Exploring the Feasibility of Co-construction Among Mathematics Teachers and Teacher Educators: Analysis of Discourse in a Product-based Teacher Professional Development Program

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Using a product-based teacher professional development workshop in the Just Do Math program as a case, this study investigated the discourse between teachers and teacher educators from three perspectives, namely focus, form, and flow, to see how the two cohorts communicated in the co-construction, whether the co-construction is feasible to produce innovative teaching materials, and what the feasible dialogical modes are for this co-construction. The participants included four experienced university-based mathematics teacher educators and 38 mathematics teachers. The discourse from the eight 3-hour sessions of the workshop was collected. The findings included that both teachers and teacher educators focused more on students' development of concepts than on learning motivation, even if learning motivation was one main problem to be solved. Feasible dialogical modes for co-construction in teacher professional development workshops in the context of East Asian educational cultures were revealed, which could be described as teacher educator-centred but teacher-focused.

Keywords • mathematics teacher education research • co-construction • mathematics learning task design • teacher professional development program • university-based teacher educator • discourse

Introduction

It is well-known that education is culturally shaped and mathematics classes in East Asia are characterised by teacher-led instruction and passive student learning (Leung, 2001; Leung et al., 2006). With the rising trend of educational reforms in recent decades, however, student active learning activities such as small-group cooperative learning, hands-on tasks, and student discussions, have been promoted and practiced in mathematics classrooms (Wang & Hsieh, 2017). In the present era, mathematics classes in East Asia are featured with not only teacher idea-elaboration but also student active learning (Kaur, 2009; Wang & Hsieh, 2017). Nevertheless, "What is the situation when the teachers become the learners, and teacher educators, who usually stand for and promote those educational reforms, become the instructors? Are workshops in teacher professional development (TPD) programs in East Asia teacher educator-led or learner-centred?" Concerning the international stream, co-learning among participants in TPD programs has been discussed in the literature for decades (Bragg & Lang, 2018; Horn, 2010; Little, 1990; Rigby et al., 2020). Some studies have further discussed the co-construction of knowledge or identity among participants, including teachers and teacher educators (Ngcoza & Southwood, 2015; Orland-Barak, 2006). In these programs, teachers are not receivers but active participants who engage productively in developmental activities, and teacher educators are not transmitters but facilitators who advance teachers' professional development and operate as co-learners who gain insight into their research and their professional assistance to teachers (Chen et al., 2018; Zaslavsky & Leikin, 2004). It follows that questions worthy of investigation are: "Is it feasible for a TPD workshop in East Asia to adopt this co-construction and co-learning partnership perspective and achieve its objectives? How is this co-construction and co-learning, and what characterises this feasibility?"

The large-scale program Just Do Math (JDM), launched in Taiwan, adopted the co-construction and co-learning partnership perspective in TPD to implement its learner-centred and active learning ideas for the learning of all-level personnel (Lin et al., 2018). JDM aims to deal with the problems in East Asian countries revealed by the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). TIMSS and PISA findings indicate that although East Asian countries outperform their Western counterparts in mathematics achievement, their students have notoriously low levels of interest and confidence in learning mathematics. In addition, they have a high percentage of low performers whose mathematics proficiency is lower than the baseline level (Mullis et al., 2012, 2016; Organisation for Economic Co-operation and Development [OECD], 2013, 2016). To counter these problems, JDM introduced three core ideas to help enacting student active learning grounded in an enactivist perspective on learning (Yang et al., 2022), namely:

- developing student fundamental prerequisite mathematical ideas before class rather than providing remedial instruction after they fail to learn (Core 1),
- helping students construct concrete references for mathematical concepts through operating manipulative representations (Core 2), and
- increasing students' motivation for mathematics learning through gamified activities (Core 3).

The teaching materials developed in JDM are called mathematical grounding activity (MGA) modules, which must reflect the three core ideas. In Taiwan, the design of MGA modules is innovative because it deviates from the mathematics textbooks and handouts that teachers typically design, reflecting conventional teaching in Taiwan. The educational philosophy in Taiwan is based on traditional Confucian culture, emphasising the development of students' concepts and skills to help them succeed in examinations (Leung, 2001).

MGA module designer workshops, a type of teacher professional development in JDM, have two objectives: to increase teachers' competence in designing tasks that embed the JDM core ideas and to produce MGA modules. Workshop leaders were teacher educators from universities specialising in mathematics education; they are expected to not only facilitate teachers' learning in task design (Zaslavsky & Leikin, 2004) but also co-construct modules. None of the teachers and teacher educators had experience in designing MGA modules. The two cohorts co-constructed MGA modules using their relatively strong backgrounds in practical and theoretical fields, respectively. The two cohorts were expected to have an equal relationship. The teachers were not merely receivers who wrote down the teacher educators' ideas; they were the prominent writers and had the power to decide the final versions of the modules. In addition, the teacher educators actively initiated ideas and suggestions for a design or respond to teachers' questions in discussions. Discussion, negotiation, and communication between the two cohorts to reach an agreement on module design were critical and expected. Since both cohorts were unfamiliar with designing MGA modules, no cohort took the dominant role in designing them. JDM was underpinned by the ideas of co-learning and facilitated co-constructing partnerships between teachers and teacher educators in TPD.

Many studies on teachers' design tasks have considered the modification of current textbooks and materials rather than developing original teaching materials embedded with innovative ideas that deviate from conventional mathematics teaching (Bardy et al., 2021; Pepin et al., 2017; Remillard, 2000). Studies that focused on the co-construction of teachers and teacher educators to design tasks have been even fewer, not to mention within East Asia. The differences between teachers and teacher educators in their knowledge, expertise, practical experience, and their perceived position in the education community, could make the expected co-construction challenging to accomplish. In the East

Asian education community, learners are deemed to highly respect instructors and theoretical-based knowledge is valued (Hsieh e.t al, 2018; Leung, 2006). Therefore, to move the field forward in its understanding of how these cohorts with different perspectives function to co-construct innovative and unfamiliar tasks, how they communicate and negotiate various ideas based on their experience to develop interpersonal reasoning, and how they can be pushed outside their comfort zones to develop new ideas (Orland-Barak, 2006), it is essential to explore the discourse among these cohorts. Accordingly, the present study investigated the discourse between teachers and teacher educators in a product-based TPD program, the MGA module designer workshops in JDM, in which co-construction to produce innovative and unfamiliar tasks is expected from three perspectives. The following research questions were addressed:

RQ1. What core ideas of JDM do mathematics teachers and teacher educators focus on during the co-construction of tasks in the MGA module design process?

RQ2. What do forms of discourse reveal about the co-construction of tasks during the MGA module design process?

RQ3. What do flows of discourse reveal about the co-construction of tasks during the MGA module design process?

While exploring the discourse between teachers and teacher educators, using the JDM TPD program as an instrumental case study, the findings could help us better understand how the two cohorts communicate in the co-construction process to produce innovative tasks, whether the co-construction between the two cohorts is feasible, and identify the dialogical modes for this co-construction. The lenses of focus, form, and flow were employed in alignment with the objectives and methods of JDM implementation. By examining the focus, we aim to gauge the degree to which workshop participants engage with the three core ideas that JDM identifies as solutions to Taiwan's educational challenges. The notion of co-construction embodies JDM's emphasis on learner-centred, active learning strategies for personnel at all levels. Through exploring forms and flows, we gain insights into the collaborative dynamics between mathematics teachers and teacher educators in a Confucian-heritage East Asian culture, such as the division of labour of the two cohorts, and the rules for their actions and interactions (Engeström, 1987).

Theoretical Background

Designing tasks serves as a method to draw teachers' attention to vital elements of mathematics teaching and student learning, thereby playing an essential role in teacher learning (Watson & Sullivan, 2008). Zaslavsky (2005, 2008) posited that teachers' constructive engagement in tasks can enhance their mathematical and pedagogical power. Such experiences offer teachers opportunities to recognise challenges, disturbances, confrontations, extensions, and alternative strategies derived from the teaching context. The use of mathematics tasks in TPD has attracted increasing attention in the literature. Arbaugh and Brown (2005) researched an exercise in which teachers sorted and examined mathematics tasks and demonstrated that the teachers' thoughts toward and use of tasks in class improved. As well, Boston and Smith (2011) revealed that teachers' abilities to select high-level mathematical tasks were higher after they attended a TPD program focused on the selection and implementation of tasks.

Some studies that investigated more than teachers' manipulation of existing tasks, required teachers to design their own tasks. These studies suggested that teachers can develop their competencies and revise various aspects of existing materials to encourage students' cognitive development in mathematics. For example, Fleming et al. (2015) trained teachers to revise the tasks employed in their current teaching materials to better reflect the characteristics of rich mathematical questions and discovered that the cognitive demand level in the revised tasks was higher. Lee and Özgün-Koca (2016) asked teachers to revise tasks by adding high-level, thought-provoking questions. Although their study did not obtain conclusive evidence that the teachers had succeeded in developing high-quality tasks, it identified four facets to their task revisions, namely: context, concept, procedure, and format. In another study, Kumar and Subramaniam (2015) asked teachers participating in a TPD program to develop unit plans that extended beyond the textbooks with respect to representation and context. They discovered that the teachers shifted from focusing on rules and computational fluency to developing students' understanding and reasoning skills. The studies, however, did not provide much information on teachers' designing original tasks outside their comfort zones. They also lacked a focus on increasing student learning motivation as well as bridging and developing mathematical concepts. In addition, the importance of cooperation between teachers and leaders in TPD programs has emerged from observations of researchers and evaluations collected from teachers in the studies though how these two cohorts cooperate is not elaborated on specifically in these studies.

In contrast to most studies, which have focused on the professional development of teachers in task-design-related TPD programs, Zaslavsky and her colleagues (2004, 2008) proposed a three-layer model to address the professional growth of mathematics teacher educators through practice and reflection in TPD programs. The model defines the roles of mathematics teachers and teacher educators as designers of students' mathematics tasks and teachers' professional development tasks, respectively; both groups are also reflective practitioners of their designs (Schön, 1987). The teacher educators in the studies conducted by Zaslavsky and her colleagues were in-service teachers, and Chen et al. (2018) subsequently modified their model to investigate university-based teacher educators with a research background in mathematics education. Although the teacher educators in that study did not directly design students' mathematics tasks, they helped teachers design such tasks with innovative and unfamiliar ideas from a theoretical perspective. Consequently, the teacher educators grew professionally in terms of mathematics task design by observing and reflecting on students' work on the tasks designed by the teachers.

The professional development of teachers not only benefits from teacher participation in task design, but also from the additional aspects of task design that would be missing without their involvement (Jones & Pepin, 2016). This is also true for teacher educators with a research background, especially in the design of innovative tasks with features that are unfamiliar to them and their learners (mathematics teachers). Also, Coburn et al. (2013) and Penuel et al. (2011) promoted the research–practice partnership on the basis of cooperation between teachers and researchers.

To demonstrate the relationships among mathematics teachers and teacher educators in a TPD program in which co-construction is expected when producing tasks reflecting critical innovative features (the three core ideas of JDM), the present study adapted the frameworks of Zaslavsky and Leikin (2004) and Chen et al. (2018), and the result is illustrated in Figure 1 (Appendix). In this framework, the engagement and reflection of mathematics teachers in the design and practice of MGA modules offer opportunities to enhance their professional development and students' learning. Consequently, mathematics teacher educators have a dual role. Besides their engagement in and reflection on the teacher professional development activities (the MGA module designer workshops of JDM), they collaborate with mathematics teachers in the design of MGA modules. Drawing from Zaslavsky's (2005) insights, we posit that this co-construction of innovative tasks can, through interactions and communication between the two groups, bring forth cognitive conflict, doubt, and perplexity, but also create pathways to address and overcome these challenges.

Research Methods

Design of MGA Module Designer Workshops

JDM incorporated social constructivism into all its activities. In the MGA module designer workshops, there was the expectation that participating teachers and workshop leaders (university-based teacher educators) have equal status. The aim was for members from each cohort to comfortably express their ideas or disagreement, negotiate with others to reach an agreement, and subsequently co-construct MGA modules. The modules were designed for student learning activities in Fun-Math Camps, which take place on weekends and during summer and winter vacations and are taught by MGA teachers trained by JDM (Wang et al., 2021). Participants in these camps were primary and secondary school students. Student learning in the Fun-Math Camps was based on the idea of learning by doing and exploring together with other participants. The aim was for students to actively and freely discover and develop mathematical ideas in gamified activities through cooperation and negotiation with their peers and teachers. In these situations the teachers are co-constructors rather than transmitters of the ideas (Beck & Kosnik, 2006).

The MGA module designer workshops were product-based. As designers, all participating teachers were required to develop at least one module reflecting the three core ideas of JDM (Core 1 to 3), though they were permitted to collaboratively develop multiple modules. The workshops consisted of five stages: module evaluation, theory learning, module designing, instructional design implementation, and reflection and modification (Figure 2). The five stages provided MGA designers with three critical approaches for professional development: learning from experts, learning from the community, and learning from teaching experience. These approaches enabled designers to acquire insight from experts, peers, students, and themselves (Ball & Cohen, 1999; Sowder, 2007). The workshops required all MGA designers to participate in eight sessions, each lasting at least 3 hours. Mathematics teachers and teacher educators could possibly discuss privately at other times. Additionally, teachers were responsible for allocating their own time to implement their instructional designs.

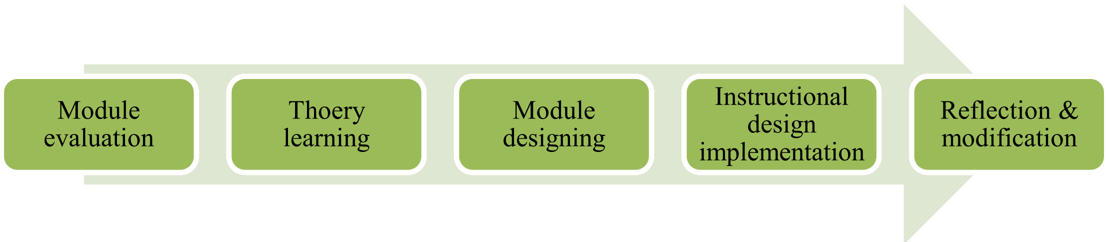

Figure 2. Stages in the MGA module designer workshops.

The MGA designer activities in the module evaluation stage involved the analysis and evaluation of previously designed modules, including assessment of which modules reflect the core ideas of JDM and which modules require modification. MGA designers discussed their opinions with other designers and the workshop leaders through whole group discussions. These processes enabled the designers to learn to identify the primary aspects that must be addressed in MGA module design.

In the theory learning stage, MGA designers were exposed to theories central to the core ideas of JDM. Piaget's theory of constructivism (1952) and Ausubel's theory of meaningful learning (1961) were introduced to help designers identify fundamental prerequisite mathematical ideas. Bruner's (1966) three types of representation—namely (from least to most abstract) enactive, iconic, and symbolic—were introduced to help designers develop learning activities that use manipulatives. Designers were encouraged to employ concrete and non-linguistic representations—such as manipulatives, physical movements, and pictographs—at the beginning of instruction to improve student understanding (Thompson & Rubenstein, 2000; Wang & Hsieh, 2017). To increase students' motivation to learn mathematics, approaches relevant to intrinsic motivation, including individual and interpersonal motivation, were introduced (Hsieh, 2006). Approaches to increasing individual motivation involved challenge, curiosity, control, and fantasy. Approaches to improving interpersonal motivation involved cooperation, competition, and recognition.

In the module design stage, MGA designers and workshop leaders held small-group discussions to identify the mathematical concepts, ideas, procedures, and skills that students considered difficult and the abilities that students should develop. Subsequently, they identified the fundamental prerequisite ideas that must be reinforced. Designers then enacted the first step of creating learning activities or games that develop these fundamental prerequisites. During the design process, in addition to the sessions held by JDM, designers and leaders constantly engaged in private discussions to exchange design ideas and ensure that the activities were aligned with the core ideas of JDM.

In the instructional design implementation stage, MGA designers instructed students in a classroom setting and observed and analysed the students' learning. They assessed whether students understood the fundamental prerequisite ideas while participating in the gamified learning activities, whether they were interested and willing to engage in the activities, and whether they had sufficient opportunities to operate manipulative representations. The designers modified their modules in accordance with the information they collected during these experiments.

In the reflection and modification stage, which comprised the final two sessions, MGA designers presented their modified versions of the MGA module to the whole group, based on insights gained from instructional design implementation. The leaders and other designers discussed with the presenters and provided ideas for modifying the modules. The designers reflected on the suggestions and modified their designed modules accordingly. Most designers have repeatedly modified their modules to perfect their design before the final two sessions were scheduled.

Participants and Data Collection

The research reported in this article is related to the discourse in one MGA module designer workshop held in Taipei, Taiwan. The participants were four workshop leaders (experienced university-based mathematics teacher educators) and 38 MGA designers. The discourse included small-group and whole-group discussions from the eight 3-hour sessions conducted on Friday afternoons. That time slot was mandated by the government as designated time for mathematics teachers to engage in teacher professional development. The workshop was video recorded, and the corresponding dialogue was transcribed. We also recorded field notes while observing interactions in the workshop. We interviewed leaders and designers when there was a need to clarify their contributions to and evaluations of the MGA modules, with the aim of confirming our interpretations of the discourse analysis.

Data Analysis

We browsed, read, reread, and interpreted the transcripts of dialogues from the workshop and finally selected the data collected in the final two sessions (the reflection and modification stage) for analysis. In the sessions, nine designers who completed their instructional design implementation reported on their designs (another ten designers completed their instructional design implementation or finalised their designs after this stage and subsequently, submitted their modules). At this final stage, both the MGA designers and workshop leaders had a comprehensive understanding of the modules. Because the designers had completed their instructional design implementation at this point, the discussion in the final sessions reflected both the theoretical and practical facets of the process. The data collected in these two sessions contained sufficiently rich information to use discourse analysis to analyse the dialogue. All documents were analysed by two coders to ensure reliability. Both coders had participated in JDM since its launch. When discrepancies occurred, the coders discussed the codes until a consensus was reached.

The perspectives of focus, form, and flow were adopted from the literature to analyse the discourse. The first perspective, focus (Huizinga et al., 2015), was used to explore the emphasis of discourse on the three core ideas of JDM: fundamental prerequisite mathematical ideas, manipulative representations, and motivation to learn mathematics. The second perspective, form (Kontkanen et al., 2016), was used to investigate the categories of collaborative dialogue, namely statement, clarification, comparison, analysis, assertion, argument, and synthesis (Table 1). The transcripts were deconstructed into discourse units of consecutive statements, and a change in idea denoted a new unit (Wilson et al., 2017). For example, the following is a continuous dialogue.

Leader:	Which competency indicator [of the curriculum guidelines] does this correspond to?
Designer:	5-s-? I can't remember which one, but there is this one, fifth grade [...]
Leader:	Don't book merchants usually provide teaching aids?
Designer:	Its teaching aid is strips of paper.

The first two utterances pertain to competency indicators and belong to one unit, while the remaining two utterances are about teaching aids and belong to another unit.

A total of 309 units were identified in the discourse of the final two sessions. Each unit was coded according to the focus and form of discourse. Some units were assigned more than one code, and some units were assigned no codes for focus. For example, one designer provided the following statement regarding the time required for implementing the module activity after his instructional design implementation: "I found that this may need a whole class session to complete. They [the students] need a whole class session to induce and discover, and I did not actually provide sufficient time." This statement was not related to any of the three core ideas of JDM.

We combined several related units to comprehensively analyse the discourse from the third perspective, flow. For example, a leader may have responded to a designer's question raised several discourse units earlier. In such cases, the units of the leader and the designer were combined to investigate the discourse flow. For this perspective, we adopted functional dualism (Lotman, 1988; Peressini & Knuth, 1998) and coherence (Huizinga et al., 2015). In functional dualism, discourse is categorised into the univocal or dialogic mode (Lotman, 1988; Peressini & Knuth, 1998). Univocal discourse primarily comprises speakers conveying information to an audience, whereas dialogic discourse is an active approach that uses discourse as a thought device for exchanging ideas and creating new meaning. Coherence was used to determine whether leaders and designers fulfilled each other's expectations for discourse. More than two participants could have joined the discussion to address a single problem. While the discourse flow from some participants might be coherent, that from others might be incoherent. In addition, the percentage of each category of focus and form was calculated based on the total number of units, and the percentage of each type of flow was also calculated based on all the flows identified.

Table 1
Dialogue Categories for Exploring Forms of Discourse

Forms of discourse	Description	Example
Statement	Recall, describe, and demonstrate circumstances and normal situations that have occurred.	A designer introduced a previous MGA module: "During the first phase of module design in JDM, Teacher Sherry designed one module about forming triangles, using the geometric sticks to assemble one-colour, two-colour, and three-colour triangles."
Clarification	Clarify discourse content to improve understanding of the direction of discourse.	A designer answered a workshop leader's question to improve her understanding as follows: The leader asked, "Don't book merchants usually provide teaching aids?" and the designer answered, "Its teaching aid is strips of paper."
Comparison	Identify similarities and differences.	A workshop leader compared two teaching strategies: "For him, exploring and discovering, the [thinking] intensity of exploration to him, might not necessarily be lower than answering your question."
Analysis	Deconstruct discourse into smaller components to improve understanding of characteristics revealed.	A workshop leader detailed a mathematical concept by decomposing all the cases: "Here involves a logic. Typically, our textbooks state that the sum of the two smaller sides is greater than the third side. This is a logic, right? Small plus medium is greater than large. Large plus medium naturally is greater than small. Large plus small will also be greater than medium. There's a logic inside this. Therefore, its 'any' comes from this logic."

Forms of discourse	Description	Example
Assertion	Assert that something is true or correct.	As she began introducing her module, a designer asserted students' understanding to support her design of the present module: " Students always don't quite understand the word 'any,' and they also find it hard to understand why the sum of two sides should be greater than the third side."
Argument	Provide reasons to persuade others that something is true or correct.	Based on her teaching experience, a designer provided reasons to argue with a workshop leader who suggested an activity to help students grasp the idea of "any": "But 'any,' we later brought it into formulas. When it comes to formulas, the students can't think of it; they can't possibly list that kind of formula."
Synthesis	Combine several ideas to reach a conclusion.	After suggesting various ideas for student activities, the leader synthesised those concepts and concluded: "Therefore, both sides must be established simultaneously, and not just one side."

Note. The examples are taken from the discourse about an MGA module relating to the side lengths of triangles.

Results

Discourse Focus

Discourse focus was to explore the discourse related to the three core ideas of JDM: fundamental prerequisite mathematical ideas, manipulative representations, and motivation to learn mathematics. Table 2 shows that the discourse emphasised the development of fundamental prerequisite mathematical ideas (Core 1; 55%) and constructing a concrete reference for mathematical concepts through operating manipulative representations (Core 2; 49%). The discourse on Core 1 focused on which fundamental prerequisites would be developed in students and the essence of certain fundamental prerequisite mathematical ideas. The following dialogue shows an example, in which the designer highlighted that the essence was the development of concepts ("What we want is students' concepts ...") rather than the skills to perform accurate measurement ("... asking them to calculate any small piece.") in a discussion on an MGA module relating to similar figures.

(1-1) Designer E: You can see these [sides of the rectangle] are 6 cm and 4.01 cm, and then you may ask how to deal with these [...] Drawing pictures would result in errors. It is not possible to draw them exactly the same, so it's all right if there's a little difference. What we want is [the development of] students' concepts rather than asking them to calculate any small piece.

Table 2
Percentages of Various Types of Discourse Focus

	Designers	Leaders	All participants
Core 1: Fundamental prerequisite mathematical ideas	18%	36%	55%
Core 2: Manipulative representations	28%	21%	49%
Core 3: Motivation to learn mathematics	4%	2%	7%

Discourse pertinent to Core 2 involved discussion and negotiation on the use of manipulative representations. The following dialogue shows an example in a discussion on an MGA module relating to the side lengths of triangles. The discourse focused on changing students' experience of operating manipulative representations.

(2-1) Designer B: To save time, we could divide the whole class into three groups; one group finds acute triangles, one group finds right triangles, and one group finds obtuse triangles [...]

(2-2) Leader B: The students would then be disadvantaged in that some of them would not have experience with right triangles and some would not have experience with acute triangles. Students would be forced to learn through the experiences of others. I think that allowing each group to identify one set for each type of triangle would be sufficient. Each group forms one set of acute triangles, one set of right triangles, and one set of obtuse triangles. Every group then has three types of experience and reports to the whole class. Each group does the same thing but with various lengths [...]

In this dialogue, the designer intended to let each group of students focus on assembling one type of triangle to save time; however, the leader suggested that he rearranged the tasks for each group to give students various experiences ("Each group forms .... Every group then has three types of experiences"), but the quantity of each type of triangle would be reduced ("... allowing each group to identify one set for each type ...").

As indicated in Table 2, the designers mostly focused on Core 2, whereas the leaders focused mostly on Core 1. This indicates that the leaders cared more about which mathematical ideas should be developed in students, whereas the designers cared more about the process of operating manipulative representations. Increasing mathematics learning motivation through gamified activities is a core idea of JDM (Core 3), but it was rarely the focus of discourse (7%). Every time a topic related to Core 3 was mentioned by a leader or designer, the topic was changed by the subsequent speaker. In the following dialogue, one leader asked, "Will the children like this?" after the designer introduced a gamified activity, but no one provided any feedback to the leader's question. The designer continued explaining the fundamental prerequisite in the module.

(3-1) Designer E: [...] The person who has built more houses wins the game.

(3-2) Leader C: Will the children like this?

(3-3) Designer E: The development of the concept regarding similarity is not required. I just want the students to develop the concepts of enlargement and reduction.

Discourse Form

Investigation of discourse form was to explore the categories of collaborative dialogue, namely statement, clarification, comparison, analysis, assertion, argument, and synthesis. Table 3 shows that assertions (26%) and statements (24%) were the most common forms of discourse, followed by analysis (20%), clarifications (18%), and arguments (15%). Comparisons and synthesis rarely occurred (3% and 2%, respectively). Statements primarily originated from designers introducing their modules and modifications after instructional design implementation (Designers: 18%; Leaders: 6%). Assertions were made mostly by leaders when they evaluated the designers' modules and gave opinions on how they could be modified (Leaders: 23%; Designers: 3%). In some cases, leaders provided examples to support their assertions to help designers understand. These instances increased the likelihood that designers would modify their modules compared with when leaders voiced assertions composed of only abstract concepts. For example, in a discussion on an activity aiming to help students understand that not any three side lengths can create a triangle, one leader asserted that many activities can be used to help students and then provided several examples, such as "each member in a group of four students takes three sticks randomly [...] this group may form only three triangles, another group may form four triangles, and maybe one group forms no triangles." Another leader asserted a principle that "the method of thinking is very important. The key to mathematics teaching is to help students learn how and where to focus, in addition to teaching materials and knowledge" but did not provide examples of activities that can achieve this goal. The designer used the first leader's statements to think and then provided feedback. She responded,

> This is also one approach [to exploring the conditions]. Discussing those that cannot be arranged into triangles. Then, students will share their own experiences, similar to what I wrote in the module. The students will then say [....].

The designer modified her module accordingly, but she did not respond to the second leader's statements or make any modifications based on them.

Table 3
Percentages of Various Types of Discourse Form

	Designers	Leaders	All participants
Assertion	3%	23%	26%
Statement	18%	6%	24%
Analysis	7%	13%	20%
Clarification	8%	11%	18%
Argument	9%	6%	15%
Comparison	1%	2%	3%
Synthesis	1%	1%	2%

Clarification occurred when participants did not understand the designers' introduction of their module or the leaders' evaluations or suggestions. Argument by leaders occurred primarily when leaders used theories of mathematics education or their own experiences to persuade the designers to modify their modules. The leaders used two categories of reason: student cognition and the nature of mathematics (e.g., logical and abstract language use, the essence of each mathematical field, and literacy that should be developed in mathematics). Designers' arguments occurred primarily when they explained their reasons for certain designs or persuaded leaders to retain the current versions of their modules. The designers' reasons had four categories: student learning situations from their teaching experiences, practical considerations regarding teaching (e.g., required time for the activity, and relevance to exams), the core ideas of JDM, and the perspectives obtained from the leaders. The following dialogue shows an example in a discussion on helping students construct the concept pertinent to the Pythagorean Theorem.

(4-1) Leader C: If you can write the side length and the area on the back of the square, they don't have to measure them.

(4-2) Designer B: What I provide has cells on it.

(4-3) Leader C: They still have to calculate it.

[...]

(4-4) Designer B: But this is for lower secondary students...they are lower secondary students.

(4-5) Leader C: I'm not saying that it's difficult, but it wastes time...

(4-6) Designer B: Actually, I ask students to memorise the square of 29, since a combination [of triangle side lengths] in the Pythagorean Theorem is 20, 21, and 29...

(4-7) Leader B: What are the mathematical ideas you want to develop in students? You can give them sufficient information because calculation isn't important. Let them focus on what we want.

In this dialogue, the leaders attempted to persuade the designer to provide the areas on the manipulatives to reduce the calculations required of students (4-1) and focus on student exploration (4-5 and 4-7), whereas the designer attempted to persuade the leaders to retain the current version of the module by describing their understanding of the students' situation on the basis of their teaching experiences (4-4) and their intention to help students succeed on the exams (4-6). The two cohorts intended to persuade each other using their different perspectives based on their various experience and expertise.

Discourse Flow

Investigation of discourse flow was to explore the functional dualism and coherence. Four types of discourse flow were identified (Table 4). Most of the discourse are of the dialogic coherent and the univocal incoherent types. The following exemplified the two types of the highest percentages.

Table 4
Percentages of Various Types of Discourse Flow

	Coherence	Incoherence	Total
Dialogic discourse	42%	3%	44%
Univocal discourse	3%	53%	56%
Total	44%	56%	

The following example presents a dialogic coherent discourse on a module to develop the concept that the sum of the lengths of any two sides of a triangle is larger than the length of the third side by enabling them to manipulate geometric sticks.

(5-1) Designer A: […] Manipulations simply enable students to see […] students still don't understand that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. Assembly is one thing, but students may not develop an understanding of the knowledge itself. Second, […] Most students think that any three geometric sticks can be used to form a triangle.

(5-2) Leader A: Regarding your remark that most students believe that any three sticks can form a triangle […] if each member in a group of four students takes three sticks randomly […], this group may form only three triangles, another group may form four triangles, and maybe one group forms no triangles […] the students would then understand that some conditions must exist in which a triangle cannot be formed […]

(5-3) Designer A: This is also one approach [to exploring the conditions] […] similar to what I wrote in the module […] Students would induce that three sticks of the same colour can form a regular triangle, which is not the case. Students would then discover that if the combined length of the two shorter sticks is less than that of the third one, a triangle can't be formed. This is the first experience we want them to have […]

(5-4) Leader A: […] Suppose that I have two sides of certain lengths and ask the students to determine the range of the length of the third side that they can use to form a triangle. This activity focuses on how short the third side can be. Suppose that the unit of measurement for the sides is 1 in centimetres; the students can then perceive the [length of the third side in terms of] natural numbers, increasing from 1 cm, 2 cm…

Leader A comprehended Designer A's thoughts ("Regarding your remark that…" in 5-2), responded to Designer A's concerns of "Assembly is one thing…not develop an understanding of the knowledge itself" and "Most students think that any three geometric sticks can be used to form a triangle" in 5-1, and suggested feasible learning activities that could fulfill Designer A's expectations and provide meaningful support ("Suppose that the unit of measurement for the sides is 1…natural numbers…" in 5-4 and "each member in a group of four students takes three sticks randomly …" in 5-2). Designer A's response in 5-3 ("… which is not the case … a triangle cannot be formed") demonstrated her using Leader A's narrative as a thinking device. The final version of Designer A's module, which incorporated the two suggested activities, demonstrated Designer A's understanding and acceptance of Leader A's suggestions. Furthermore, Designer A's remarks in 5-3 revealed her style of thinking. First, Designer A recalled her module in which she asked students to form triangles with three geometric sticks using one, two, and three colours. She then considered the activity suggested by Leader A in 5-2 to anticipate her students learning in it. The designer's engagement in thinking had to start with the familiar; subsequently, the designer could then understand the leader's ideas. In this case, Leader A listened to Designer A and gave the designer time to complete her thinking process. This example demonstrates coherent leader–designer discourse that comprises two-way interactive dialogue.

The following example presents a univocal incoherent discourse that resulted in the designer not modifying the module accordingly. The discourse was focused on a module to help students construct concrete images for enlarging and shrinking graphs to prepare them for lessons on similar figures. The learning activity prompts each group of students to draw a card showing the national flags of various countries and then rescale the flag.

(6-1) Designer F: Will the groups that draw the cards of the same national flag draw the same [sized] flags? Is it possible that, for example, two groups take cards of a Swiss flag and draw the flag with different scales?

(6-2) Designer E: […] There are no duplicate cards in my design […]

(6-3) Designer F: Suppose that there were, then one group could enlarge [the flag] by a factor of four and the other could enlarge [the flag] by a factor of five to produce flags of different sizes.

(6-4) Designer E: I think that I can let the students do this […] to determine whether the two images are the same or similar. For these two groups, I think that there are many questions for them to discuss and clarify through discussion rather than by me telling them the answers […]

(6-5) Leader D: Let's not talk about the game. Take out the images [of the flags]. I would like to say something. For example, in these images, the main objective is to recognise the images. What is the key to recognising the images? After we humans learn geometry, the key to recognising images is identifying basic elements that we have learned during geometry class. How do you describe this image? A rectangle that is cut into three pieces. Many will discuss the various sizes of the three pieces or various approaches to cutting. You would have a description of the basic elements of the image and how these elements are assembled [….] A complex geometric image is actually composed of basic images. This is a very basic notion. In fact, this is a fundamental mathematical concept. However, it isn't discussed in our geometry textbooks [….] Therefore, this learning material has two levels. The first level is recognition, and the second level is picturing figures similar to those learned at the lower secondary school level […]

(6-6) Designer G: Do we have to consider the cultural connotations or meanings of the national flags?

(6-7) Designer E: I've surfed the web. There's a lot of information […]

As revealed in 6-1 through 6-4, the designers discussed possible modifications to the module (allowing students to enlarge the flags to different sizes) to improve student understanding and facilitate their learning of similar figures. Their discussion was interrupted when Leader D introduced knowledge on the features of geometry learning, in 6-5. Leader D did not fulfill the designers' expectations. Although Leader D mentioned foundational concepts in geometry, his ideas were not related to the module, and his comment interrupted the thoughts. Thus, the designers could not use Leader D's remarks as a thinking device to extend the concepts; instead, they directly transitioned to the subsequent topic discussed in 6-6 and 6-7, and Designer E's modification to the module was unrelated to Leader D's comment. The discourse revealed the incoherent and univocal nature of the dialogue.

Discussion and Conclusion

Before discussing possible implications for the research on TPD programs, which emphasise designing teaching materials through co-construction, we would like to recall the limitations of the present study, which indicate the interpretations of the evidence with care. The participating teachers in this study were those who had participated in the MGA module designer workshops. For eligibility to participate in this type of workshops, the teachers were required to have participative experience in another type of JDM workshops where they were trained to teach mathematics using MGA modules. Therefore, the teachers in the study already understood the JDM ideals before they attended the designer workshops, supported the promotion of those ideals, and were willing to learn more and do more by designing innovative and unfamiliar tasks. Consequently, the feasible dialogical modes revealed in this study may not be generalisable to all types of intentions to conduct co-construction between teachers and teacher educators. However, the findings based on this research background might suggest the importance of teachers' understanding and endorsement of the ideals promoted by the TPD programs to the success of attaining the program objectives (Locke et al., 1998; Patrick, 2022).

The discourse focus is mainly on developing fundamental prerequisite mathematical ideas (Core 1) and constructing a concrete reference for mathematical concepts through operating manipulative representations (Core 2), rather than on increasing motivation for mathematics learning through gamified activities (Core 3). This suggests a greater focus of the participants on student cognitive achievement than their affective development—even in a program that aimed to improve students' low performance in the affective facets of learning mathematics (Wang et al., 2021). Two following two reasons can possibly explain this observation. First, leaders and designers may have believed that identifying the fundamental prerequisite mathematical ideas that require reinforcement and arranging manipulative representations for student learning are more critical than planning enjoyable activities and games to motivate student learning, reflecting East Asian identity in mathematics proposed by Leung (2001). Second, helping students understand mathematics and offering them opportunities to engage in hands-on activities are effective approaches to increasing their learning motivation (Wang & Hsieh, 2016). In addition, the leaders (teacher educators) cared more about which fundamental prerequisite mathematical concepts should be developed. In contrast, the designers (teachers) cared more about the process of operating manipulative representations to construct concepts. This highlights the distinct perspectives of the two cohorts along the spectrum: from the theoretical stance of teacher educators to the practical orientation of mathematics teachers. It underscores the potential benefits of implementing co-construction (Jones & Pepin, 2016).

Concerning discourse form, the leaders employed assertion and analysis much more than the designers did, whereas the designers employed arguments slightly more than the leaders did. These findings indicate that the leaders were accustomed to providing ideas to the designers without offering reasons, and they attempted to instruct the designers using detailed analysis. In contrast, the designers seemed to feel obligated to provide reasons for their decisions. In addition, the discourse flow results revealed that a substantial amount of discourse was of the univocal and incoherent types. From the perspective of Engeström's (1987) expansive model of the human collective activity system, these findings may relate to the rules applied to the perceived positions and unequal power between the two cohorts (teacher educators and teachers) and the division of labour in which teacher educators provide guidance and suggestions, and teachers ask for the teacher educators' approval. This revealed difficulties in implementing co-construction between teachers and teacher educators. Behind this dynamic lies the influence of Confucian culture. In this cultural context, a teacher's role is highly esteemed as they are considered to possess knowledge and truth. Challenging a teacher's ideas is seen as impolite (Chan & Chan, 2005). Leung (2001) pointed out that even those adopting a facilitative role rather than an instructive role in East Asian mathematics classrooms are deemed inadequate if they do not have solid knowledge. This situation is particularly pronounced between teachers and their teachers (teacher educators), given the high regard for theoretical knowledge in Chinese society where university-based teacher educators are viewed as representatives of theory and authorities of knowledge (Hsieh et al., 2018).

Furthermore, several pragmatic reasons may also explain these findings. The leaders' typical practice of training teachers may have influenced their behaviours more than their co-construction trial with designers to produce MGA modules. In addition, the phenomenon may have been pertinent to the MGA module designer workshops being part of a product-based TPD program for producing MGA modules. The leaders were under time constraints to cooperate with the designers to produce the modules. Another possible reason is related to module acceptance by JDM. All modules are evaluated by experts invited by the JDM to decide whether they can be accepted for future promotion. Teachers' willingness to participate in the workshops might decrease if their modules are rejected after all their efforts. Some interviewed leaders mentioned the pressure of balancing the co-construction process, time limitations, and production of acceptable modules. Regardless of the reasons, the assertive, incoherent, and univocal nature of the discourse revealed shortcomings that leaders can consider when developing their skills as teacher educators.

The discourse flow findings revealed that when the leaders and designers conversed at a coherent level (fulfilling each other's expectations for discourse)—that is, when they had the same objects from the perspective of the expansive model of the human collective activity system (Engeström, 1987)—they modified MGA modules more easily. Furthermore, concrete examples of feasible approaches can enable designers to understand the leaders' thoughts and modify their modules accordingly, despite the leaders employing the discourse form of assertion. In contrast, introducing theories without concrete examples in learning activity design does not encourage designers to modify their modules. The designers' thought patterns were based on familiarities, such as their teaching experience, or the potential thoughts and learning behaviours of their students; teachers use these experiences or thoughts to understand the ideas and suggestions of leaders. Leaders can help by providing designers with suggestions based on their thought patterns. The findings, however, also uncovered that, in addition to serving as a foundation, the designers' teaching experiences strongly influenced their module designs. When designers made decisions based on their teaching experience, persuading them to modify their modules was difficult.

Although co-construction and co-learning among participants are increasingly considered key to successful teacher professional development, not all implementations can attain objectives, produce productive outcomes, or fulfill teachers' needs (e.g., Darling-Hammond et al., 2017; Little, 2003; Rigby et al., 2020; Supovitz, 2002). For example, some teachers in studies by Hargreaves (1991, 1994) and Patrick (2022) described certain collaboration processes heavily controlled by leaders. The present study also revealed similar difficulties regarding teachers and teacher educators working collaboratively on equal standing to co-construct teaching materials. The following evidence, however, shows the feasibility (though not perfection) of co-construction: the different focuses of the teachers and the teacher educators respectively reflecting their practical and theoretical perspectives, the higher percentage of the teachers' discourse form of argument than that of the teacher educators, and almost half of the discourse being of dialogic type. By analysing the discourse between teachers and teacher educators using the MGA module designer workshops in JDM as an instrumental case study, we better understand how teachers and teacher educators communicate during teacher professional development processes in an East Asian culture, which is affected by both traditional Chinese-rooted and Western-influenced educational characteristics (Wang & Hsieh, 2017).

Furthermore, this study revealed feasible dialogic modes between teachers and teacher educators that make co-construction possible to produce teaching materials that are innovative and unfamiliar to the two cohorts. Previous studies suggested that perspectives of instructional quality in mathematics classrooms could vary in various cultures (Bryan et al., 2007; Wang et al., 2021), and the present study's findings might reveal the extension of this idea to the quality of TPD. As discussed earlier, we did not consider the two cohorts on equal standing in co-construction. However, regarding the participants' learning, the two cohorts evaluated the workshops positively and confirmed their successful learning (see Lin et al., 2018, Wang et al., 2021 for details). Regarding the production of MGA modules, almost all the trials of designing were successful. Although studies in the Western context indicated teachers asking for autonomy in TPD (e.g., Hargreaves, 1994; Patrick, 2022), the feasible dialogical modes for co-

construction in TPD in East Asia might be characterised as teacher educator-centred but teacher-focused, similar to Kaur (2009) described good mathematics teaching in this area.

Based on the human nature to shift from a state of disequilibrium to equilibrium (Piaget, 1952), Zaslavsky (2005) emphasised the importance of addressing uncertainties—such as cognitive conflict, doubt, and perplexity—in mathematical tasks. He suggested that social interactions can facilitate learners in acquiring mathematical knowledge. In these social interactions, discourse is a critical component, whether viewed from a philosophical perspective (on the origins and construction of human knowledge) or from a psychological angle (pertaining to an individual's development of understanding). This holds true for the learning of mathematical and mathematical pedagogical knowledge by mathematics teachers and teacher educators. However, resolving mathematical pedagogical tasks is rooted more in shared meanings among participants than in solving mathematical tasks. These shared meanings can differ substantially across cultures. Within different cultures, the focus, form, and flow of discourse for co-construction can vary. Likewise, the uncertainties brought about by discourse in mathematical pedagogical tasks and their apt resolutions can diverge. Further exploration of these differences across cultures can contribute to the development of TPD knowledge in our field.

Studies exploring the co-construction between teachers and teacher educators to design innovative tasks are still scant. Effective methods that TPD programs can employ to overcome the difficulties revealed by the present study require further investigation. The feasible dialogical modes identified in this study can be referenced by hosts of TPD workshops, and their applicability in various cultures could be studied further.

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Ethics Declarations

Ethical approval

This research met the requirements of ethical approval for research in the field of mathematics education in Taiwan at the time the study was conducted. Informed consent was obtained from all participants for the publication of their data.

Competing interests

The authors declare there are no competing interests.

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References

Arbaugh, F., & Brown, C. A. (2005). Analyzing mathematical tasks: A catalyst for change? *Journal of Mathematics Teacher Education, 8*(6), 499–536.

Ausubel, D. P. (1961). In defense of verbal learning. *Educational Theory, 11*(1), 15–25.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). Jossey-Bass.

Bardy, T., Holzaepfel, L., & Leuders, T. (2021). Adaptive tasks as a differentiation strategy in the mathematics classroom: Features from research and teachers' views. *Mathematics Teacher Education and Development, 23*(3), 26–53.

Beck, C., & Kosnik, C. (2006). *Innovations in teacher education: A social constructivist approach.* State University of New York Press.

Boston, M. D., & Smith, M. S. (2011). A ''task-centric approach'' to professional development: Enhancing and sustaining mathematics teachers' ability to implement cognitively challenging mathematical tasks. *ZDM-Mathematics Education, 43*(6-7), 965–977. https://doi.org/10.1007/s11858-011-0353-2

Bragg, L. A., & Lang, J. (2018). Collaborative teacher educator reflection as an approach to redesigning a mathematics education assessment task. *Mathematics Teacher Education and Development, 20*(3), 80–101.

Bruner, J. (1966). *Toward a theory of instruction.* Belknap Press.

Bryan, C. A., Wang, T., Perry, B., Wong, N.-Y., & Cai, J. (2007). Comparison and contrast: Similarities and differences of teachers' views of effective mathematics teaching and learning from four regions. *ZDM-Mathematics Education, 39*(4), 329–340. https://doi.org/10.1007/s11858-007-0035-2

Chan, K.-L., & Chan, C. L. W. (2005). Chinese culture, social work education and research. *International Social Work, 48*(4), 381–389. https://doi.org/10.1177/0020872805053461

Chen, J.-C., Lin, F.-L., & Yang, K.-L. (2018). A novice mathematics teacher educator–researcher's evolution of tools designed for in-service mathematics teachers' professional development. *Journal of Mathematics Teacher Education, 21*, 517-539. https://doi.org/10.1007/s10857-017-9396-9

Coburn, C. E., Penuel, W. R., & Geil, K. E. (2013). *Research-practice partnerships: A strategy for leveraging research for educational improvement in school districts.* William T. Grant Foundation.

Darling-Hammond, L., Hyler, M. E., Gardner, M., & Espinoza, D. (2017). *Effective teacher professional development*. Learning Policy Institute.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research.* Orienta-Konsultit.

Fleming, A., Roble, A., Yao, X., & Brosnan, P. (2015, Nov). *Building pedagogical capacity through task design and implementation.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI.

Hargreaves, A. (1991). Contrived collegiality: The micropolitics of teacher collaboration. In J. Blasé (Ed.), *The politics of life in schools* (pp. 46–72). SAGE Publications.

Hargreaves, A. (1994). *Changing teachers, changing times: Teachers' work and culture in the postmodern age*. Cassell.

Horn, I. S. (2010). Teaching replays, teaching rehearsals, and re-visions of practice: Learning from colleagues in a mathematics teacher community. *Teachers College Record*, *112*(1), 225–259. https://doi.org/10.1177/016146811011200109

Hsieh, F.-J. (2006). The foundations of theories. In Department of Mathematics and Information Education, National Taipei University of Education (Ed.), *The collection of the results in the classroom observation workshops of teacher professional development for advisory groups of lower secondary mathematics teachers* (pp. 5). Department of Mathematics and Information Education, National Taipei University of Education.

Hsieh, F.-J., Lu, S.-S., Hsieh, C.-J., Tang, S.-Z. & Wang, T.-Y. (2018). The conception of mathematics teachers' literacy for teaching from a historical perspective. In Y. Li & R. Huang (Eds.), *How Chinese Acquire and Improve Mathematics Knowledge for Teaching* (pp.37-56). Brill: Sense.

Huizinga, T., Handelzalts, A., Nieveen, N., & Voogt, J. (2015). Fostering teachers' design expertise in teacher design teams: conducive design and support activities. *Curriculum Journal, 26*(1), 137–163. https://doi.org/10.1080/09585176.2014.990395

Jones, K., & Pepin, B. (2016). Research on mathematics teachers as partners in task design. *Journal of Mathematics Teacher Education*, *19*(2–3), 105–121. https://doi.org/10.1007/s10857-016-9345-z

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore Grade 8 classrooms: A juxtaposition of teachers' practice and students' perception. *ZDM-Mathematics Education, 41*(3), 333–347. https://doi.org/10.1007/s11858-009-0170-z

Kontkanen, J., Kärkkäinen, S., Dillon, P., Hartikainen-Ahia, A., & Åhlberg, M. (2016). Collaborative processes in species identification using an internet-based taxonomic resource. *International Journal of Science Education, 38*(1), 96–115. https://doi.org/10.1080/09500693.2015.1129469

Kumar, R. S., & Subramaniam, K. (2015). From "following" to going beyond the textbook: Inservice Indian mathematics teachers' professional development for teaching integers. *Australian Journal of Teacher Education, 40*(12), 86–103. https://doi.org/10.14221/ajte.2015v40n12.7

Lee, H.-J., & Özgün-Koca, S. A. (2016). Professional development for mathematics teachers: Using task design and analysis. *Current Issues in Education,19*(2), 1–17.

Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*(1), 35–51. https://doi.org/10.1023/A:1017936429620

Leung, F. K. S. (2006). Mathematics education in East Asia and the West: Does culture matter? In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural tradition: A comparative study of East Asia and the West* (pp. 21–50). Springer.

Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. J. (Eds.) (2006). *Mathematics education in different cultural traditions: A comparative study of East Asia and the West*. Springer.

Lin, F.-L., Wang, T.-Y., & Yang, K.-L. (2018). Description and evaluation of a large-scale project to facilitate student engagement in learning mathematics. *Studies in Educational Evaluation, 58*, 178-186. https://doi.org/10.1016/j.stueduc.2018.03.001

Lin, P.-J., & Li, Y. (2009). Searching for good mathematics instruction at primary school level valued in Taiwan. *ZDM-Mathematics Education, 41*(3), 363–378. https://doi.org/10.1007/s11858-009-0175-7

Little, J. W. (1990). The persistence of privacy: Autonomy and initiative in teachers' professional relations. *Teachers College Record*, *91*(4), 509–536. https://doi.org/10.1177/016146819009100403

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teachers College Record*, *105*(6), 913-945. https://doi.org/10.1111/1467-9620.00273

Locke, E. A., Latham, G. P., & Erez, M. (1988). The determinants of goal commitment. *Academy of Management Review*, *13*(1), 23–39. https://doi.org/10.2307/258352

Lotman, Y. M. (1988). Text within a text. *Soviet Psychology, 24* (3), 32-51. https://doi.org/10.2753/RPO1061-0405260332

Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *TIMSS 2011 International results in mathematics.* TIMSS & PIRLS International Study Center, Boston College.

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 International results in mathematics.* TIMSS & PIRLS International Study Center, Boston College.

Ngcoza, K., & Southwood, S. (2015). Professional development networks: from transmission to co-construction. *Perspectives in Education, 33*(1), 1–12.

Organisation for Economic Co-operation and Development. (2013). *PISA 2012 Results in focus: What 15-year-olds know and what they can do with what they know.* https://www.oecd.org/pisa/keyfindings/pisa-2012-results-overview.pdf

Organisation for Economic Co-operation and Development. (2016). *PISA 2015 results (Volume I): Excellence and equity in education*. http://dx.doi.org/10.1787/9789264266490-en

Orland-Barak, L. (2006) Convergent, divergent and parallel dialogues: knowledge construction in professional conversations. *Teachers and Teaching: Theory and Practice, 12*(1), 13–31. https://doi.org/10.1080/13450600500364547

Pang, J. (2009). Good mathematics instruction in South Korea. *ZDM-Mathematics Education, 41*(3), 349–362. https://doi.org/10.1007/s11858-009-0169-5

Patrick, S. K. (2022). Collaborating for improvement? Goal specificity and commitment in targeted teacher partnerships. *Teacher College Record, 124*(1), 164–190. https://doi.org/10.1177/01614681221086104

Penuel, W. R., Fishman, B. J., Cheng, B. H., & Sabelli, N. (2011). Organizing research and development at the intersection of learning, implementation, and design. *Educational Researcher, 40*(7), 331–337. https://doi.org/10.3102/0013189X11421826

Pepin, B., Gueudet, G., & Trouche, L. (2017). Refining teacher design capacity: Mathematics teachers' interactions with digital curriculum resources. *ZDM-Mathematics Education*, *49*(5), 799–812. https://doi.org/10.1007/s11858-017-0870-8

Peressini, D. D., & Knuth, E. J. (1998). Why are you talking when you could be listening? The role of discourse and reflection in the professional development of a secondary mathematics teacher. *Teaching and Teacher Education, 14*(1), 107–125. http://dx.doi.org/10.1016/S0742-051X(97)00064-4

Piaget, J. (1952). *The origins of intelligence in children.* International University Press.

Remillard, J. T. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text. *The Elementary School Journal, 100*(4), 331–350. https://doi.org/10.1086/499645

Rigby, J. G., Andrews-Larson, C., & Chen, I.-C. (2020). Learning opportunities about teaching mathematics: A longitudinal case study of school leaders' influence. *Teachers College Record*, *122*(7), 1–44. http://dx.doi.org/10.1177/016146812012200710

Schön, D. A. (1987). *Educating the reflective practitioner.* Jossey-Bass.

Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester, Jr., (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 199–204). National Council of Teachers of Mathematics.

Supovitz, J. A. (2002). Developing communities of instructional practice. *Teachers College Record*, *104*(8), 1591–1626. https://doi.org/10.1111/1467-9620.00214

Thompson, D. R., & Rubenstein, R. N. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. *Mathematics Teacher, 93*(7), 568–574.

Wang, T.-Y., & Hsieh, F.-J. (2016). What teachers should do to promote affective engagement with mathematics—from the perspective of elementary students. *Proceeding of the 40th Annual Conference of the International Group for the Psychology of Mathematics Education*, Hungary (Vol. 3, pp. 371-378).

Wang, T.-Y., & Hsieh, F.-J. (2017). Taiwanese high school students' perspectives on effective mathematics teaching behaviors. *Studies in Educational Evaluation, 55*, 35-45. https://doi.org/10.1016/j.stueduc.2017.06.001

Wang, T.-Y., Lin, F.-L., & Yang, K.-L. (2021). Success factors for a national problem-driven program aimed at enhancing affective performance in mathematics learning. *ZDM - International Journal on Mathematics Education,53*, 1121-1136. https://doi.org/10.1007/s11858-021-01285-8

Watson, A., & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh & T. Wood (Eds.), *Tools and processes in mathematics teacher education* (pp. 109–134). Sense Publishers.

Wilson, P. H., Sztajn, P., Edgington, C., Webb, J., & Myers, M. (2017). Changes in teachers' discourse about students in a professional development on learning trajectories. *American Educational Research Journal, 54*(3), 568–604. https://doi.org/10.3102/0002831217693801

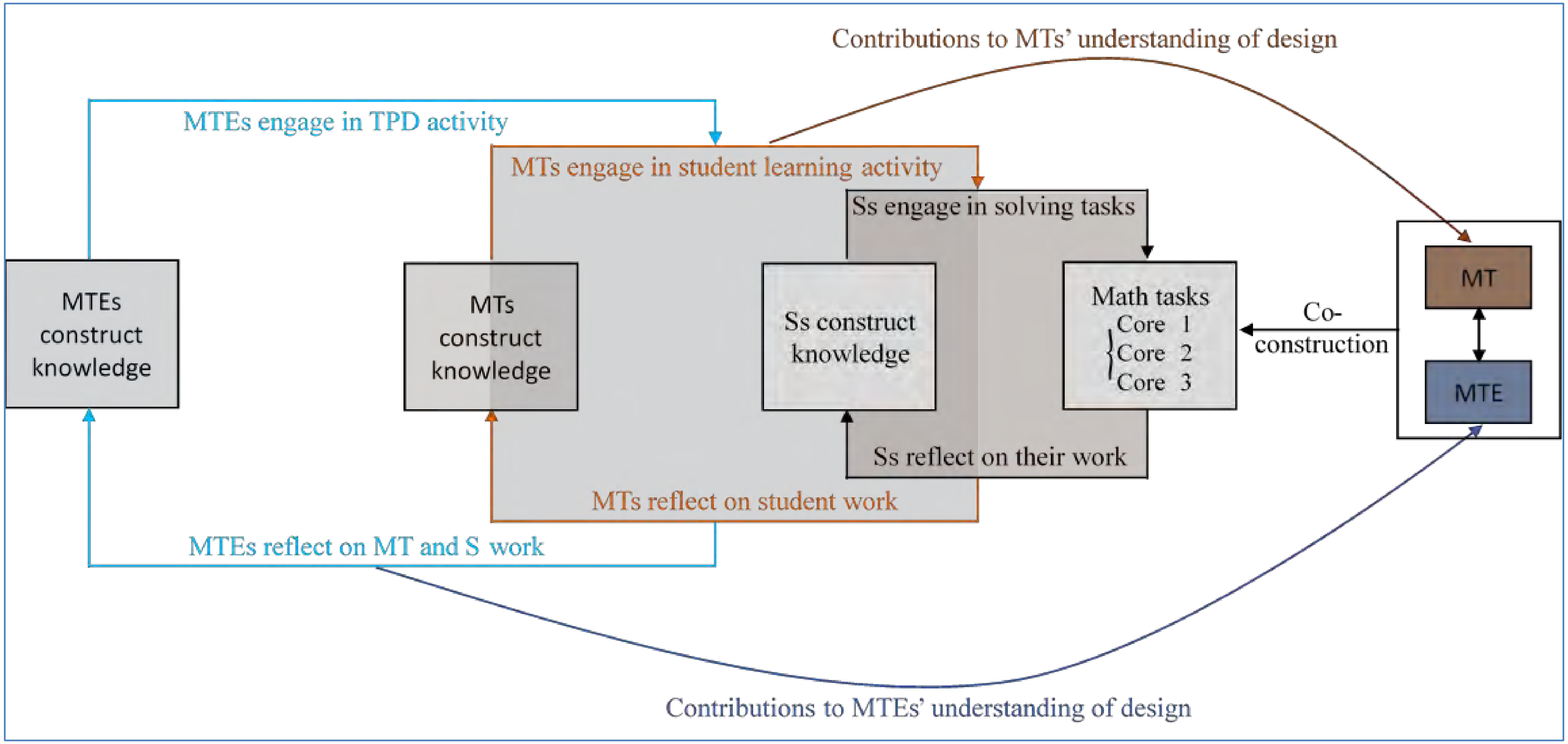
Yang, K-L., Lin, F-L., & Tso, T-Y. (2022). An approach to enactivist perspective on learning: mathematics-grounding activities. *Asia-Pacific Education Researcher, 31*(6), 657-666. https://doi.org/10.1007/s40299-021-00616-3

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, 60(3), 297–321. https://doi.org/10.1007/s10649-005-0606-5

Zaslavsky, O. (2008). Meeting the challenges of mathematics teacher education through design and use of tasks that facilitate teacher learning. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 93–114). Sense Publishers.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth through practice. *Journal of Mathematics Teacher Education*, 7(1), 5–32. https://doi.org/10.1023/B:JMTE.0000009971.13834.e1

Appendix



Note: Abbreviations: S, student; MT, mathematics teacher; MTE, mathematics teacher educator.

Figure 1. Mathematics teachers and teacher educators' co-construction of tasks in product-based TPD.