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# THE INVESTIGATION OF BEST PRACTICES ON SYMBOLIC MATHEMATICAL COMMUNICATION: A COMPARATIVE STUDY IN JAPAN, LAO PDR, AND THAILAND

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## ABSTRACT

**Purpose –** This comparative study sought to identify best practices concerning symbolic mathematical communication between primary school teachers and students in Japanese, Laotian, and Thai classrooms.

**Method –** The target groups were 18 teachers and 671 students in Grade 1 to Grade 6 mathematics classrooms in Japan, Lao PDR, and Thailand. A total of 18 classrooms were inspected, one from each grade; thus, the unit of analysis was a classroom. Research instruments included a video recorder, a camera, and field notes. The data was gathered by videotaping, photographing, and taking notes. A descriptive analytics method was used to examine the data, following Pirie's mathematical communication framework (Pirie, 1998).

**Findings –** Based on cultural norms and educational approaches in each country, the country-specific practices of symbolic communication were found to differ significantly among the mathematics teachers from the three countries. Owing to the education system in Japan placing a strong emphasis on discipline and respect, their teachers were found to focus on students' symbolic explanations, particularly allowing students to elaborate on the meaning of complex mathematical ideas and concepts using symbolic communication. Laotian teachers tended to explain the answers using symbolic mathematical communication. This is because the country-specific practices in the Laotian local context concerning mathematical concepts are contextualized to relate to students' daily lives and experiences, making abstract symbols more meaningful. Finally, Thai teachers were found to focus on the students' answers rather than the learning process or operations. This suggests that Thai teachers often focus on memorising and repeating of mathematical procedures and formulas.

**Significance –** The study findings offer a substantial understanding of the role of culture in education by investigating symbolic communication in mathematics classrooms in Japan, Laos, and Thailand. This implies the potential to improve teaching practices, enhance student learning experiences, and promote cultural sensitivity and inclusion in educational settings.

**Keywords:** Best practices, comparative study, interactive and collaborative learning, primary school students, symbolic mathematical communication, symbolic notation, mathematics teachers.

## INTRODUCTION

The mathematical communication standard emphasizes the importance of primary school students communicating their mathematical thinking coherently to peers and teachers (National Council of Teachers of Mathematics, 2000). As a result, primary school students should be given support, encouragement, and opportunities to engage in oral communication and to learn to communicate mathematically (Whitin & Whitin, 2003). Consequently, teachers should aim to develop a link between mathematics and language (Stigler & Hiebert, 2004). Following this line of reasoning, teachers should utilize such a link to help their students communicate their ideas by elaborating upon what they already know (Schmidt, 2004). Such mathematical communication can assist primary school students in clarifying their thinking and sharpening their understanding as they try to make sense of communication (Cooke & Buchholz, 2005).

According to Kharde (2016), mathematics has its symbols and rules of grammar, and mathematics is written in a symbolic language designed to express mathematical thoughts. Therefore, we can usually read the mathematical expression used in symbolic communication in any language. Wilson (2019) emphasised the importance of good symbolic mathematical communication in providing opportunities for students to display their ideas and strategies to solve mathematical problems accurately. This was further supported by Maulyda et al. (2020). Maulyda et al. (2020) explained that when students can convey their ideas about solving mathematical problems appropriately, it can assist the teacher in recognising any conceptual or procedural mistakes experienced by students. Although students may communicate mathematical ideas incorrectly, the teacher can understand the idea behind the student's answers and detect any mistakes on the part of students.

On the other hand, if students have good ideas but cannot communicate them appropriately, the teacher will have difficulty determining the correctness of mathematical ideas conveyed by the students (Cristobal & Lasaten, 2018). This implies that students who possess symbolic mathematical communication skills can help to communicate their ideas clearly, a feature that will implicitly impact the teacher's evaluation of the student's answers. Likewise, correct mathematical ideas may be judged wrong if they are not properly communicated because of the incorrect interpretation of the teacher who evaluates the students' answers (Ramadhani & Dwi, 2018).

Pirie (1998) identified six modes of mathematical communication:

i. Ordinary language refers to the use of language or phrases that are commonly used in everyday life.
ii. Mathematical verbal language means the use of terms connected to mathematics (whether speaking or writing) and which are related to mathematics.
iii. Mathematical symbols that are used in symbolic language.
    Visual representation, which allows individuals to envision clear images.

iv. Unspoken but shared assumptions, based on which students can demonstrate mutual understanding without using words and
v. Quasi-mathematical language involving individuals employing language that is understandable in that context but not to outsiders.

Symbolic mathematical communication is one of the six modes of mathematical communication. Specifically, Pirie (1998) defined such communication as using symbols, equations, and mathematical notations to represent mathematical concepts and ideas. Symbolic mathematical communication is utilized in several mathematics fields, such as algebra, calculus, geometry, and statistics. In this study, the researchers defined *symbolic mathematical communication* as using mathematical symbols and notation to represent mathematical concepts, equations, and relationships. Therefore, this form of communication is seen as a powerful tool that allows students to communicate complex ideas succinctly and precisely. The theories described in the following section highlight the importance of symbolic mathematical communication in facilitating the exchange of mathematical knowledge, promoting abstraction and generalization, ensuring precision and clarity, enabling compact representation, and fostering creativity and innovation in mathematics.

Symbolic mathematical communication, the so-called Universal Language of Mathematics Theory, can be seen as a universal mathematical language that transcends cultural and linguistic barriers (Hoffer, 2009). While spoken and written languages can vary across different regions and countries, mathematical symbols and notation remain consistent and universally understood by mathematicians worldwide. This universality enables efficient and effective communication of mathematical ideas across borders, facilitating collaboration and advancing mathematical knowledge (Hoffert, 2009).

Symbolic mathematical communication facilitates abstraction and generalization in mathematics (Mitchelmore & White, 2007). Mathematicians can represent mathematical concepts and operations concisely and abstractly by using symbols. Such abstraction allows them to focus on the underlying structures and relationships rather than on specific numerical values, leading to generalisations that apply to various situations. Using symbols enables mathematicians to communicate these generalizations effectively, fostering the development of mathematical theories and frameworks (Mitchelmore & White, 2007).

Symbolic notation provides a precise and unambiguous way of expressing mathematical ideas. The use of symbols eliminates the ambiguity that can arise from the use of natural language, where words can have multiple interpretations (Vance, 2021). This precision and clarity aids in conveying mathematical concepts accurately and enables rigorous reasoning and logical deductions. This is because symbols have defined meanings and rules of manipulation, ensuring that mathematical expressions are interpreted consistently (Vance, 2021).

Symbolic mathematical communication allows the compact representation of complex mathematical concepts and equations. Mathematicians can express intricate relationships and operations concisely by considering mathematical ideas and using concise symbols and notation (Fu et al., 2022). This compactness simplifies the presentation of mathematical arguments, making it easier to comprehend and work with complex mathematical structures. Such compact representation also enhances the ability to manipulate and transform mathematical expressions efficiently.

Symbolic mathematical communication provides a platform for creative expression and exploration in mathematics. The rich and versatile mathematical symbols and notation allow mathematicians to express a various ideas and concepts. This expressive power and creativity enable the development of new mathematical frameworks, notations, and techniques. Following this line of reasoning, symbolic communication encourages innovative thinking, enabling mathematicians to communicate and build upon each other's work, leading to discoveries and advancements in the field (Grégoire, 2016).

The discussion above shows that symbolic mathematical communication involves using symbols, formulas, and equations to represent mathematical concepts and operations. This has caused a challenging aspect of learning mathematics, particularly for primary school students who may encounter problems using mathematical symbols. Previous researchers (e.g. Cristobal & Lasaten, 2018; Kharde, 2016; Maulyda et al., 2020; Ramadhani & Dwi, 2018; Wilson, 2019) have identified some learning problems associated with symbolic mathematical communication, such as misinterpretation of symbols, lack of understanding of notation, inability to translate word problems into symbols, difficulty in expressing ideas clearly, overreliance on memorization, and lack of feedback and guidance.

Even though symbolic mathematical communication plays a fundamental role in enabling students to express complex concepts succinctly and precisely, certain challenges are associated with symbolic mathematical communication, such as those mentioned above. For example, students may misinterpret the meaning of mathematical symbols or confuse similar-looking symbols, leading to errors in their calculations or solutions - the so-called problem of misinterpretation of symbols (Wilson, 2019). Another problem is that students may need help understanding the meaning and purpose of notation, such as superscripts, subscripts, and parentheses in mathematical expressions (Cristobal & Lasaten, 2018). Moreover, students may need help translating word problems into symbolic expressions, which requires a deep understanding of the underlying mathematical concepts and operations. This problem is exacerbated by students' inability to translate word problems into symbols (Maulyda et al., 2020). In addition, students may need help expressing their mathematical ideas and solutions using symbols, particularly if unfamiliar with the appropriate notation and conventions, as highlighted by Kharde (2016). On top of that, students may rely too heavily on memorising formulas and equations without fully understanding their meaning and purpose, which can lead to difficulty in applying them in different contexts (Inprasitha, 2022). To address these problems of symbolic mathematical communication, teachers can provide students with clear explanations of notation and conventions, offer opportunities for practice and feedback, and encourage students to develop their strategies for communicating mathematically using symbols (Cristobal & Lasaten, 2018; Kharde, 2016; Maulyda et al., 2020; Ramadhani & Dwi, 2018; Wilson, 2019).

According to Steinbring et al. (1998), symbolic communication practices vary within each country based on regional, ethnic, and individual differences. Symbolic communication in a Japanese primary mathematics classroom includes nonverbal cues, group harmony, respect for authority, and visual aids (Isoda & Nakamura, 2010). Japanese teachers often use nonverbal cues, such as nodding or eye contact, to show attentiveness and understanding. Moreover, Japanese students may be encouraged to work collaboratively to solve problems to promote group harmony and cooperation. In addition, students are expected to show respect for the teacher's knowledge and authority, which may lead to a more formal classroom setting than occurs in other countries. Japanese classrooms often use visual aids, diagrams, and concrete examples to facilitate understanding and enhance symbolic representation.

As reported by the Ministry of Education and Sport (Lao PDR, 2015), the education system in Laos has different dynamics than Japan's. These are as follows. Symbolic communication in a Lao mathematics classroom could involve interaction, the local context, respectful questioning, and hands-on learning. Lao classrooms prioritize interactive learning methods to engage students actively. Mathematics concepts are contextualized to relate to students' daily lives and experiences, making abstract symbols more meaningful. Students are encouraged to ask questions respectfully, as direct questioning of the teacher is considered impolite. Teachers are encouraged to incorporate hands-on activities and real-life problem-solving to enhance students' understanding of mathematical concepts.

Inprasitha (2022) stated that Thai education emphasises the hierarchical relationship between students and teachers. Symbolic communication in a Thai mathematics classroom can be generalised in terms of features, namely "*wai*" and respectful gestures, memorization and repetition, a teacher-centred approach, and respectful tones. Teachers and students in Thailand perform the traditional Thai "*wai*", or the use of respectful gestures as a sign of reverence to the teacher and each other. Thai classrooms often focus on memorisation and repeating mathematical procedures and formulas. The teachers play a central role in the classroom, and students are expected to listen attentively and avoid challenging the teacher's authority openly. Students must use a polite and respectful tone when interacting with the teacher and peers during class discussions.

## Research Objectives

Based on the learning problems associated with symbolic mathematical communication, a comparative study of its practices across different countries, schools, and 18 classes in Japan, Laos, and Thailand was undertaken. The researchers aimed to gain valuable insights into successful approaches. It involves comparing 18 mathematics classrooms (Grades 1 to 6) from the three countries and analyzing their characteristics to gain a deeper understanding of their symbolic mathematical communication. Consequently, this study could inform the adaptation and implementation of best practices in various contexts, leading to improved mathematical communication and enhanced teaching and learning experiences. Specifically, the objective of this research was to identify best practices for symbolic mathematical communication on the part of primary school teachers and students from Japan, Laos, and Thailand.

The focus of this comparative study was on the best practices of symbolic mathematical communication. However, while this can be valuable, it is important to acknowledge the study's limitations. There are four reasons that the researchers have to highlight limitations to this study, namely cultural context, educational systems and policies, language barriers, and context-specific challenges. While the study was limited for these reasons, focusing specifically on identifying the best practices of symbolic mathematical communication could still yield valuable insights and suggest improvements in a particular context. Therefore, the researchers could develop a more nuanced understanding of the transferability and adaptability of practices and make informed decisions when implementing them. It was vital to consider the unique characteristics and needs of the target context and ensure that the identified country-specific practices align with the specific goals and constraints of the study. Hence, this comparative study enabled the researchers to identify country-specific practices from different countries and apply them in various contexts. By studying how other human societies or educational organizations teach symbolic mathematical communication, the researchers could learn from their successes and failures and improve practices in each context.

## METHODOLOGY

### Research Design

The research design involved using a comparative study to compare three groups of mathematics classes from Grade 1 to Grade 6 in three countries, namely Japan, Lao PDR, and Thailand. The aim was to determine the similarities and differences between them (Pongboriboon, 2010; Yin, 2014). In symbolic mathematical communication, the comparative study involved comparing different methods of teaching symbolic language or examining the impact of symbolic notation on student performance in different mathematics classes. The researchers employed a qualitative research design using classroom observation, which involved systematically observing and analysing classroom interactions, behaviours, and dynamics to gain a deep understanding of symbolic mathematical communication processes and phenomena (Schoenfeld, 1985). Descriptive analytics was conducted to illustrate the details of mathematical communication based on Pirie's (1998) framework for analysing symbolic mathematical communication. This approach focuses on exploring the subjective experiences, perspectives, and social interactions in terms of symbolic mathematical communication within the three classroom settings in Japan, Lao PDR, and Thailand.

## Target Group

The target groups were 18 teachers and 671 students in Grade 1 to Grade 6 at three public primary schools. The first school was located near the Naruto University of Education in Japan, involving six teachers and 240 students. The second school was a demonstration school affiliated with Khon Kaen University's Faculty of Education, Thailand, involving six teachers and 217 students. The third school was a demonstration school under the administration of Savannakhet Teacher Training College in the Lao PDR, with six teachers and 214 students participating in the study. Both demonstration schools serve multiple purposes, including providing practical training and research opportunities for future teachers, conducting educational research, and implementing innovative practices. Due to the Coronavirus pandemic of 2019 (COVID-19), a range of constraints were introduced by the researchers. Specifically, the researchers could only access some primary schools in the countries concerned.

Consequently, it was impossible to use probability sampling methods. As a result, convenience sampling was the only feasible option for the researchers when selecting participants. Such selection was based on easy accessibility and convenience, as the schools being researched were either near or on the campus of the researchers' affiliated institutions. To mitigate potential biases and limitations associated with employing convenience sampling, the researchers had a clear definition of the target population, in that the selected schools must adhere to the learning culture about symbolic mathematical communication of the nation concerned to ensure the sample was at least somewhat representative of the population of interest. Following this reasoning, each school being researched had to have one class per grade. This means that there were six classes per country, multiplied by three countries, making a total number of 18 classes.

**Study Tools and Data Collection**

The study team could not travel abroad to collect data due to the COVID-19 outbreak. The following information outlines the differences in how the research tools were used to collect data in each country. The researchers used video recordings of instructional sessions that were between 30 and 60 minutes in length as a general guideline to collect observation data to achieve the research objectives. Data were gathered through videos of classroom activities, photographs of teaching activities, and field notes on the modes of mathematical communication. The data collection methods were as follows:

The data collection tools were used at the public schools around the Naruto University of Education in Japan; in Grade 1 to Grade 6, primary school mathematics lessons consisted of class videos, with one lesson for each class. They were filmed during open class activities in the Academic Years of 2017-2019 in the Naruto University Demonstration School classes. In addition, the fourth author was permitted by Naruto University of Education to record class videos for educational and research purposes. These classes had been videotaped for ordinary educational and research objectives and had been taught under normal circumstances.

In the Lao PDR, data was collected in the Demonstration School of the Savannakhet Teacher Training College by the third author. A video recorder, image recorder, and field notes were utilized to collect data during the first semester of the Academic Year of 2021-2022 between May and June, in which teaching was being carried out under normal circumstances. For teachers and students in the target group's classes, video recordings, photographs, and field notes were taken of the mathematical communication behaviours between teachers and students in the mathematics classes of Grades 1 to 6. There was one lesson per class, totalling six lessons.

In Thailand, a video recorder, a camera, and field notes were used for data collection at the Khon Kaen University Demonstration School. Recorded videos, photographs, and field notes on the mathematical communication behaviours of teachers and students were made in mathematics classes from Grades 1 to 6. There was one lesson per class with six lessons, which were held in Semester 2 of the Academic Year 2021-2022. Moreover, due to the COVID-19 epidemic, these were hybrid classes that used both onsite learning and online learning, which were conducted under the supervision of the first and second authors at the Faculty of Education of Khon Kaen University in Thailand.

## Data Analysis Framework and Analysis Methods

The researchers utilized Pirie's (1998) modes of mathematical communication analysis framework, which consists of ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions, and quasi-mathematical language. However, the researchers chose to focus on only one mode of mathematical communication from this framework, namely symbolic communication, because symbolic mathematical communication is the only mode of mathematical communication that involves a balanced approach that incorporates multiple modes of communication that might be particularly effective in learning mathematics.

The researchers used the classroom as the unit of analysis. This allowed the researchers to examine the contextual factors influencing symbolic mathematical communication, including classroom dynamics, teaching approaches, and student engagement. Moreover, the researchers also analyzed such practices within the classroom context to provide insights into how these practices were implemented, their effectiveness in facilitating mathematical communication, and their impact on student learning outcomes. In this way, the researchers could gain a detailed understanding of how teachers and students engage in symbolic mathematical communication, the instructional strategies employed, and the interaction patterns that support effective communication.

Triangulation was conducted as a validation process using multiple sources in the form of instructional videos, photographs of teaching activities, and field notes relating to the symbolic communication mode. The different data collection methods were used to verify and cross-validate the findings. For example, the researchers compared the information obtained from the classroom videos with that from field notes to ensure consistency. The participants' names, photographs, videos, or other personally identifiable information were anonymized. The data was analyzed by utilizing a descriptive analytics method based on the analyses of the class videos, the protocol about teaching and learning, and pictures of the teaching and learning activities according to Pirie's analytic framework of modes of mathematical communication (Pirie, 1998), which classified symbolic mathematical communication as a communication style that communicates mathematical concepts using mathematical symbols. Symbolic languages are primarily expressed through the use of symbols. It is a crucial method of mathematical communication and inevitably involves learning mathematics.

## FINDINGS AND DISCUSSION

### Overview of Symbolic Mathematical Communication in Japan, Laos, and Thailand

After investigating the mode of symbolic mathematical communication of primary school teachers and students in Japanese, Lao PDR, and Thai mathematics classrooms, the mode of symbolic mathematical communication of both teachers and students was discovered. As illustrated in Table 1, Japanese and Thai mathematics classrooms demonstrated the mode of symbolic mathematical communication used throughout every grade of primary school. On the other hand, in mathematics classes in the Lao PDR, the mode of symbolic mathematical communication was mainly found in four grades, excluding Grade 2 and Grade 4.

**Table 1**

*A Summative Finding of Symbolic Mathematical Communication in Japan, Laos, and Thailand*

| Mode of Symbolic Mathematical Communication | The Presence of Mode of Symbolic Mathematical Communication | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Japanese Classes | | | | | | Lao PDR Classes | | | | | | Thai Classes | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 |
| Symbolic Language | / | / | / | / | / | / | / | - | / | - | / | / | / | / | / | / | / | / |

### Findings of Country-specific Practices of Using Symbolic Communication in Mathematics Classrooms

The findings about the 18 scenarios of symbolic mathematical communication observed in the three countries revealed several country-specific practices, namely integration of visual representations, explicitly teaching symbolic notation, real-world contexts, interactive and collaborative learning, individual and cultural diversity, use of technology, formative assessment, and metacognition and reflection.

## Integration of Visual Representations

The comparative study revealed that Japanese and Thai mathematics teachers effectively use visual representations such as diagrams, graphs, and symbols to enhance their students' understanding of mathematical concepts. This implies that visual aids can make abstract concepts more accessible and provide multiple ways for students to approach problem-solving. The following pieces of evidence illustrate the effective use of visual representations as one of the best practices for using symbolic communication.

### *Japanese Grade 3 Class Scenario*

Problem situation: Takumi is going to hold a party. He will buy six cans of orange juice at a unit price of 70 Yen and six oranges at 30 Yen. So, the question is, what is the total price?
To calculate the total price, we need to multiply the unit price of each item by the quantity purchased and then sum the individual prices.

The teacher asked the students to find out the total cost.

*Teacher:* *How did you get this solution? Please explain.*
*Student:* *We have six cans of juice with a unitary price of 70 yen.*
*Teacher:* *So, does this figure stand for this mathematics sentence? This is the juice, so I may write here 70 yen per can. 70,70,70,70,70,70 (The teacher writes the number 70 under the pictures of the cans of juice).*
*Student:* *The total of six, so the total cost of buying all six cans of juice is 420.*
*Teacher:* *We have six oranges as a unitary price of 30, 30, 30, 30, 30, 30 (The teacher writes the number 30 under the pictures of oranges).*
*Student:* *Therefore, to get the total price of oranges it is 30 ×6, so this mathematics sentence corresponds to the figure. This picture corresponds to this mathematics sentence. Then I add two of them 420 + 180 = 600*

**Figure 1**

*The Finding of Grade 3 Japanese Student's Symbolic Mathematical Communication*

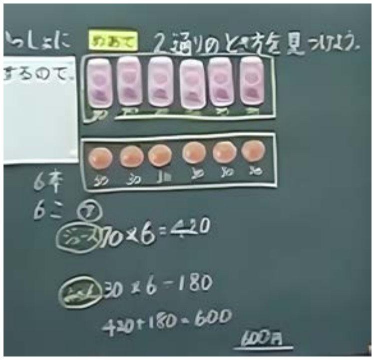



Figure 1 displays that the students explain their solution, and the teacher writes the numbers and symbolic sentences to show the total price. The teacher and students integrate the visual representation by using symbolic language to communicate the answer. This is a mode of symbolic mathematical communication.

***Thai Grade 6 Class Scenario***

Problem situation 1. Write the explanation of how to find the area of the square in the following picture (refer to Figure 2a) and Problem Situation 2. Write the explanation of how to find the area of the kite in the following picture (refer to Figure 2b).

*Figure 2a*. Problem Situation 1

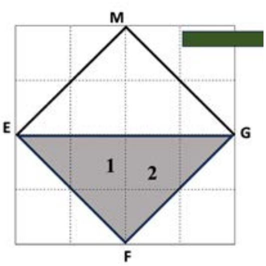
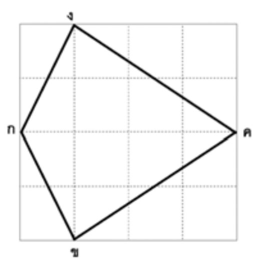



*Figure 2b*. Problem Situation 2



After the teacher had discussed with the students how to find the area of squares, she posed the next situation by asking the students to find the area of the kite. Students were encouraged to think independently and explain their ideas accordingly. The students elaborated on their ideas while the teacher wrote the students' ideas on the blackboard. The teacher tried to guide the students to write the symbolic expressions as $\frac{1}{2}$ x product of the length of the diagonals on the blackboard. The protocol analysis result reveals that students' symbolic language was found in the following verbatim data:

*Teacher: Umm... ขง (Thai symbol) and กค (Thai symbol) are lines. You have to understand how to calculate the area by providing information on operating with diagonals. The area is calculated by squaring the length of one of its sides, so the area is A = s^2, where s represents the length of a side.*
*Student: Multiply*
*Teacher: Ok…this will be the reference for the formula of the area of the kite. How to get the area of the kite, one-half…*
*Student: One-half multiplied by the product of the length of diagonals.*

## Explicitly Teaching Symbolic Notation

Teachers who explicitly teach and explain symbolic notation and mathematical language enable students to communicate their ideas more precisely. A clear understanding of symbols and mathematical language helps students articulate their reasoning and solutions effectively. The findings demonstrated that most of the Laotian teachers involved in this study explicitly taught symbolic notation as best practice when using symbolic communication compared to the teachers in the other countries.

### *Laotian Grade 1 Class Scenario*

Problem situation: Let's create a problem story related to 8 – 3 = 5 by looking at the picture.

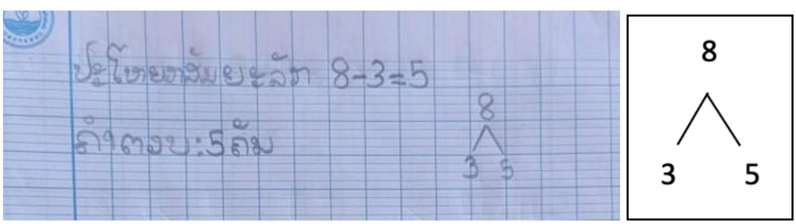
The findings revealed that the teacher encouraged the students to elaborate the problem story according to what they perceived from the picture and encouraged the students to write symbolic sentences in their notebooks as illustrated in the following communications protocol:

*Student: There are three red tricycles and eight blue tricycles. How many more blue tricycles are there than red tricycles?*
*Teacher: How can we write your story using symbolic expressions? Try to write it.*
*Student: Eight minus three equals five*

Figure 3 indicates that the students used a diagram to decompose eight to three and five and wrote the symbolic expression as 8 – 3 = 5. This evidence indicates that a mode of symbolic mathematical communication was used in the Grade 1 Laotian class.

**Figure 3**

*The Finding of Grade 1 Laotian Student's Symbolic Mathematical Communication*




***Laotian Grade 3 Class Scenario***

Problem situation: Think of a way to find the number and fill in the mathematics sentence:

① $8 \times \square = 56$ ② $\square \times 6 = 30$

The findings indicated that the students shared their ideas to elaborate on the problem according to what they perceived from the picture. The teacher encouraged the students to write the symbolic sentences in their notebooks as illustrated in the following communications protocol:

*Teacher: Find the number to fill in the mathematics sentence: $8 \times \square = 56$. What number is multiplied by eight equals fifty-six?*

*Student: Seven*
*Teacher: How do you know?*
*Student: I remember it from the multiplication table*
*Teacher: Come and write it on the board.*

Figure 4 reveals that the teacher presented the problem situation in a symbolic sentence $8 \times \square = 56$ for students to find the answer. The student not only needed to answer but also had to give the method that they used to get the answer. This evidence implies that the teacher and students used symbolic language to communicate mathematically.

**Figure 4**

*The Finding of Grade 3 Laotian Student's Symbolic Mathematical Communication*


**Real-world Contexts**

The Thai teacher, in this situation, is connecting symbolic communication to real-world contexts and everyday experiences to help students see the relevance of mathematics in their lives. The teacher relates symbols to practical applications to foster greater engagement and interest in mathematics. The findings revealed that this teacher possesses the symbolic communication skills to relate mathematical concepts to real-world contexts as best practices when using symbolic communication.

***Thai Grade 1 Class Scenario***

Problem situation: Let's tell a story about 7+8
The teacher reviewed the storytelling activity about 2+5 with students as a whole class activity. The teacher wrote the students'

story involving the symbolic expression 2+5 on the blackboard. The teacher then presented the activity "Let's tell a story about 7+8". The students then wrote the story. The teacher asked a student to volunteer to present their story. The teacher allowed the students to conduct a class discussion and arrive at a conclusion.

**Figure 5**

*Whole Class Discussion and Comparison of Students' Answers*

The observation findings reveal that the student used symbolic language to express himself appropriately in corresponding to the above problem situation. He did this as follows:

> *Teacher: Let's tell the story about 7+8*
> *Student: (Write 7+8 = 15 on the worksheet)*

**Figure 6**

*The Finding of Grade 1 Thai Student's Symbolic Mathematical Communication*


Figure 6 illustrates that the student could express his answer using symbolic language as he wrote the symbolic mathematical expression after he considered the fruit pictures to compose his story accordingly. This implies that the student was using the mode of symbolic mathematical communication in real-world contexts.

### *Thai Grade 5 Class Scenario*

Problem situation: Mother buys paprika chili $\frac{1}{4}$ kg less than pepper chili $\frac{3}{10}$ kg. How many chilies has the mother bought?

**Figure 7**

*Posing an Open-ended Problem and Finding of Grade 5 Thai Student's Symbolic Mathematical Communication*

The observation findings revealed that the teacher asked her students how to write the symbolic expression that derived from " $\frac{3}{10}$ add $\frac{1}{4}$ ". Figure 7 shows that the students could express their ideas appropriately using the mode of symbolic mathematical communication. The protocol analysis result reveals that the students' symbolic language was found in the following verbatim data:

*Teacher: How do you write the symbolic expression?*
*Student:* $\frac{3}{10}$ add $\frac{1}{4}$

*Teacher: (Teacher opens slide to show the symbolic expression* $\frac{3}{10}$ add $\frac{1}{4}$ *= [ ] on the screen)*

### Interactive and Collaborative Learning

Teachers encourage interactive and collaborative learning environments in promoting symbolic communication among students as best practice. Students engaged in discussion, group work, and sharing ideas to allow them to learn from each other and to construct their understanding collaboratively. The results revealed that most Thai mathematics teachers actively engaged in the Thailand Lesson Study incorporated the Open Approach model, prioritizing interactive and collaborative learning. This model has been proven as best practice in Thailand because it establishes a positive, student-centred, and active learning-based school culture in line with teachers' beliefs about innovations (Inprasitha, 2022).

***Thai Grade 2 Class Scenario***

Problem situation: The length problem
The teacher posed a problem to onsite and online students involving a length comparison. The teacher then provided them with time to solve the problem. The teacher and students then discussed the students' ideas.

**Figure 8**

*Posing an Open-ended Problem and Students' Self-learning*

The finding of the protocol analysis reveals that students' symbolic language was found in the following verbatim data:

*Teacher: We can find the length of the blue ribbon using subtraction. Can you tell me your answers?*
*Students: One hundred and ninety-eight less one hundred and thirty-nine equals the difference (students wrote: 198-139 = [  ])*

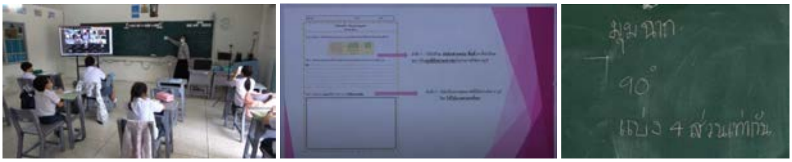
According to the above verbatim protocol and students' written work, the results indicate that the student's mode of mathematical communication involved the use of symbolic language because they express themselves verbally and could write mathematical symbolic sentences accordingly.

***Thai Grade 3 Class Scenario***

Problem situation: What is a right angle?
The teacher demonstrated folding a piece of paper to form a right angle. The students then use right-angled paper to measure the size of the angle of anything around them.

**Figure 9**

*Posing an Open-ended Problem and Students' Self-learning, and Finding of Grade 3 Thai Student's Symbolic Mathematical Communication*




The protocol analysis findings were derived from the teacher and students' interactions during the class discussion (refer to Figure 9). The findings imply that the teacher and students interacted using symbolic mathematical communication, as shown:

*Teacher: Do you know the name of this angle?*
*Student: Right angle*
*Teacher: Based on this piece of paper, where is a right angle?*
*Student 1: Both of them are at the same angle.*
*Teacher: OK, how many degrees are there in a right angle?*
*Student 2: A right angle has 90 degrees.*
*Teacher: (Write $90^0$ on the blackboard)*

## Individual and Cultural Diversity

One of the best practices for using symbolic communication is recognizing and valuing individual and cultural diversity in mathematics classrooms. This helps to create an inclusive learning environment, allowing the teacher to be sensitive to different cultural interpretations of symbols and communication styles and fostering a positive classroom atmosphere. The findings indicate that Japanese teachers are more alert than from the other countries included in the study to individual and cultural diversity among their students.

### *Japanese Grade 1 Class Scenario*

Problem situation: There are 12 birds on the tree. Out of the 12, eight birds flew away. So, how do many birds remain in the tree?

*Teacher: Please write the mathematics sentence or any additional information based on what we have discussed, and then conclude.*
*Student: (walks to the front of the class and writes the mathematics sentence on the blackboard. She then writes the solution). So, the mathematics sentence is 'twelve minus eight equals four.'*

**Figure 10**

*The Finding of Grade 1 Japanese Student's Symbolic Mathematical Communication*

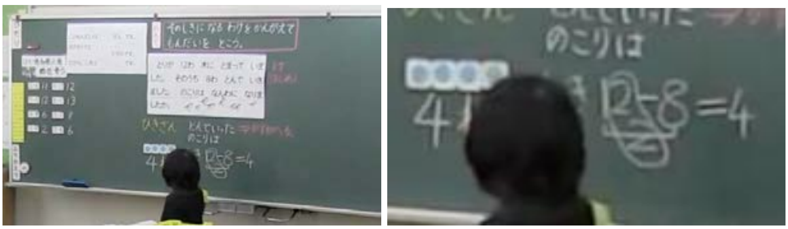



Figure 10 demonstrates that the student wrote the symbolic language as "12 - 8 = 4", in which eight was subtracted from ten, with two remaining. Add the remaining two to the original two, and the answer is four. This answer is based on the decimal notation system.

***Japanese Grade 2 Class Scenario***

Problem situation: In the beginning, 24 children play in one place, and more friends join the group. In total, there are 35 children. Therefore, how many children joined the group? The teacher wanted the students to find the number of children who joined the group by using a tape diagram.

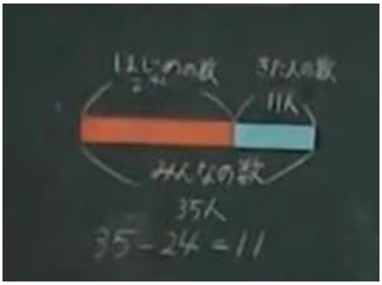
The observation results revealed that students tried to find the answer by adding the number of children. They wrote mathematical sentences on the blackboard. The teacher then asked the students to explain the meaning of the mathematical sentences. Figure 11 demonstrates that the teacher used a tape diagram to confirm the student's mathematical ideas. The following transcript indicates that the teacher and students were using symbolic language.

*Teacher: Could you explain the mathematical sentence that you have applied?*
*Student: 35-24, I think it is ok.*

*Teacher: Are you sure? 35-24 = 11, is that correct?*
*Student: The answer is exactly 11.*

**Figure 11**

*The Finding of Grade 2 Japanese Student's Symbolic Mathematical Communication*




## Use of Technology

Teachers integrating technology, such as interactive whiteboards, computer software, or graphing calculators, can enhance communication in Thai and Lao PDR mathematics classrooms to develop best practices. Technology allows for dynamic representations and facilitates discussions on complex concepts, as displayed in the Laotian Grade 5 and Thai Grade 4 classrooms considered in this study.

### *Laotian Grade 5 Class Scenario*

Problem situation: Let's find out how to divide the fractions shown:

1) $\frac{4}{5} \div 3$ 2) $4 \div \frac{2}{9}$ 3) $\frac{1}{4} \div \frac{6}{5} \times 0.18$

The teacher revised the students' prior knowledge about the principles of the multiplication and division of fractions. Students were given sufficient time to answer these three questions based on their understanding. The teacher then summarized the principles in the mathematical symbolic format as shown in the following communications protocol:

*Teacher: If we want to divide fraction by fraction, how do we do it? Who can answer?*
*Student: We can divide fractions by multiplying the numerator by the numerator and multiplying the denominator by the denominator.*

*Teacher: That's correct. Is the division of fractions, not the multiplication of fractions?*
*Student: Teacher, it is inverting multiplication.*
*Teacher: How do we invert multiplication? We can see that it is multiplied by the reciprocal of the divisor of the fractions, as in this formula*

$$\frac{b}{a} \div \frac{c}{d} = \frac{b}{a} \times \frac{d}{c} = \frac{b \times d}{a \times c}$$

**Figure 12**

*The Finding of Grade 5 Laotian Student's Symbolic Mathematical Communication*

Figure 12 shows that the teacher wrote the principles of dividing fractions into symbolic mathematical sentences as $\frac{b}{a} \div \frac{c}{d} = \frac{b}{a} \times \frac{d}{c} =$. This evidence implies that the teacher communicated with his students using symbolic mathematical communication.

***Thai Grade 4 Class Scenario***

Problem situation: The teacher has 0.9 meters and 0.3 meters of paper tape. They are put together continually. How many meters are there?

**Figure 13**

*Posing an Open-ended Problem and Finding of Grade 4 Thai Student's Symbolic Mathematical Communication*

The observation findings revealed that the teacher discussed with her students how to add decimals and requested that they explain their ideas. The students described and explained their ideas. Thus, the teacher wrote students' ideas on the interactive blackboard. Figure 13 shows that the teacher asked questions and encouraged her students to write the symbolic expression. The evidence from Figure 13 shows that students used the mode of symbolic mathematical communication in responding to the teacher's problem situation. The findings imply that the teacher and students interacted using symbolic mathematical communication, as shown:

*Teacher: How do you write the symbolic expression?*
*Student: 0.9 add 0.3 equal the space.*
*Teacher: (Write 0.9 + 0.3 = [ ] on the interactive blackboard)*

## Formative Assessment

The findings showed that the Japanese teachers assessed their students' symbolic communication skills through formative assessment to provide the teachers with valuable feedback. This enables teachers to identify areas for improvement and tailor instruction to meet students' needs. This best practice when using formative assessment is shown in Japanese Grade 4 and Laotian Grade 6 classrooms.

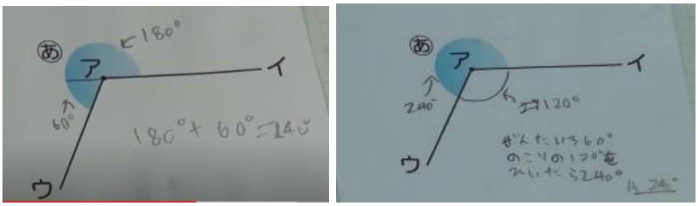
### *Japanese Grade 4 Class Scenario*

Problem situation: The teacher showed the angle that students had to measure and then asked them to measure the angle by using addition and subtraction.

The qualitative results revealed that the students tried to find the size of the angles by themselves. The students used a protractor to measure the size of the angles, and then they wrote the number and degree symbol to show their answers. Additionally, the students wrote the method to find the angles. Figure 14 below shows that the student wrote the size of angles as $60^0$, $120^0$, and $180^0$. Moreover, they wrote the symbolic sentence $180^0 + 60^0 = 240^0$ to explain the size of determined angles. This implies that students were using symbolic modes of mathematical communication.

**Figure 14**

*The Finding of Grade 4 Japanese Student's Symbolic Mathematical Communication*



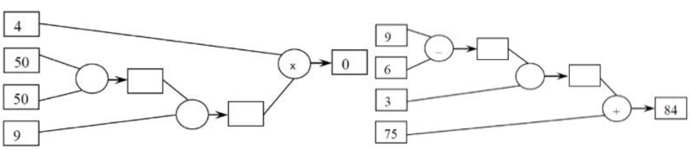

***Laotian Grade 6 Class Scenario***

Problem situation: Situation 1 – Write the bracket into a symbolic sentence to ensure correctness.

| | |
|---|---|
| ก . $8+6\times2=28$ | ข . $7+8\div8=8$ |
| ค . $20+12\div3=24$ | ง . $13-1\div4=3$ |
| จ . $10+2\times6-4=18$ | ส . $6-3+9\times0=0$ |
| ช . $12+4\div24\div3=2$ | ย . $2\times6-4+4=4$ |

**Situation 2** – Sriwan buys 4 pencils at 1000 Kib each. Also, he buys 3 notebooks at 1500 Kibs each. Additionally, he buys 1 ruler, which costs 2000 Kib. What is the total cost?

**Situation 3** – Fill in with the symbols **+, -,** × in circles to get the number in the squares.




The teacher assigned the problem situations 2 that required students to find out the total price of each item followed by the total prices in symbolic mathematical sentences according to the following communications protocol:

*Teacher: What is the total price?*
*Student: 10500 Kib.*
*Teacher: Are you sure 10500 Kib is the correct answer?*

*Student: Yes.*
*Teacher: Please write in mathematics a symbolic expression to show how you get this answer.*
*Student: 4 ×1000 + 3 ×1500 + 2000 = 10500*
*Teacher: Is that correct?*
*Student: Yes.*

**Figure 15**

*The Finding of Grade 6 Laotian Student's Symbolic Mathematical Communication*

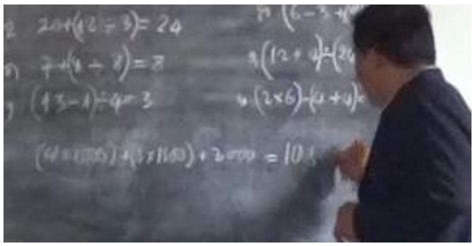


Figure 15 demonstrates that the teacher wrote the mathematics symbolic expression as (4×1000) + (3×1500) + 2000 = 10500, conforming to the student's answer. This evidence implies that the teacher used symbolic language as a mode of mathematical communication.

**Metacognition and Reflection**

The findings revealed that Japanese teachers encourage metacognition and reflection regarding symbolic communication. This is considered one of the best practices to help students become more aware of their thought processes and communication strategies. This self-awareness can lead to improved problem-solving and communication skills. The following Japanese class scenarios demonstrate the best practices of metacognition and reflection.

***Japanese Grade 5 Class Scenario***

Problem situation: The teacher told students would like to make Calpis juice using water and Calpis liquid. There are several ratios for several intensities of Calpis juice. The favourite ratio is water 50 to Calpis 20. If we have 150 ml of Calpis liquid, how much water do we need to make this intensity?

After the students explained their ideas about how to find the volume of water needed to mix with the Calpis liquid, the teacher extended the students' ideas by writing the protocol and picture on the blackboard.

*Student: So, I wanted to find out how much the volume of water was more than Calpis liquid, so I divided by 20, 50 divided by 20 and I found that it was 2.5. Now I added the relationship between 50 and 20, I used 2.5 to find out the water amount using 150, so 150 times 2.5.*

*Teacher: So, what is the difference between the two numbers? So, how much does 2.5 work? If divided by 20 at once water could be 2.5 that was the reason why we tried to multiply by 2.5 to 150. So, what are your excellent ideas? So, 2.5 times more than the other. Therefore, this 2.5, we call a ratio value.*

Figure 16 demonstrates how students explained the ideas while the teacher wrote the ratio 50:20 as the ratio symbol. The students then elaborated on the ratio value being 2.5, derived from the calculation of 50 divided by 20. This implies that the teacher and students used symbolic language to communicate their ideas.

**Figure 16**

*The Finding of Grade 5 Japanese Student's Symbolic Mathematical Communication*


***Japanese Grade 6 Class Scenario***

Problem situation: The students have to find out how to determine the volume of the triangular prism using their previous knowledge and the guidelines in the worksheet. The students can then share their different ideas with the class to find the volume of the triangular prism.

After the students solved the problem of finding the volume of the triangular prism, they explained their idea using the base area multiplied by the height. The teacher emphasized the students' ideas by writing mathematical symbols on the blackboard as follows:

*Student: So, this is 4 times 3 divided by 2, which corresponds to the area of the base. Then, multiply by the height. First, I have to find the area of the base followed by multiplying it with the height. Therefore, 4 multiplied by 3 divided by 2 multiplied by 5.*
*Teacher: (Write 4×3 ÷ 2×5 on the blackboard)*

**Figure 17**

*The Finding of Grade 6 Japanese Student's Symbolic Mathematical Communication*

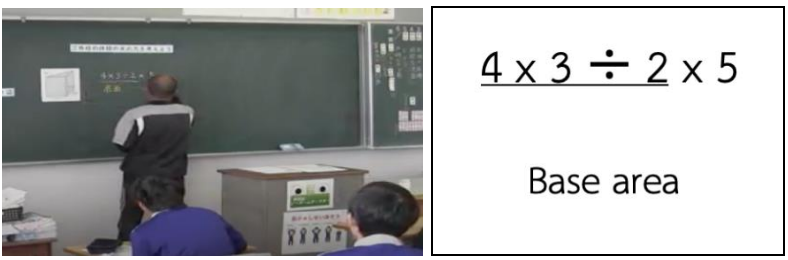



Figure 17 elucidates how the students explained their ideas about the calculation needed to find the volume of the triangular prism. The teacher wrote the student's ideas using a mathematical symbol "4× 3÷2× 5". This implies that the teacher and students used symbolic language to communicate their ideas.

The above findings from the Grade 1 to Grade 6 Japanese mathematics classes revealed that symbolic language was used to help students develop a deeper understanding of mathematical concepts, and thus improve their problem-solving abilities. In other words, the Japanese teachers in this study focused on the students' symbolic expression. They encouraged students to elaborate on the meaning of symbolic language to deepen their understanding of mathematical concepts. The best practices of these Japanese teachers in terms of symbolic communication are the integration of visual representations, individual and cultural diversity, formative assessment, and metacognition and reflection.

The results of the Laotian Grade 1, 3, 5, and 6 classrooms indicated that students could translate expressions into symbolic language. The results imply that students had a better understanding of mathematical concepts than those who relied solely on verbal explanations. The best practices of Laotian teachers in terms of symbolic communication involve explicitly teaching symbolic notation, using technology, and using formative assessment.

The results from Grade 1 to Grade 6 Thai mathematics classes can be concluded by stating that symbolic mathematics communication is an essential tool for Thai mathematics teachers to communicate complex mathematical ideas and to solve real-world problem situations. The best practices of the Thai teachers in this study in terms of symbolic communication are the integration of visual representations, the use of real-world contexts, interactive and collaborative learning, and the use of technology.

## CONCLUSION

The overall findings of this comparative study among the three countries of Japan, Lao PDR, and Thailand suggest that the use of symbolic language in mathematics classes has a positive impact on students' understanding of mathematical concepts and their ability to solve mathematical problem situations. This study has made a significant contribution to the field of symbolic communication between primary school teachers and students in Japanese, Laotian, and Thai mathematics classrooms by revealing several best practices, namely the integration of visual representations, explicitly teaching symbolic notation, engaging with real-world contexts, interactive and collaborative learning, individual and cultural diversity, the use of technology, formative assessment, and metacognition and reflection.

Nevertheless, the Japanese mathematics classes utilized highly symbolic mathematical communication compared to Laotian and Thai mathematics classes. The Japanese teachers were able to focus on the students' symbolic expression and encouraged them to elaborate on the meaning of symbolic language to deepen their understanding of mathematical concepts. This could have been because the Japanese mathematics programmes, which have a student-centred approach, had introduced the students to several problem-solving methods and presented the problem-solving ideas to their full potential. The class management revealed in this study was similar to the approach towards mathematics instruction in Japanese classrooms proposed by Takahashi (2006), which focused on students solving problems on their own. Once they had solved problems, the students would bring several ways and solutions to the problems for discussion in class. At the same time the teachers would provide opportunities for students to improve their mathematical abilities and skills.

Furthermore, Stigler and Hiebert (1999) pointed out that when teaching mathematics in Japan, the teachers play a minor part in allowing students to come up with solutions. The concepts and thought processes of the students were discussed, and the class learned together. Likewise, Khaing et al. (2007) discovered that Japanese mathematics classrooms had adopted the Openness Approach to Mathematical Problem Solving as a teaching method, that increased mathematics communication. Students were able to express themselves with several problem-solving ideas. This allowed students to use a variety of means of mathematical communication. Moreover, the researchers would like to suggest that best practices involve the use of mathematical symbols, equations, and notation to represent mathematical concepts and relationships as a means of building a solid foundation in mathematical understanding (Zulhelmi & Anwar, 2021). For example, teachers should provide clear explanations of mathematical symbols and notation. This should be followed by teachers demonstrating how to read and write mathematical expressions, equations, and formulas correctly. On top of that, teachers can use visual aids, real-life examples, and hands-on activities to reinforce understanding. In conclusion, the best practices of Japanese teachers in terms of symbolic communication are the integration of visual representations, individual and cultural diversity, formative assessment, and metacognition and reflection.

On the other hand, Thai mathematics classes are found to be the second in terms of using symbolic language in their mathematics classes. Thai teachers are found to use symbolic mathematics communication as an essential tool to communicate complex mathematical ideas and solve real-world problems. Regarding mathematics classes in Thailand, which had been taught during the COVID-19 epidemic in the form of hybrid learning using both onsite and online instruction, it was found that most of the classes had utilized four main modes of mathematical communication: ordinary language, mathematical verbal language, symbolic language, and visible representations. This could be related to the fact that most classrooms had been managed in a manner that attempted to shift from a teacher-centred to a student-centred approach, resulting in a broader range of mathematical communication techniques between teachers and students. This is consistent with the past research results from Inprasitha et al. (2012), Pimkaew (2007), Premprayoon (2007), and Thinwiangthong (2012), all of whom found that Thai mathematics classes tended to use a teaching approach that focused on solving students' problems and on encouraging teachers and students to use a variety of modes of mathematical communication.

However, according to Chiangkul (2016), some Thai mathematics classes are taught conventionally. He stated that some Thai classes still emphasize lectures, demonstrations, and explanations, as well as memorization. In these classes, problems are not solved by students, nor are the student's learning processes focused on. Furthermore, due to changes in instructional management during the COVID-19 outbreak, which required social distancing measures, students could not work face-to-face. As a result, some modes of mathematical communication were, therefore, challenging for students to employ. An unexpected finding was that Thai teachers introduced physical objects such as blocks, counters, or geometric shapes to represent abstract concepts. These helped students visualize mathematical ideas, making it easier to transition to symbolic representations later. The best practices of Thai teachers in terms of symbolic communication are the integration of visual models, the use of real-world contexts, interactive and collaborative learning, and the use of technology.

The findings about the Laotian mathematics classes showed that the symbolic mode of mathematical communication was not very popular, and researchers could not find the figurative language being used in Grades 2 and Grade 4. However, the results showed that students could translate expressions into symbolic language. The results imply that students had a better understanding of mathematical concepts than those who relied solely on verbal explanations. In addition, the results of the Laotian mathematics classes primarily utilized three modes of mathematical communication: ordinary language, mathematical verbal language, and symbolic language. Most of the mathematics teaching and learning in the majority of Laotian classrooms was based on a teacher-centred method, even though some classes had attempted to make the transition to problem-solving. However, teachers rarely allowed students to articulate their problem-solving thoughts fully. As a result, the educational process did not encourage students to think through various methods.

Consequently, students only used a few modes of mathematical communication. This is consistent with the results of a study by Thinwiangthong (2012), who found that mathematics classrooms in the Lao PDR did not focus on students solving problems, but rather on lecturing and transferring knowledge to students. The students were given no opportunities to discuss and argue. Following this line of reasoning, the best practices shown by Laotian teachers in terms of symbolic communication are explicitly teaching symbolic notation, the use of technology, and formative assessment. The researchers suggested that Laotian teachers should provide ample opportunities for practice involving symbolic communication, while repetition and regular practice could help solidify the understanding of symbols and mathematical notation.

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