Characterizing students' beliefs about mathematics as a discipline

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To fully possess mathematical competence and to understand its relevance, importance and aesthetics, it is essential to be aware of aspects of mathematics not only as a school subject but also as a scientific discipline. In a systematic literature review, the theoretical characterization of compulsory school students' beliefs about mathematics as a discipline is investigated, as well as the empirical tendencies in the nature of their actual beliefs. Furthermore, the valuation of these beliefs is addressed. The 18 included studies demonstrate a clear pattern in applying a dualistic/relativistic spectrum when characterizing and analysing students' beliefs about mathematics as a discipline, with students generally possessing dualistic beliefs, which is in contrast to what is favourable to their learning.

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1 Introduction

Mathematics is part of the education of all compulsory school students around the world. To be mathematically competent can be defined in many ways, but it is widely studied and generally agreed among researchers of mathematics education that the way in which students perceive the subject is an important factor for their motivation, their learning process, and their approach to mathematical problems etc. (e.g., Furinghetti & Pehkonen, 2002; McDonough & Sullivan, 2014). Several definitions of students' mathematics-related beliefs have been presented in existing literature (e.g., Underhill, 1988; Kloosterman, 1996; Op't Eynde et al., 2002), but not all of these definitions include the dimension of mathematics that exceeds the school subject (e.g., Op't Eynde et al., 2002). However, to fully possess mathematical competence and to understand its relevance, importance and aesthetics, it is essential to be aware of aspects of mathematics not only as a school subject but also as a *scientific discipline*, as pointed out by Niss & Højgaard (2011). The latter could also be characterized as the *nature* of mathematics, and includes the role of mathematics in the world, the development of mathematics, the methods used by mathematicians and the philosophy of mathematics, to name a few examples. Skemp (1976) already noticed the importance of students' beliefs about mathematics as a discipline in his distinction between instrumental and relational understanding, as did Schoenfeld (1985) with the introduction of the term "mathematical world view". Making students aware of mathematics in the world and as a discipline can both provide justification for the school subject and a sense of relevance as well as gives the subject a meaningful context. All of which may increase their motivation and benefit their learning.

As part of a larger PhD project aiming to develop students' beliefs about mathematics as a discipline, I wish to form an understanding of how students' beliefs about mathematics as a discipline or the nature of mathematics have been described and characterized in existing research. In this paper, I therefore approach the subject through a systematics literature review, serving two purposes: 1) to provide information of how students' beliefs about mathematics as a discipline can be categorized and analyzed, and 2) to detect tendencies in the nature of compulsory school students' beliefs about mathematics as a discipline.

2 Method

In order to select relevant literature in a systematic manner, certain criteria have been taken into account:

A. Only studies concerning students in primary and secondary school have been included in the review, as students on tertiary educational levels in many cases will have chosen a certain educational path and thereby have a bias in regard to their interest in mathematics.
B. Although there is a wide representation of studies concerning students' beliefs about mathematics in general, this review is restricted to address students' beliefs about mathematics as a discipline or the nature of mathematics. However, some studies addressing students' beliefs about "what is mathematics?" are included in the review as they cover the essence of beliefs about the nature of mathematics. As "mathematics as a discipline" is an ambiguous term, it might be characterized in several ways, including a variety of content. Thus, studies that cover only parts of mathematics as a discipline (e.g., beliefs about the history of mathematics, the role of mathematics in the world or beliefs about problem-solving) have not been included, as such a search might exclude content that some literature considers part of mathematics as a disci-

pline. Furthermore, this study investigates students' beliefs about mathematics as a discipline as an overall concept, not their beliefs on the individual parts or perspectives on this.

C. Only literature published within the last 20 years have been included in the performed searches under the assumption that previous relevant and important literature will be cited in more recent studies. Hence, some references with several citations in the selected studies, or references appearing to be relevant, have also been assessed and included, if fitting the inclusion criteria.
D. To ensure the validity of the included literature, only peer-reviewed studies have been included.
E. As determined by Furinghetti and Pehkonen (2002), the concept of beliefs is not clearly and somewhat unambiguously defined. Even though the term beliefs is the most commonly used in the resulting studies (15 studies), other terms are used as synonyms: conceptions (in 2 studies), views (in 3 studies), and images (in 1 study). These terms may on the other hand cover aspects that are not relevant to this study, which have been considered during the subsequent screening processes. An elaboration of how these concepts are interrelated or defined in the studies will not be a part of this paper. To leave room for the actual focus of this paper, they will instead all be considered as similar, if not identical, and quite closely related notions that in essence cover the same phenomenon.

Williams and Leatham (2017, p. 377) define the 20 most important journals in mathematics education. To cover these, I conducted searches in ERIC[1] and in Web of Science[2]. Furthermore, proceedings from the MAVI 16–35, PME 24–43 and CERME 2–11 conferences have been manually searched.

[1] Search string in ERIC: noft((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT ("STEM" OR teacher*)) AND la.exact("English") AND PEER(yes) AND pd(>20001231) (April 22, 2021)

[2] Search string in Web of Science TS=((belief* OR view* OR perception* OR conception* OR image* OR understanding*) AND math* AND (discipline OR nature) AND (student* OR pupil* OR child*) AND (school OR primary OR secondary) NOT (STEM OR teacher*)) (April 22, 2021)

2.1 Summary of search process

The search in databases resulted in 292 studies imported into the review software Covidence, whereof 7 duplicates were removed. 285 studies were screened against title and abstract using the above-mentioned criteria A through E (Table 1). This resulted in a further exclusion of 275 studies. 10 studies were imported from MAVI proceedings, 5 studies from CERME proceedings and 8 studies from PME proceedings. Hence, a total of 33 studies were full-text screened, leading to the exclusion of 26 studies, whereof 5 were excluded due to criterion A (wrong sample group), 20 due to criterion B (not mathematics as a discipline), and 1 due to criterion B (wrong aspect of affect). Finally, 7 studies were included in this review as well as 11 relevant references cited in the 7 included studies.

Table 1. Overview of review process and studies excluded based on inclusion criteria. Exclusions related to criteria C and D tool place in step 1.

Step 1:	292 references imported from databases for screening 7 duplicates removed (285 remaining)
Step 2:	285 studies screened against title and abstract 275 studies removed (10 remaining)
Step 3:	23 studies imported from conference proceedings for full-text assessment
Step 4:	23 + 10 = 33 studies assessed for full-text eligibility 26 studies excluded: 5 (criterion A); 20 (criterion B); 1 (criterion E). (7 remaining)
Step 5:	11 studies included from snowballing
Step 6:	In all 18 studies included

2.2 Analysis

The research and findings in the included studies have been analyzed from three perspectives. (1) Characterization of what constitutes beliefs about mathematics as a discipline is synthesized from nine studies. Where some researchers apply existing frameworks or categories to their data, others develop their own framework. Studies that do not present a clear framework, categorization or definition are not included in this perspective. (2) Thirteen of the studies present empirical findings that indicate what kind of beliefs students actually seem to possess. These findings are presented in the second section. (3) Eight of the included studies concern the quality of students' beliefs, strongly indicating that there are beliefs about mathematics that are considered "appropriate", "healthy" or "ideal" – beliefs that are preferable to others and thus should be pursued in the students' learning and their cognitive development. Hence, there are also beliefs that are inappropriate and undesired, generally because they do not support the students' learning, motivation, critical sense etc. This sort of ranking of beliefs is the theme of the third section.

3 Results concerning the characterization of beliefs about mathematics as a discipline

The characterization of what actually constitutes and is included in students' beliefs about mathematics as a discipline can be approached in different ways. One option is to present a set of categories or issues that define this form of beliefs, and thus list the content of mathematics as a discipline, as done in three of the included studies. These are presented in the following section. The subsequent section describes eight studies that apply another option, namely to describe the characteristics or quality of a person's beliefs within a spectrum.

3.1 Content of beliefs about mathematics as a discipline

Based on existing research, Borasi (1993) categorizes beliefs about mathematics in four categories: two concerning mathematical *activity* (nature and scope), and two concerning mathematical *knowledge* (nature and origin). Grouws (1996) operates with similar categories in his framework for analyzing students' conceptions of mathematics, but with a different definition of the dimensions of mathematical knowledge (*composition*, *structure* and *status*) and mathematical activity (*doing* mathematics and *validating* ideas in mathematics). Furthermore, Grouws (1996) adds the dimensions of *learning* mathematics and the *usefulness* of mathematics.

By including beliefs about the learning of mathematics, Grouws (1996) relates the discipline of mathematics to an educational context. However, in Jankvist's (2015) expansion of Op't Eynde et al.'s (2002) model of students' mathematics-related beliefs (Figure 1), he argues that where the original model concerns beliefs about mathematics in a school setting, the added dimension of mathematics as a discipline concerns issues related to *non-school* settings.

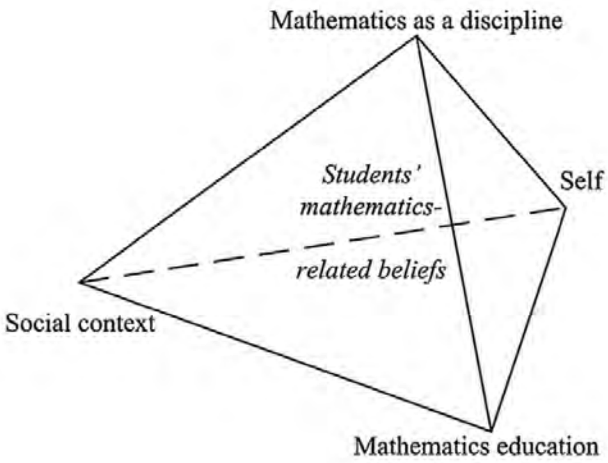

Figure 1. Jankvist's expansion of "Constitutive dimensions of students' mathematics-related belief systems" (Jankvist, 2015, p. 45). The bottom triangle constitutes the original model (Op't Eynde et al., 2002, p. 27).

The dimensions of the original model are (1) beliefs about mathematics education (beliefs about mathematics as a subject, mathematical learning and problem solving and mathematics teaching); (2) beliefs about the self (beliefs about self-efficacy, control, task-value and goal-orientation); and (3) beliefs about the social context, normally the classroom (beliefs about the social norms in the class, i.e., the role and the functioning of the teacher and the role and the functioning of the students aa well as beliefs about the socio-mathematical norms in the class). Included in the fourth dimension concerning mathematics as a discipline are beliefs about mathematics as a *pure science*, an *applied science*, a *system of tools for societal practice* as well as the *philosophical and epistemological nature* of mathematical concepts, theories etc. However, the dimensions of the belief system are interdependent. Beliefs about the issues connected to mathematics as a discipline are to a large degree developed within a school setting and only in the interplay between the three other dimensions. Therefore, the fourth dimension is placed outside the triangle, turning the model into a tetrahedron instead of a square, thus making the three original dimensions the basis on which the fourth is build. Inspired by Spangler (1992), Jankvist further characterizes this category of beliefs through a set of questions (Jankvist, 2015, p. 45):

> [H]ow, when and why mathematics came into being; if mathematics is discovered or invented; where mathematics is applied; if it has greater or lesser impact on society today than previously; if mathematics can become obsolete; what mathematicians do; if mathematics is a scientific discipline.

3.2 A spectrum of beliefs

Another way to characterize students' beliefs about mathematics as a discipline concerns the characteristics of the beliefs, or perhaps rather the quality or spectrum of how one perceives the nature of mathematics. This characterization is found in eight of the included studies. The majority presents spectra that range from seeing mathematics as a static, rigid and rule-based discipline, to a dynamic, relativistic and applicable "science of patterns", as described by Schoenfeld (1992, p. 334):

> At one end of the spectrum, mathematical knowledge is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and the relationships among them; knowing mathematics is seen as having mastered these facts and procedures. At the other end of the spectrum, mathematics is conceptualized as the "science of patterns," an (almost) empirical discipline closely akin to the sciences in its emphasis on pattern-seeking on the basis of empirical evidence.

A similar spectrum is described in Borasi (1993) and Grouws (1996), who characterize the range of mathematics-related beliefs from *dualistic* to *relativistic*, relying on the framework of Oaks (1989). Each of the aforementioned seven dimensions used by Grouws to define beliefs about mathematics as a discipline is described as a continuum with two poles (Table 2), illustrating the extremes of the spectrum.

Table 2. Dimensions for the conceptions of mathematics and their poles on a range from dualistic to relativistic (my extraction from Grouws, 1996).

Dualistic	**< ------------------------ >**	**Relativistic**
	1. composition of mathematical knowledge	
facts, formulas and algorithms	< ------------------------ >	concepts, principles and generalizations
	2. structure of mathematical knowledge	
collection of isolated pieces	< ------------------------ >	coherent system
	3. status of mathematical knowledge	
static entity	< ------------------------ >	dynamic field
	4. doing mathematics	
results	< ------------------------ >	sense-making
	5. validating ideas in mathematics	
outside authority	< ------------------------ >	logical thought
	6. essence of learning mathematics	
memorizing	< ------------------------ >	constructing and understanding
	7. usefulness of mathematics	
school subject with little value in life	< ------------------------ >	useful endeavour

In his study of secondary school students, Grigutsch (1998) also finds that the development of students' views of mathematics can be seen as two contrasting poles (*schema-orientation* (aspects S and RS below) and *process/application-orientation* (aspects P and A)). In essence, these two poles resemble the dualistic and relativistic perspectives, but Grigutsch characterizes the spectrum between them with five different aspects, thereby enabling a more detailed analysis of students' beliefs (Grigutsch, 1998, pp. 174-176):

> F: The Formalism-Aspect (mathematics as logical and precise thinking)
> P: The Process-Aspect (mathematics as a method for considering, understanding and solving problems)
> A: The Application-Aspect (mathematics as useful in daily life)
> S: The Schema-Aspect (mathematics as a collection of rules and procedures)
> RS: The Rigid Schema-Orientation (mathematics is learned (memorized) only to pass exams)

However, the Formalism-Aspect is not easily placed between the dualistic and relativistic poles, and thus Grigutsch's (1998) framework may also be perceived as different aspects that describe students beliefs from a perspective that do not operate within a spectrum.

Gattermann et al. (2012) distinguish between two contrasting categories in their study of students' epistemological beliefs in mathematics, namely *naïve* and *sophisticated* beliefs, the latter being more closely related to deep-processing learning. To measure the students' beliefs, they use a questionnaire composed by items from existing large-scale assessment tools such as PISA and TIMSS. The students' beliefs are assessed within six different conceptual aspects that are closely related to those of Grigutsch (1998). Three of them are related to the category of naïve epistemological beliefs and constitute a conception of mathematics similar to the dualistic view mentioned above: (1) rigid schemes ("exercises in mathematics always have only one right solution"), (2) schematic conception (mathematics as a collection of calculation methods and rules) and realistic conception ("all mathematical problems have already been solved"). Likewise, the aspects related to sophisticated epistemological beliefs resemble the relativistic view: (4) relativistic (mathematics as a coherent system), (5) processes (mathematics can be discovered and constructed by oneself) and (6) relevance/application (relevant for everyday life).

Several of the studies (Gattermann et al., 2012; Grady, 2018; Grevholm, 2011; Halverscheid & Rolka, 2006) rely on a categorization of beliefs (or views) that adds a third category, as presented by Ernest (1989) (instrumentalist, Platonist and problem-solving view) or the corresponding notions of Dionne (1984) (traditional, formalist and constructivist perspective). I shall therefore briefly summarize the essence of Ernest's notions. Parallel to the aforementioned dualistic view of mathematics, the *instrumentalist* view is characterized by perceiving mathematics as "a set of unrelated but utilitarian rules and facts" (Ernest, 1989, p. 249). In the *Platonist* view, mathematics is characterized as "a static but unified body of knowledge" (ibid.), while the *problem-solving* view, similar to the relativistic view, characterizes mathematics as "a dynamic, continually expanding field of human creation and invention" (ibid.). As was the case with the framework presented by Grigutsch (1998), Ernest's notions are not necessarily defined within a spectrum, but rather as a characterization of three different perspectives on mathematics that are not opposites.

An alternative approach to students' beliefs are introduced by Grady (2018). In her study, she presents a framework for describing and analyzing students' enacted conceptions of the nature of mathematics from their behaviour instead of the often used self-report data. From behavioural indicators, the framework can be used to assess to what degree students conceive mathematics as *sensible*, which is defined as viewing mathematics as "a coherent, connected system that can be reasoned about and used to describe and reason about the world at large" (Grady, 2018, p. 127). As the reader might notice, this definition has common features with both Oaks' relativistic view, Grigutsch's process/application-orientation and Ernest's problem-solving view. The degree to which students' hold such a conception is assessed based on four dimensions of behaviour (as well as the students actually stating that mathematics makes sense): 1. strategizing (e.g., discussing methods or seeking alternative solutions), 2. expecting explanations (e.g., justifying, reasoning and inquiring), 3. expecting/seeking connections (within mathematics and to other contexts), and 4. assuming authority (e.g., inventing problems of their own or checking answer with an alternative strategy). The framework thereby contributes to an understanding of the action-oriented dimensions of students' conceptions of the nature of mathematics.

Having established several frameworks for characterizing students' beliefs about mathematics as a discipline, next step is investigating what kind of beliefs students' actually hold when studied empirically, both in relation to the frameworks as well as in the form of concrete examples of students' beliefs.

3.3 Results concerning students' actual beliefs about mathematics as a discipline

Based on experience, discussions with teachers and students (Garofalo, 1989) and existing research (Schoenfeld, 1992), the selected literature offers concrete examples of beliefs about the nature of mathematics that typically are held by students. One of them is that mathematical problems only have one correct answer and that it can only be solved using the correct rule, formula or procedure, usually shown by the teacher. Thereby mathematics will most likely be perceived as a fragmented set of rules and formulas that must be memorized and applied appropriately. This is connected to another typical belief related to the nature of mathematics: that mathematics is not created by "ordinary" people, but must be transferred (normally from teacher or textbook to student) and memorized. Hence, students feel unable to produce mathematics on their own (Garofalo, 1989), and a deeper understanding of the rules and formulas becomes irrelevant, as does formal proof (Schoenfeld, 1992). Furthermore, mathematics is typically believed to have little relevance to the real world but is merely seen as a school subject (ibid.). These are generally beliefs that can be said to reflect what in the previous section is characterized as a dualistic perspective, which is confirmed by Underhill (1988) in his review of mathematics learners' beliefs. In general, students at all ages emphasize memorization and algorithms as important in mathematics, which foster what Skemp (1976) categorizes as instrumental learning, not relational understanding.

Regarding empirical findings, the majority of the studies included in this review, present results that confirm such a tendency. For example, Halverscheid and Rolka (2006) find that half of 28 fifth grade students hold an instrumentalist view of mathematics. Kloosterman (1996, 2002), Grootenboer (2003) Grevholm (2011) all find that students' beliefs about mathematics as a discipline are generally linked and maybe even limited to numbers and calculations, although there are indications that students do not seem to have given much consideration to mathematics as a discipline (Kloosterman, 2002, Grevholm, 2011).

Even though most research thus points to students beliefs about mathematics as a discipline being rather traditional/instrumental/dualistic, some of the included studies show a slightly more complex picture of students' beliefs. McDonough (1998) shows in her in-depth study of two third grade students' engagement in mathematical procedures that students' beliefs about the nature of mathematics might be more complex, subtle and broad than reported in other research studies. In ten one-to-one interviews with each student over a period over five months, the nature of mathematics is discussed through e.g., photographs of both school and non-school activities, personal definitions for mathematics and ending the sentence “Math is like…”. In both cases, what first appears to be a simple and clear classification of beliefs turns out to be quite complex and ambiguous during the analysis of the data collected. For example, numbers initially appear significant for one of the students, but during subsequent discussions, it becomes clear that she puts more emphasis on measurement and estimating and mainly relates mathematics to non-school activities.

Gattermann et al. (2012) find in their aforementioned study that the 145 secondary school students in average score relatively high on scales addressing the sophisticated epistemological beliefs. Their scores on naïve beliefs are proportionally low. However, the contrary applies to the aspects of relativistic conception (low score) and schematic conception (high score). Thereby, these students do not perceive mathematics as a coherent system, but rather a collection of exact methods and rules for calculation, even though they find mathematics process-oriented and useful in daily life.

According to Schoenfeld (1989), 230 mathematics students in grades 10-12 hold apparently contradicting beliefs about mathematics as a discipline. In their responses to a questionnaire concerning their mathematics-related beliefs, including their view on mathematics as a discipline, the students for example state that mathematics is a discipline of creativity, logic and discovery, but at the same time emphasize the importance of memorization in the learning of mathematics. The students generally separate school mathematics from abstract mathematics, and Schoenfeld suggests that the reason might be that the students’ behavior is driven by their experiences with mathematics rather than what they are told or what they value as “appropriate” beliefs. Likewise, both Schoenfeld (1992) and Garofalo (1989) stress that students to a large extent form their beliefs based on their experiences in the classroom, and that these beliefs thereby is a reflection of how mathematics is presented, performed and evaluated in the educational system. Grootenboer (2003) confirms this by pointing out that the views of students’ in his study are firmly grounded in school experiences.

A noteworthy result is presented in Grouws (1996), who compares the conceptions of mathematics of 55 talented and 112 average high school students. Here, he finds that while the two groups generally see mathematics as a dynamic and useful field, there are noteworthy differences in their conception of doing and learning mathematics. Where the average students – as in Gattermann et al. (2012) – largely follows the above described dualistic way of perceiving mathematics as a discrete system of facts and procedures based on memorization, the talented students view mathematics from a more relativistic perspective. They tend to see it as "a field composed of a system of coherent and interrelated concepts and principles, which is continuously growing. Doing mathematics is a sense-making process in which one must rely on personal thought and reflection to establish the validity of that knowledge" (Grouws, 1996, p. 31). A corresponding result is found by Grigutsch (1998) where the process/application-orientation is increasingly significant among 12th grade students in the high-performance class, compared to the students in the basic level class, who tend to have a more schema-oriented view of mathematics. In lower grades involved in the study (grade 6 and 9), the two poles of the beliefs spectrum are not as distinct, and the students beliefs seem to be more of a mix of the five different aspects in Grigutsch's framework (cf. the previous section). Both of these studies indicate that certain beliefs might be related to high performance in mathematics and thus are preferable and worth striving for. This issue is unfolded in the following section.

4 Are some beliefs better than others?

The overall impression from the studies included in this review is quite clear when it comes to what sort of beliefs to aim for in the teaching of mathematics, and which are considered unfavourable. In relation to the spectrum of beliefs, there seems to be consensus that beliefs belonging to the dualistic pole are not conducive for students' leaning, motivation or self-concept. According to Borasi (1993) and Schoenfeld (1992), such beliefs impoverish mathematics and do not reflect its nature. In contrast, the relativistic end of the spectrum is seen as unambiguously beneficial. For example, Gattermann et al. (2012) finds that sophisticated epistemological beliefs are more related to a higher degree of self-concept and performance compared to naïve beliefs. This corresponds with the results of Grouws (1996), who as mentioned relates the relativistic perspective to talented students, and Grigutsch (1998), who connects the process/application-orientation to both high performance, motivation and a positive self-concept in mathematics.

Spangler (1992) presents 11 open-ended questions aimed both at assessing students' beliefs about mathematics on the one hand, and at making the students aware of own their own beliefs. These questions indirectly indicate that the ideal beliefs belong to the relativistic end of the spectrum. For example, students are encouraged to consider the possibility that different answers to the same mathematical problem can be equally correct, that mathematics is more than memorization or computation, and that mathematics is used in many non-school situations, fields and careers. The behavioural indicators in the framework presented by Grady (2018) illustrate what kind of behaviour that is connected to such a perception (see previous section).

The ideal beliefs about mathematics as a discipline are in Jankvist (2015) presented as beliefs that are held evidentially (cf. Green, 1971), i.e., beliefs supported by evidence from examples, experience, reasoning etc. Evidentially held beliefs are more likely to be changed with reason or through reflection. Thus, mathematics education should aim for developing students' *reflected* image of mathematics as a discipline by providing opportunities for experiences and reflection. In a didactical perspective, Jankvist (2015) relates three types of mathematical overview and judgment to the development of students' beliefs. The three forms of overview and judgment are described in the Danish mathematics competencies framework, the so-called KOM-report (Niss & Højgaard, 2011, 2019), and concern: (1) the actual application of mathematics in other subject and practice areas; (2) the historical evolution of mathematics, internally as well as in societal context; (3) the nature of mathematics as a subject. These have a certain equivalence to one of the visionary aims set up by Ernest (2015) for school mathematics with an intention to "contribute to students' mathematical confidence, mathematical creativity, social empowerment and broader appreciation of mathematics" (Ernest, 2015, p. 189). Especially the latter of these aims (broader appreciation of mathematics) is related to the students' beliefs about the nature of mathematics and requires an increased awareness of the following aspects (p. 191):

- mathematics as a central element of culture, art and life, present and past, which permeates and underpins science, technology and all aspects of human culture.
- the historical development of mathematics, the social contexts of the origins of mathematical concepts, its symbolism, theories and problems.
- mathematics as a unique discipline, with its central branches and concepts as well as their interconnections, interdependencies, and the overall unity of mathematics.
- the way mathematical knowledge is established and validated through proof [...], as well the limitations of proof.
- a qualitative and intuitive understanding of many of the big ideas of mathematics (pattern, symmetry, structure, proof etc.)

Where the first aspect can be seen as a parallel to the first form of overview and judgment concerning the application of mathematics, Ernest's second aspect resembles overview and judgment about the historical evolution of mathematics. The last three aspects are all included in the third form of overview and judgment concerning the nature of mathematics as a subject.

5 Concluding remarks

The review of the literature shows a clear pattern in the research on students' beliefs about mathematics as a discipline. Regarding the first purpose of this study—to provide information of how students' beliefs about mathematics as a discipline can be categorized and analyzed — the majority of the included studies place this dimension in a non-school setting, addressing aspects of mathematics in the "real world". The characterization of these beliefs overall relies on a more or less nuanced version of the dualistic/relativistic framework, spanning from perceiving mathematics as a static body of facts and procedures to be memorized, to viewing it as a dynamic, coherent, sense-making system that plays an important role in the world and in life. The second purpose regarded tendencies in the nature of compulsory school students' beliefs about mathematics as a discipline. Here, the findings in the included studies show that students in general tend to possess beliefs belonging to the dualistic end of the spectrum, with an emphasis on numbers, calculations and memorization.

As more than one researcher underline, is worth noting that students largely base their belief on their experiences in the classroom, and the results of this review indicate that quite few of these experiences include aspects connected to mathematics as a discipline. Consequently, it must be considered and taken into account which beliefs are favourable to students' learning and appreciation of mathematics. Again, the literature is quite clear in their recommendation of beliefs belonging to the relativistic end of the spectrum. In conclusion, a comprehensive change in students' beliefs is required, ensuring that they are based on experiences that represent mathematics in a more relativistic perspective as well as on evidence and reflection.

It is striking, how relatively few studies were found concerning primary and secondary students' beliefs about mathematics as a discipline in the search for literature. The reason for this might be found in the search strategy. Broader criteria for age group, time span or object (e.g. mathematics in general) might have led to a higher number of relevant hits. On the other hand, as seen in the analysis, mathematics as a discipline can be characterized in multiple ways and with various content. A search strategy addressing the individual issues might also lead to an increased base of results. Nevertheless, the reasons for the apparently low interest in the field must be considered, especially because of the importance of the students' beliefs to their learning and educational well-being as well as the contrast in their actual beliefs and the beliefs considered ideal.

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Note: Studies included in the systematic review are indicated by an asterisk (*).

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