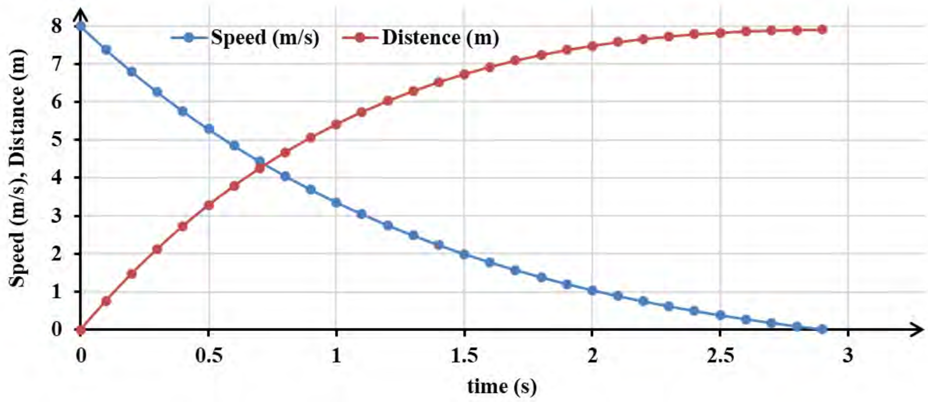
$$v = -2.5333e^{-0.1551t} + 10.5332e^{-0.6449t} = 0 \longrightarrow \frac{e^{-0.1551t}}{e^{-0.6449t}} = \frac{10.5332}{2.5333} \longrightarrow e^{0.4898t} = \frac{10.5332}{2.5333}$$

$$0.4898t = \ln\frac{10.5332}{2.5333} \longrightarrow t = \frac{1}{0.4898}\ln\frac{10.5332}{2.5333} \approx 2.9095(s)$$

$$x(2.9095) = 16.3332(e^{-0.1551\times 2.9095} - e^{-0.6449\times 2.9095}) \approx 7.9(m)$$

The boat stopped about 2.9 seconds and travelled about 7.9 m since the power-off.

c. Plot the speed and distance curves against time with an interval of 0.1 seconds.



(Source: Author)

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