Mathematical Symbols in Academic Writing: The Case of Incorporating Mathematical Ideals in Academic Writing for Education Researchers

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Lin Li*

Faculty of Education, University of Windsor

Abstract

Mathematical symbols, such as those embodying quantum concepts, are indispensable for conveying complex ideas and relationships in academic writing. However, some education researchers and students keep a distance from anything mathematical: algebraic equations, geometrical reasoning, or statistical symbols. How to lower the access threshold for this type of mathematical narrative and reveal the meanings of a range of quantum conceptions to modern educators thus becomes a real problem. Using the pendulum motion equation as a reference point, I argue in this article for the advantages of academic English or French writing genres that fuse a range of mathematical symbols of quantum concepts and conceptual change. Such writings help demonstrate how incorporating the idea of probability (a) refines the debate among conceptual, verbal, and mathematical academic writing; (b) allows new conceptions that draw on the insights from quantum cognition-supported theories; (c) helps explain students' understanding of mathematical symbols; and (d) offers a new taxonomy for categorizing academic writings.

Keywords: academic writing, quantum concepts, probability, taxonomy, mathematical narrative

* li81@uwindsor.ca

Introduction

In 2022, the federal government of Canada published its position statement, *Canada's National Quantum Strategy*, characterizing the nation as a quantum pioneer and envisioning a new strategy supported by three pillars: research, talent, and commercialization (Government of Canada, 2022). The statement called for provincial and international collaboration to reinforce such a three-pronged foundation. However, the consulted stakeholders in this national strategy share a growing concern about the talent pool challenge that is likely to pose a counterproductive problem in implementing the future-shaping agenda. In this aspect, they noted that "(r)epresentation in key fields relevant to the quantum sector is currently imbalanced, so Canada is missing out on a critical supply of new ideas and talent in a highly competitive international market" (p. 21). One way to restore the balance as liberal arts educators is to promote quantum educational programs, especially those using mathematical symbols and their epistemic significance in academic activities.

Even as advocated by the federal position statement, quantum-related conceptions tend to stir mixed feelings in some education researchers when contemplating the involvement of the underlying mathematical structures and their written expressions (Haven & Khrennikov, 2017). Education researchers and students often keep a distance from anything mathematical: algebraic equations, geometrical reasoning, or statistical symbols, ignoring their essential role in academic writing and embracing a mathematical symbol-free writing style (Hutto, 2013; Hutto & Myin, 2014). At the other end of the attitude spectrum toward mathematical expressions, some education researchers try to fill up a page with an array of mathematical or statistical equations without a clear explanation of the underlying conceptual structure familiar to most readers (Hõhn, 2017). Either writing style is unsatisfactory for the purpose of instruction, especially when understanding quantum conceptions is one of the long-term goals of instructional activities. How to lower the access threshold for this type of quantum-conception-embedded written narratives and reveal the meanings of a range of such mathematical functions, symbols, parameters, numbers, and units to modern education researchers and students thus becomes a real problem. I take a language teaching perspective (Benesch, 2001) to address the challenge by assuming classic and modern mathematical symbols as a new visual language (Mazur, 2014) with its own lexis, grammar, and sentence structures. Language instruction strategies (Canagarajah, 2011) can be borrowed to examine the academic writings about the quantitative written elements or units of quantum conceptions and their combination in the context of the unique narrative flows.

More importantly, I argue in this article for the advantages of academic English or French writing genres that fuse a range of mathematical symbols of quantum concepts (Martínez-Mingo et al., 2023; Pothos & Busemeyer, 2013) and conceptual change (Amin & Levrini, 2017).

Such writings help demonstrate how incorporating the idea of probability (a) refines the debate among conceptual, verbal, and mathematical academic writing; (b) allows for new conceptions that draw on the insights from quantum cognition-supported theories; (c) helps explain students' understanding of mathematical symbols; and (d) offers a new taxonomy for categorizing academic writings. In each of these aspects of academic writing research, achieving self-acceptance of mathematical symbols frees education researchers' creativity, pointing to a promising way to express themselves flexibly with mathematical signs. Throughout this article, I establish a reference framework by referring to the quantitative aspect of simple pendulum motion (Baker & Blackburn, 2005; Matthews, 2005) for illustration. The case of pendulum motion is chosen because it is paradigmatic in explaining classical quantum mechanics (Kuhn, 1970).

This article starts with an introduction to the notion of a simple pendulum. Next, conceptual, symbolic, and mathematical elements in academic writing are differentiated, followed by a description of the nuts and bolts of quantum conceptions and interview data regarding students' understanding of the pendulum motion equation. The results of this led to a discussion of the need for a new taxonomy of academic writings with mathematical symbols in the context of making mathematical ideals explicit. Finally, the pedagogical implications of the taxonomy are detailed.

Writing About an Idealized (Mathematized) Simple Pendulum in Motion

According to Matthews (2005), the Renaissance Italian mathematician, astronomer, and engineer Galileo di Vincenzo Bonaiuti de' Galilei's contribution to modern science can be summarized as "the novel methodology of idealization" (p. 209). Since then, the idealization has been an exercise in the mind field or a conceptual space (Amin & Levrini, 2017; Gärdenfors, 2014), which is often conventionally written in a visual language: algebraic or geometrical symbols. In Matthews's (2005) words, "Galileo's laws of pendulum motion could not be accepted until the empiricist methodological constraints placed on science by Aristotle, and by common sense, were overturned" (p. 209). The well-known algebraic relationship between the length of time for a pendulum swing and the string length was not discovered by Galileo. Instead, it was the Dutch mathematician Christiaan Huygens who derived the mathematical equation $T = 2\pi \sqrt{l/g}$, which is a perfect case of this type of idealized conception of a physical phenomenon. In such a written narrative of the visual language, the actors or "heroes" of their narratives are mathematical elements such as a constant and a few approximates, such as 2, π (a fixed numerical value), and g (another variable constant measuring the acceleration due to gravity at a location). However, these Renaissance pre-calculus natural philosophers' time-measuring attempts can only be told when the observation of pendulum motion is set at one fixed observation location on the earth's surface due to the g-related variances and other boundary conditions. It is important because the countless errors and variances can be ignored by assuming mathematical ideals rather than relying on what really happens.

In terms of the new rhetorical devices of the visual language, the mathematical relationship in this equation can be characterized as a scaler product, which features a multiplying connection between the constants and unknown or varying values. Furthermore, there is a co-varying relationship connoting the same change in the quantities of the two variables on the left and right sides of the equal sign. Finally, the often-ignored aspect of this written mathematical narrative is its nature as an approximation to a simple harmonic oscillator swinging within a small release angle of the suspended weight. Out of such a boundary condition, the predictive power of such an idealized mathematical construct is lessened; thus, a new full-fledged all-terms-included equation of pendulum motion would be needed for predicting its time-keeping behaviour. In this sense, this classical model of such a paradigmatic deterministic system should be considered quasi-deterministic or probabilistic. The same idealized or mathematized process, through the new visual language, can also be said about understanding the quantum conceptions and constructing their academic written narratives.

Conceptual, Symbolic, and Mathematical Elements in Academic Writing

For education researchers and students, the science concepts they might learn in liberal arts programs are different from those embodying mathematical ideals, defined or expressed mathematically. Take two straightforward concepts: the period of a pendulum and force, for example. Intuitively, time is associated with a process with a duration or a before-or-after relation, whereas force is associated with the experience of pushing or pulling an object. However, the physics identification and mathematical calculation of the period define it as a product with a coefficient of 2π and a square root of a quotient *(l/g)*. Similarly, a Newtonian force is expressed as another product of two terms: mass (m) and a change in velocity ($\frac{dv}{dt}$), as expressed in the equation form below:

$$F = m \times \frac{dv}{dt}$$

In the language of mathematics, this equation tells us clearly that the force exerted on an object is directly proportional to the mass of the object and the acceleration it experiences.

In introducing the rise of the mechanical view, Albert Einstein and Leopold Infeld (1966) wrote:

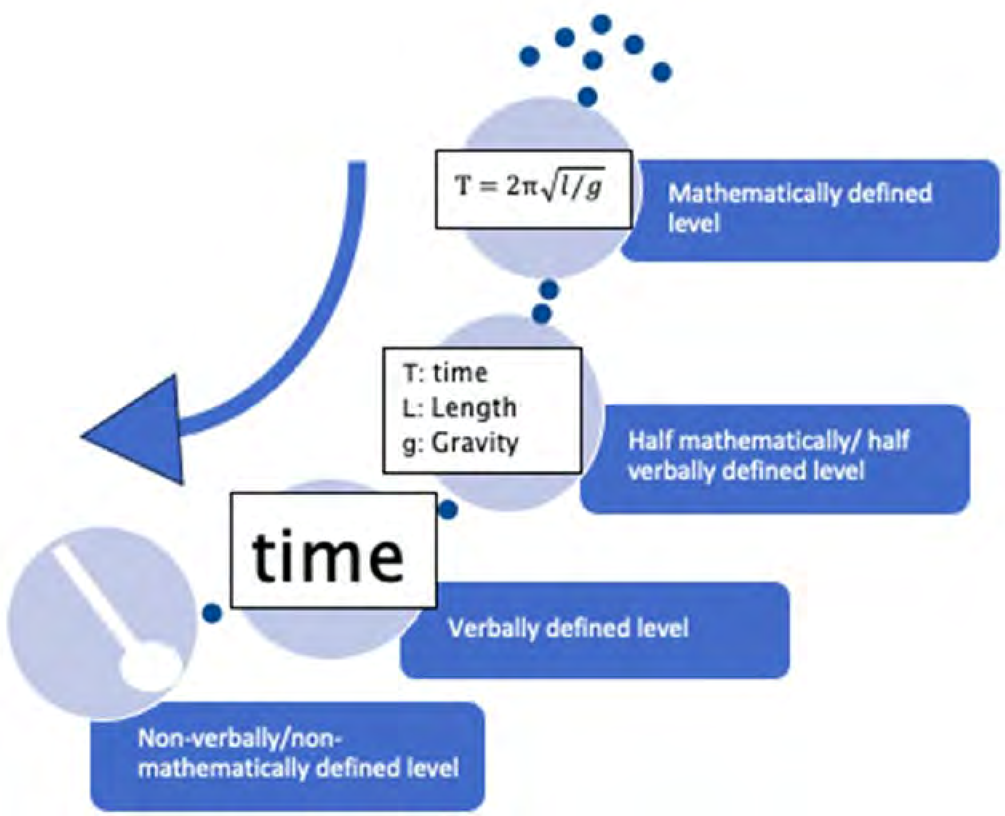
> When and where we observe a change in velocity, an external force, in the general sense, must be held responsible. Newton wrote in his Principia: An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line. This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis*

inertiae only. Impressed forces are of different origins; as from percussion, from pressure, from centripetal force. (p. 11)

The mathematical idealization of force is translated into the verbal descriptions penned repeatedly since Galileo's time and the mathematization of physics. In this sense, a concept can be conceptualized as a mental construct defined half-mathematically and half-verbally. Similarly defined physics concepts can easily be found in conceptual physics textbooks or the reviewed conceptual change studies. Kim et al. (2018) categorized seven mathematically defined physics concepts to illustrate their ontological and epistemological functions in science education. Figure 1 is an illustration of conceptual, verbally symbolic, and mathematical elements at various levels of academic written narratives using the pendulum motion example.

Figure 1

Illustration of the Conceptual, Symbolic, and Mathematical Elements for Characterizing Pendulum Motion



To dramatize the effects of differentiating these elements, a pictorial and symbolic mixed artwork is created to foreground the conception of mathematically defined physics concepts, especially in the case of learning pendulum motion. The keywords of the new taxonomy in Figure 1 characterize a knowledge system of pendulum motion phenomena, statements, data, and a structural realist's theory about them (Chalmers, 1999; Matthews, 2015; Rowbottom, 2019; Worrall, 2007). At the bottom of the upward swing (i.e., at the non-verbally/non-mathematically defined level), the physical objects, natural processes, and simple events of our world occurred naturally, without the involvement of any form of symbolic processing in any language. Moving up a bit, it is the human psychological aspects of observing what has happened in the world and the symbolism in English or other languages.

Next, human perception-driven statements or scientific narratives about the experience of understanding pendulum motion are located at the verbally defined level, or propositional perception (Matthews, 2015). Following the level, error-term-characterized empirical observations are represented as half mathematically and half verbally defined as raw scientific data. At the top of the upward swing sits the structural realist's mathematical core: $T = 2\pi\sqrt{l/g}$ for small swing amplitudes and the real natural phenomenon hidden behind the veil of so-called "reality": an isochronous simple harmonic oscillator. Together, the upward movement of a pendulum acts as a conceptual linchpin for characterizing the scope of knowledge involved in learning pendulum motion, which spans from an event to propositional perception and the underlying continuous mathematical identification of this phenomenon. The downward swing by the side of the upward one shows a recurring information integration episode in a learner's mind. The dashed lines indicate the probabilistic nature of human information processing, which highlights random processes in human cognition.

Almost 11 years ago, Pothos and Busemeyer (2013) published an article in *Behavioural and Brain Sciences* and asked whether quantum probability can provide a new direction for understanding probabilistic human cognition, especially when they have to make decisions with uncertainty or under conflicts. As they note in their more recent work, the quantum conceptual models "appear to work particularly well in certain empirical cases, such as when it appears that there is interference, or contextuality" (Pothos & Busemeyer, 2022, p. 772). Before detailing this new trend in academic writing, reviewing the historical development of quantum conceptions, as explained by Canadian scholars, is informative.

Idealized Quantum Conceptions: Lexis and Grammar

The use of mathematical idealization extends beyond understanding pendulum motion; today's scientists also employ it in their research. This point is made clear by tracing the foundation of modern physics, which rests on two conceptual pillars: Einstein's general relativity theories and the conception of quantum theories (Einstein & Infeld, 1966). The origins of quantum conceptions can be traced back to the idealized blackbody radiation problem (Thagard, 1992). In explaining his own radiation energy equation, Max Planck put forward a famous hypothetical principle: the radiating energy of a subatomic oscillator can only be explained by assuming an integral multiple of a product *hf* (*h* is the constant called Planck's constant: 6.625×10^{-34} Jouleseconds. f is the frequency of radiation. Given that $E = hf$ and h is a constant, the radiation energy E is directly proportional to the frequency of the radiation). The apparently simple mathematical relationship marked the birth of the idea that radiation is quantized rather than continuous, as Maxwell's electromagnetic partial differential equations assumed. Later, the quantum idea was implemented by Einstein to explain the photoelectric effect and by Neil Bohr to explain the quantum "jump" of the electrons switching between their relatively stable orbits (Stinner, 1985). Collectively, these ideas are referred to as the classic quantum theory. This conceptual framework influences today's physicists and researchers in social, ecological, and biological sciences. Outside of academic writing, it also influences policy decision-making, such as those expressed in the position statement of *Canada's National Quantum Strategy*.

However, there are caveats to consider regarding academic writing with mathematical ideals. Take modern cognitive science researchers as an example. They do not assume quantum-like processes actually occurred in information processing systems, natural or artificial. Instead, they feed their curiosity with the mathematical structure of quantum theories, following Plato's advice on viewing geometry and numbers as the royal path to understanding reality in the information-processing brain. However, the geometry is projective in this case, sometimes even building on a complex manifold rather than on a Euclidean plane. In a feature review, Bruza et al. (2015) characterized such an effort as a new theoretical approach to a systematic understanding of human thinking and decision-making processes. Mathematically, they highlighted John von Neumann's projective geometric structure of vector spaces as a representation of probability rather than Andrey Kolmogorov's set theories. Earlier, Gabora et al. (2008) also introduced quantum formalism in the State-Context-Property (SCOP) framework with five types of elements: a set of states, a set of relevant contexts, a set of relevant properties or features, a function that characterizes the applicability of a unique feature given a specific context, and a function that describes the transition probability from one state to another one considering the context effect. Such a reconceptualization of human concepts and conceptual systems has paved the way to address the challenge of understanding insightful human conception, thus changing how to prepare academic writing in empirical cases where interference and contextuality may have to be considered.

When dealing with these cases, an academic writer might ask: Why move away from the classical or Kolmogorov probability formalism while approaching a quantum-based new conception? The conceptual change has been motivated by the recognition that certain phenomena, particularly in the realm of quantum mechanics and cognitive science, defy classical probabilistic models. Although a few key reasons help explain this departure, I focused on how to formalize a joint probability distribution over an entangled conceptual status in human cognition (Pothos & Busemeyer, 2013). According to classical Kolmogorov's definition of probability, the dynamic evolution in a conceptual space involves a transition matrix, maintaining its initial state's joint probability distribution over time. In other words, the total probability regularity still preserves, after transformation, a classic mathematical idealization still seen in today's academic writings.

In contrast, this may be the case due to the introduction of irreducible interference terms. According to the new Dirac/von Neumann's quantum probability framework, the compositionality of an entangled system precludes specifying a joint probability distribution from its componential probability distributions due to the introduction of irreducible interference. In this sense, the distance definition (i.e., the squaring of a generalized length unit) in the outcome spaces may have to be written as: $|x + y|^2 = x^2 + y^2 + x * y + y * x$, with irreducible interference terms $x * y + y * x$ included. Moreover, the specific order in a dynamic coupling context, which seems irrelevant given its classic sequential conjunction probability distribution, suddenly matters. Their products are not commutative anymore. The order of presenting two pieces of key information affects subsequent responses, showing a tendency to manifest aftereffects in weighing the last presented information more heavily. This point implies that the order of presenting task-relevant information matters for academic writers.

Geometrically, this can also be explained in a Hilbert space as a consequence of re-projecting a status vector between two basis vectors. A status vector is first projected to a basis vector; then, the projected length is further re-projected to another one. Unless it is located along the diagonal line and starting with orthogonal overlapping, the final output should be different given a different projection sequence. In other words, the new quantum-based conceptions have been established on an expanded mathematical foundation, with real and complex numbers involved in characterizing the notion of probability, which is rarely seen in today's education literature, implying birth of a new genre of academic writing.

In recent years, such quantum probability conception-based initiatives have been stressed to explain a range of mental phenomena that cannot be easily explained by the classical Kolmogorovian probability theories, such as explaining the context effect in interpreting a concept, the emerging conceptual change process given an existing intuitive conception, and the negative priming effects. Commonly, these phenomena have suggested that some aspects of human cognition cannot be explained by assuming the standard classical probability theories. For example, Gabora et al. (2008) advocated such a conception by positing a concept as a mental representation of staying in different states: grounded, super-positioned, or collapsed. When in a grounded state, what a researcher can say about it is the full potentiality of the concept, with some conceptual combinations and changes possible and others not. When a probing task is posted to a problem solver, the conditions of the problem or the constraints of the scenarios will actualize the potentiality toward a specific direction, thus becoming collapsed in the conceptual space. For those unsolved problems, the problem solver can be said to have a super-positioned conceptual state, bearing multiple choices at the same time. In particular, a creative conceptual change process is realized when an unlikely mental representation is turned into a promising candidate in a new conceptual framework conceived from another perspective. Such a quantum-inspired approach has proven suitable for explaining the real dynamics of creative conceptual change processes, which have also been characterized as co-evolving of the existing and a new conception. When writing for empirical cases where interference and contextuality must be considered, this approach offers unique advantages over other ones.

Constructing a New Discourse Flow for Understanding Mathematical Ideals: The Sentence and Beyond

Mathematical symbols mixed with verbal expressions, as illustrated in Figure 1, are particularly useful in exploring students' conceptual change learning, where interference and contextuality are always involved. In a recent mixed-methods study, I invited five interviewees to share their understanding of the simple pendulum's mathematical identity of the period: $T = 2\pi \sqrt{l/g}$. In contrast to the salient features of pendulum motion, such as the length of a string or the initial release angles, identifying the period of pendulum motion with a mathematical equation is not so obvious. Despite its abstractness, the equation $T = 2\pi \sqrt{l/g}$ plays a crucial role in the study of pendulum motion and serves as a fundamental mathematical identification for an observer to refine their understanding of oscillatory pendulum motion, though they may find it challenging to grasp this equation's significance due to its mathematical equation's construct.

When conducting the interviews, I first used a physical pendulum constructed from a white string with several silver-coloured gaskets (a metallic mechanical seal for preventing fluid or gas leakage) to demonstrate what a pendulum was and the basic terms that would be used in the interviews. Then, I focused on eliciting participants' understanding of pendulum motion. All the interviews took place in a quiet and private space and followed the same general structure to maintain consistency in data collection. The interviews were recorded and transcribed first in Mandarin, and then the transcriptions were translated into English for analysis. In the interviews, I asked the interviewees about their experiences with and understanding of the equation. One of the interview questions was about applying the interviewee's knowledge about pendulum motion to the case of adjusting a grandfather clock.

When showing the equation to interviewee #1, she responded by commenting on her previous responses (I showed the interviewee a physical pendulum and its motion at the beginning of the interview and invited her intuitive responses), "Right, it seems like what I was thinking is opposite, haha! And conversely, if its gravitational acceleration is smaller, then its period will be larger." Upon seeing the equation, she immediately checked whether her previous responses were correct. After a brief reflection, she found out the connection between these factors in determining the period:

> Interviewee #1: No, actually... out of all these variables, it only depends on the length.
> Interviewer: So, you got that just by looking at this formula?
> Interviewee #1: Really?
> Interviewer: If you were given this formula, what would you think [about…]?
> Interviewee #1: I think it's like this, because they are all constants and only L is changing.
> Interviewer: OK. That's right, you just need to see this formula, and you think the conclusion is only related to L.
> Interviewee #1: Yeah, it's interesting that it's only related to length! Haha!

At the end of the first interview, interviewee #1 added:

> I think physics is pretty amazing. Only for this one, and for the others with different weights and angles, I thought the period would become shorter, but that's not the case. It turns out that only this one [the length of a suspending string] affects it. I think physics is pretty amazing.

These final comments displayed the first interviewee's amazement at the potential instructional value of revealing a mathematical ideal through the introduction of mathematical symbols in equations. Given what she said in the interview, I tentatively conclude that, as a non-science major student, her experience with the scientific reasoning of using the mathematical equation has been enriching and rewarding, even in the straightforward case of discussing pendulum motion.

Similarly, I showed a physical pendulum and its motion to a second interviewee at the beginning of the interview and invited her intuitive responses. Also, at the end of the interview, I showed him the same mathematical equation for the period of pendulum motion and asked for his comments about seeing the mathematical equation. Without the interviewer finishing the question, the interviewee interrupted and declared, "That is, actually, only related to length. Everything else is fixed in quantities." After such a realization, the interviewee quickly solved the problem of adjusting the position of the pendulum bob to shorten the period of the pendulum motion of a slowing-down grandfather clock.

As a science-stream student in high school, interviewee #2 developed a knowledge structure aligned with the textbook-presented physics knowledge: an energy-based perspective and a force-acceleration-based one. The evidence I observed showed he could switch between force- or energy-based knowledge systems. However, his responses were fragmented when being prompted with verbally defined interview questions. For example, he first mentioned one of the mathematical symbols on the right side of the equation, and then he switched to another one without realizing how they jointly contributed to understanding the pendulum motion within its boundary conditions. The fragmented responses may indicate the actual limit of his working memory and his unique combination of mathematical symbols and verbal expressions. Only after seeing the mathematical equation did he jump to the correct conclusion without too much reasoning using either the energy-based or force-acceleration-based conceptual framework and the languages. Compared with the responses of interviewee #1, it can be said that the structure of his physics knowledge was more sophisticated, especially in using mathematical reasoning and vector-based force language to explain the pendulum motion. However, his knowledge did not guarantee the correct verbal responses (e.g., when seeing the swinging pendulum in reality) until he was presented with the equation.

When asked a verbal question about the equation, interviewee #3 was the only one who emphasized the mathematical aspect of the equation. Moreover, he described the mathematical relationship in detail: "Hmm, gravitational acceleration and pi are constant values. This period only has to do with that L. Hmm… and it's a uh… exponential function relationship, right? A half-power exponential function relationship."

Given his responses, I can tell that interviewee #3 was the only one who relied on mathematical thinking, which contrasts with the way of reasoning employed by the first two interviewees. The sampling and decision-making differences can be attributed to his training in civil engineering and background knowledge in science. He was the only interviewee who continued his mathematical and scientific training after graduating high school. Such a background has enabled his reliance on the mathematical equation and the idealization process to explain pendulum motion.

At the beginning of the interview, interviewee #4 declared that she was not good at physics. Upon seeing the mathematical equation, she noted, "Oh, physics is my weakest subject. Yes, um, I probably haven't seen it before." After being encouraged to express her intuitive understanding and her first impression of the equation, she struggled to explain the equation.

> Interviewee #4: Um, my first impression would be 2π. Pi is the circumference of a circle. Um, I only remember that r squared is... I probably only remember things about math. Um, I'm not sure what 2π means exactly, like two circumferences... but the square root part feels a bit complicated. But L represents length.
> Interviewer: Yes, it represents the length of the line.
> Interviewee #4: Length divided by weight.
> Interviewer: No, that's gravitational acceleration.
> Interviewee #4: Gravitational acceleration! Length divided by gravitational acceleration; I really don't understand that.

Her responses sharply contrast the answers of interviewee #3 to the same interview question. Whereas the latter relied on his understanding of mathematical ideals for interpreting the symbolic forms of this equation, interviewee #4 had to figure out the exact meaning of each symbol, let alone the mathematical relationship among these symbols, thus missing the opportunity of getting the point of mathematical ideals. However, when asked about the implementation question about adjusting the position of the pendulum of a slowing-down pendulum, her answer was the same as the answer of the engineering student. Given the evidence, sometimes a correct response to a probing question about a phenomenon may not always equal a proper understanding of the phenomenon without involving mathematical ideals because such a response may remain at the surface level.

At the end of the last interview, I asked interviewee #5 the same question about the mathematical equation. She commented on having forgotten the mathematical content or symbols of the equation. I also asked her whether she wanted to know more about the pendulum motion. She asked, “Did we learn about pendulum motion in middle school physics?” As I replied, “Yes,” she responded, “I don't remember anything about pendulum motion.” Despite not recollecting learning pendulum motion in high school, she answered the interview questions correctly (e.g., How do you think the angle of release affects the oscillation of the pendulum? Do you think that the length of the pendulum affects the period of the single pendulum? If yes, how?). It seemed her participation in the experiment and interview improved her understanding of pendulum motion, though she might need to understand the underlying knowledge structure centred on the mathematically idealized simple harmonic motion.

In summary, interviewees #1, #2, and #3 saw the mathematical equation as the key to understanding pendulum motion because they reversed their initial answers or expressed amazement. Interviewees #1 and #2 changed their views about the effective factors in determining the swing period of the pendulum motion, such as from highlighting the weight of the bob as a factor to the length of the string. Interviewee #3 even commented on the mathematical aspect of this equation. In contrast, it seems that interviewees #4 and #5 have lost meaningful contact with the equation after their secondary education, retreating to pure verbal expressions without any involvement of mathematical symbols and equations unless being guided. Without using and embedding them in their daily uses of the verbal language, the presented equation was insufficient for constructing their understanding around mathematical ideals. What I have observed in this series of interviews seems to support what Bruce Sherin (2001) stressed in discussing reforming introductory physics instruction:

> I challenge the assumption that in physics or any domain the conceptual and the symbolic elements (the mathematical symbols and their identities or definitions) of a practice can be separated for the purposes of instruction. Removing equations from the mix changes

> the nature of understanding. This does not imply that (introductory) physics cannot be taught without equations. However, it does imply that equation-free courses will result in an understanding of (modern) physics that is fundamentally different from physics as understood by physicists. (p. 524)

The interviewees who fused the mathematical understanding with verbal expression seem to have an advantage in explaining their experiences with the pendulum motion equation, and the equation-enhanced understanding stood the test of time and language change in terms of refocusing on the relevant factor while ignoring other irrelevant ones and reversing some responses after seeing the equation.

The Need for a New Taxonomy of Academic Writings With Mathematical Symbols

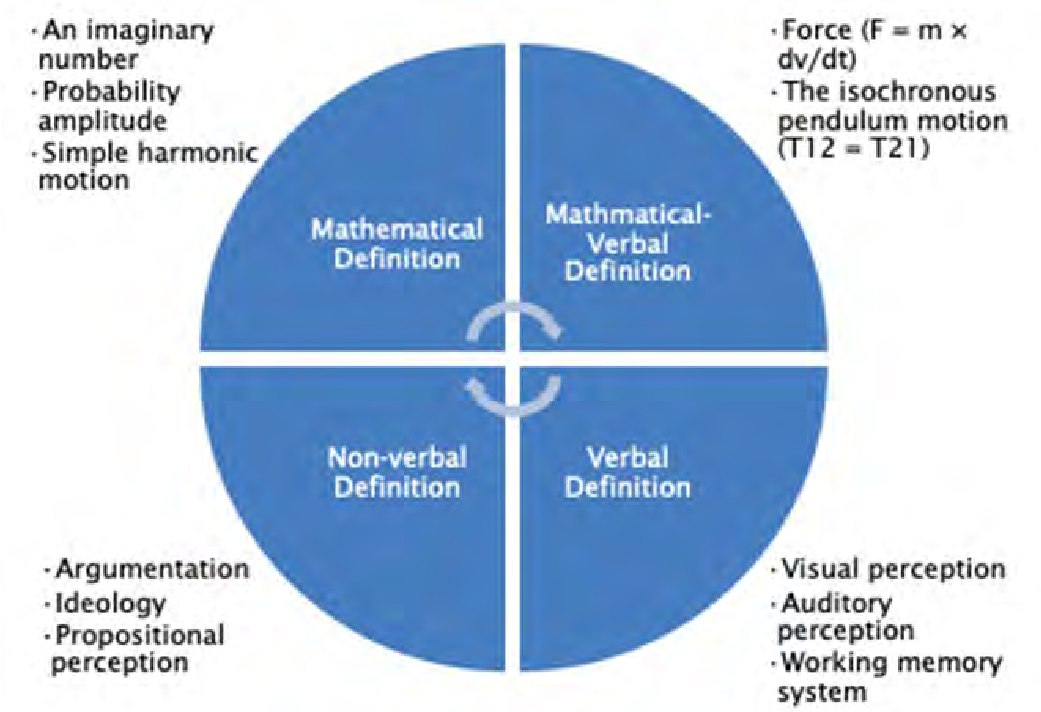
The development of any scientific field and academic writing mainly depends on an initial and rudimentary conceptualization that maximally characterizes its unique subject content and on an implicit or explicit taxonomy that categorizes the contents. For example, to most of us, a taxonomy in biology refers to a hierarchical and embedded categorical system for identifying and classifying organisms given their physical and genetic characteristics. With such a categorization, a biologist can categorize the diversity of living organisms into an ordered and accepted theoretical system of various sub-categories in a conceptual space (i.e., species, genus, family, order, class, phylum, kingdom, domain). These sub-categories embody a set of premises and organizing principles so that the stability and consistency of these categories and inter- and intra-relationships among living organisms can be represented, with the points representing animals or plants and the areas representing their connections. One of the primary goals of modern ecological science education is to help students understand the taxonomy so that they observe an environment with their mind's eyes and are ready to solve new problems even if they see abnormal data or surprising experimental results. The benefits of using taxonomy in research are also familiar to non-biologists.

One widely known example among education researchers is Bloom's (1956) *Taxonomy of Educational Objectives*. After half a century, Bloom's students Lori Anderson and David Krathwohl, who were joined by a group of educational psychologists and educators, published a revised version of Bloom's taxonomy in 2001 by adding revised categories such as remembering, understanding, applying, analyzing, evaluating, and creating, increasing interconnectedness, and the inclusion of technology. This taxonomy is important because it better aligns with contemporary educational practices. In contrast to biologists' and educators' taxonomies, another famous classification system is the *Diagnostic and Statistical Manual of Mental Disorders: DSM-5-TR* (American Psychiatric Association, 2022), highlighting its probabilistic nature in categorizing mental health-related conditions. This taxonomy is important because it covers learning and emotional problems, which instructors face daily.

Given the observed absence of mathematical symbols in academic writing for education researchers, a new taxonomy of academic writing genres (see Figure 2) is needed to differentiate their roles in categorizing academic writers' conceptions and activities.

Figure 2

A New Taxonomy of Academic Writing for Education Researchers



With this taxonomy, it becomes easier to interpret academic writings at different levels: the sensory, verbal, ideological, and mathematical descriptions with greater precision. The new taxonomy of academic writing based on mathematical symbols reconnects academic writing with its historical roots. Michael J. Crowe (1985), in his *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, depicted a hidden connection between mathematics and academic research. After introducing the history of searching for the concept of numbers, he added:

> The second tradition, that within the history of physical science, also extends back to ancient times and consists in the search for mathematical entities and operations that represent aspects of physical reality. This tradition played a part in the creation of Greek geometry, and the natural philosophers of the seventeenth century inherited from the Greeks the geometrical approach to physical problems. However in the course of the seventeenth century the physical entities to be represented passed through a transformation. This transformation consisted in the shift in emphasis from such scalar

> quantities as position and weight to such vectorial quantities as velocity, force, momentum, and acceleration. The transition was neither abrupt nor was it confined to the seventeenth century. Later developments in electricity, magnetism, and optics acted further to transform the space of mathematical physics into a space filled with vectors. (p. 1)

Such an historical connection between mathematical ideals and academic writing is rarely mentioned in academic educational studies. However, the physical and cognitive sciences were developed by introducing mathematical ideas such as the Planck constant, quantum, and quantum cognition. Avoiding these contents in education literature to attract liberal arts students to the classroom is not always the best pedagogical strategy. Instead, using the new taxonomy that features the mathematical symbol connection within academic writing and academic instruction is the first step in the right direction because raising awareness of the existence of the genre using mathematical ideals and how to use them would improve students' superficial understanding of a phenomenon and reconnect their learning in a meaningful context.

Next, the new taxonomy helps bridge the gap between educational studies, laboratory teaching, and learning sciences. As demonstrated in the interview data, the interviewees expressed their entangled and constantly changing conceptions about the mathematical identity of the period of simple pendulum motion. Their conceptions were suitable to be described with the notions of probability and the distribution of random errors. Emphasizing the mathematically defined concepts also helps establish the link with these self-reflective routines. With the aid of these interview data, it would be easier to clarify the students' pre-conceptions about scientific concepts presented in textbooks, understand solutions to problems, and demonstrate the underlying idealization principle. In this sense, the new taxonomy provides a probabilistic framework for organizing and categorizing the students' knowledge more psychologically naturally. It recognizes that their pre-conceptions may be in dynamic sampling and decision-making over overlapping knowledge distributions and that the learning is an interference-filled and conceptual change process.

In brief, the new taxonomy with a probabilistic frame of reference significantly extends Zhou's (2012) hybrid learning space by establishing meaningful connections between mathematical ideals and academic writing practices and offering opportunities to understand a whole range of academic writing genres that feature mathematical symbols embedded in equations.

The Pedagogical Implication of the Taxonomy

Learning is a probabilistic process, with students' cognition being an instance of such a process. The pedagogical implications of such a probabilistic cognitive "revolution" (i.e., the proposed taxonomy) are manifold. The probabilistic re-orientation of incorporating mathematical ideals into quantum cognition-informed academic writing can enhance domestic or international academic writers' understanding of the conceptual, symbolic, and mathematical elements in academic written narratives. By using mathematically defined conceptual tools, such as the quantum cognition framework, they can gain a deeper understanding of writing genres, such as those featuring mathematical symbols and their representation of abstract ideals that may have previously appeared challenging to comprehend. The new taxonomy allows novice writers to view verbal concepts and conceptions through a mathematics-compatible lens, which can significantly help clarify the underlying theoretical principles and the critical organizing notions. Additionally, the new taxonomy helps guide new curriculum design endeavours to bridge the gap between abstract mathematical concepts and their verbal interpretations. In the tradition of academic writing studies, literacy research has relied heavily on qualitative approaches to understanding academic written narratives, often focusing on the most important relevant factors while ignoring all other aspects of a phenomenon and overlooking the importance of idealized quantitative reasoning and its implications. By incorporating the classical and quantum probability theories into such research and academic writing through incorporating mathematical symbols and equations, researchers and students alike are more likely to appreciate a deeper understanding of the underlying mathematical structures that govern academic writing phenomena. Finally, the taxonomy deliberately promotes a positive attitude toward the interdisciplinary and comparative understanding of academic written materials since idealized mathematical modelling, a part of the mathematical idealization process, is not limited to physics or cognition research, and many other fields such as mathematical psychology, artificial intelligence, and educational assessment depend heavily on mathematical reasoning. Incorporating the new taxonomy, a cyclic process starting at either level, into researchers' teaching and writing practices helps their students develop skills necessary to apply mathematical reasoning across a wide range of disciplines where interference and contextuality must be considered.

Summary and Conclusions

Although quantum mechanisms in the physical sciences have been around for over 100 years, the real-life implications of quantum conceptions, such as in quantum communication or computation for the nation, were recently formulated. To accommodate such a stance, academic writers and researchers can promote the underlying principles of quantum mechanisms, including interference, contextuality, mathematical idealization, and their generalization in other areas. In terms of classifying academic writing in the social sciences through a quantum framework, I have proposed a taxonomy that distinguishes academic written narratives at different levels: the sensory, verbal, conceptual framework, and mathematical descriptions with greater precision. More importantly, the taxonomy highlights the probabilistic conception through its mixed elements in a half-and-half manner, with interference and contextuality presented in the process and seeing the mathematical equation as an approximation of a mathematical entity. In future academic writing practice, achieving self-acceptance of mathematical symbols would free education researchers' creativity, pointing to a promising way to express themselves flexibly, especially when considering the national strategy of promoting a quantum-technology-based future for all of us, starting with young learners in K-12 education.

The study of involving mathematical symbols in academic writing holds tremendous potential for writing about theoretical entities and empirical cases where interference and contextuality must be considered, with significant practical applications across multiple disciplines. By raising awareness of quantum-cognition-inspired written narratives, the research informs educational efforts to foster writing creativity in students and improve curriculum design through creative conceptual change writing processes. More importantly, such a reintroduction to mathematical idealization and academic writing may help advance general AI and the related technology to process mixed symbolic systems and to bridge pure verbal narratives and mathematical descriptions. Most importantly, each mathematical symbol can also serve as a rhetorical device that constantly inspires creative academic writing, let alone mathematical entities and their idealization. In conclusion, fusing mathematical symbols in academic writing represents a unique opportunity to advance knowledge at the intersection of quantitative cognitive science and the psychology of creative writing. The proposed taxonomy is not only scientifically intriguing but also contributes to enhancing the nation and the province as a quantum research hub.

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