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Relating students' proportional reasoning level and their understanding of fair games

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Abstract

This paper analyzes the relationship between proportional reasoning and understanding fair games in Costa Rican students. We conducted a quantitative and qualitative analysis of the answers to six items on comparing ratios of increasing difficulty level and another item on prize estimation in a fair game. We describe the strategies employed and the semiotic conflicts detected in 292 Costa Rican students from Grades 6 to 10 (11-16-year-olds), comparing the findings with those established in previous research. The results show an increase in the level of proportional reasoning with the grade, although the age at which the higher levels are reached is lower than that assumed by Noelting. The percentage of students applying correct strategies in the fair game problem also increases with grade, and a relationship between the understanding of fair game and the level of proportional reasoning is observed.

Keywords: Evaluation, Fair Game, Understanding Proportional Reasoning Level

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The development of statistical inference and the increase of probabilistic information in the media and professional and scientific contexts, together with its role in decision-making, requires the probabilistic education of citizens (Jones et al., 2007; Sharma, 2016). Probability has strong links to other mathematical content, such as proportionality, combinatorics, logic, and algebra (Van Dooren, 2014). This prominence has led to the teaching of probability from primary school onwards in mathematics curricula (e.g., Ministerio de Educación Pública [MEP], 2012; Ministerio de Educación y Formación Profesional, 2022).

In this paper, we focus on students' understanding of fair games, specifically their competence to transform a non-equitable game into a fair one. Games of chance originated the first ideas of probability (Batanero et al., 2005) and played a role in its teaching. They are the primary context where children become aware of randomness and perform probabilistic estimations, even before instruction (Batanero et al., 2019). Moreover, they reinforce some fundamental ideas Gal (2005) included in his probabilistic literacy model: randomness, variability, uncertainty, and independence. They also require estimating or calculating probabilities and using the probability language. In addition, Pratt (2000) suggested that understanding fairness is linked to children's conception of randomness, to which they attribute the properties of irregularity, unpredictability, and fairness. They also help develop the students' probabilistic reasoning (Hernández-Solís et al., 2021).

The Costa Rican curriculum suggests the analysis of probabilities through games of chance and problems in the student context from primary education. A learning expectation identifies equally probable events by considering the number of simple outcomes for each event (MEP, 2012). In the last year of primary education (Grade 6), students should use the intuitive ideas acquired in previous grades to compute probability according to the Laplace rule. There is no probability content in Grade 7. In Grade 8, there is a new study of all the probability ideas introduced in previous grades, increasing the difficulty of problems. In Grade 9, the frequentist definition of probability and the law of large numbers is introduced. In Grade 10, the basic rules of probability are formalized, and the properties of union and complement are used to solve probability problems. However, there is no explicit reference to situations associated with the idea of the fair game or establishing a prize according to each player's expectation of winning.

Although previous research on understanding fair games exists, as described in the background section and in Batanero and Álvarez-Arroyo (2023), we have found none that relates students' understanding of fair games to their proportional reasoning level. In a previous exploratory study (Hernández-Solís et al., 2021), we initiated an investigation on the understanding of fair games with a reduced sample of only 6th grade primary school students, disregarding the students' level of proportional reasoning. To contribute to this topic, we analyze the strategies that Costa Rican students (6th to 10th grades) use to compute the prize in a fair game, relating it to their proportional reasoning level. We also describe the main semiotic conflicts (Godino et al., 2007; 2019) found, comparing the results obtained with those established in previous research.

Theoretical Framework

We base our study on the Ontosemiotic Approach (OSA) to mathematical knowledge and instruction (Godino & Batanero, 1994; Godino et al., 2007; 2019), which assumes an ontological formulation of mathematical objects. Starting from the situation problem, the authors define the theoretical concepts of practice, object (personal and institutional), and meaning (Godino et al., 2007; 2019). A practice is "any action or manifestation carried out by someone to solve mathematical problems, communicate the solution to others, validate the solution, and generalize it to other contexts and problems" (Godino & Batanero, 1994, p. 334). According to this theoretical framework, we distinguish between the practices carried out by each person (personal practices) and those ruled by the subjects within an institution (institutional practices), e.g., the secondary school institution. In this study, we intend to analyze the students' mathematical practices when solving ratio comparison and fair game problems and compare them with those regulated by the educational institution.

In the OSA, mathematical objects emerge from the practices associated with a field of problems. This system of practices confers meaning to the object. Depending on whether the system of practices is carried out within an institution or is specific to an individual, one speaks of the institutional or personal meaning of the object. These two systems of practice intersect with what the person "knows" or "understands" about the object. On the other hand, the authors define semiotic conflicts as mathematical practices that do not correspond to those regulated by the institution. These conflicts occur when the person assigns a meaning to a mathematical object different from that the institution accepts. For example, when the student assigns equiprobability to a random experiment's events with non-equiprobable events.

The duality in the meaning and practice of mathematical objects enables the evaluation of people's (unobservable) understanding through their problem-solving practices, which are observable indicators of their knowledge and semiotic conflicts. Consequently, to analyze the participants' meaning, we analyze their practices in solving the proposed items on comparing ratios and fair game.

Ratio, Proportion, and Strategies in Comparing Ratios

Proportional reasoning supports claims about the structural relationships among four quantities (say a, b, c, d) in a context involving covariance of quantities and invariance of ratios or products. That reasoning consists of the ability to discern a multiplicative relationship between two quantities and extend the same relationship to the other pair of quantities (Lamon, 2007, p. 637). Lamon (2007) differentiates the following interpretations of the rational number (an analysis of the algebraization levels involved is reported in Burgos and Godino (2020)):

- Part-whole relationship. In this, the symbol a/b indicates the part of the whole contained in each quantity. This meaning is introduced early at school, using geometric objects, such as areas or the number line (Tourniaire & Pulos, 1985).
- Quotient. It appears when dividing one natural number by another, and the symbol a/b indicates an operation. It introduces the equivalence class of fractions representing the same rational number (Behr et al., 1983).
- Ratio. There is no reference unit, but we divide two quantities and quantify a multiplicative relationship between two magnitudes. Ratio helps introducing intensive magnitudes such as speed (Behr et al., 1983).
- Proportion. It is the equality of two ratios and involves multiplicative relationships between them. When a/b=c/d, there is a direct proportionality relationship between the two parts of the ratio. If a×b=c×d, there is an inverse proportionality relationship (Ben-Chaim et al., 2012).

In this work, we use proportion, and more specifically, mixture problems, in which we compare two ratios with the same unit of measurement of the quantities involved (number of glasses of juice or water). The following strategies have been identified in research on this subject (Tourniaire & Pulos, 1985):

- Correct strategies: a) multiplicative strategies, multiplicatively relate the terms of one ratio to those of the other, generally the numerator and denominator of each ratio or the numerators and denominators of both ratios; b) correspondence, which is valid for simple problems or when the fraction calculus is not fluent. It consists of establishing a relationship within one ratio and extending it to the second or establishing multiplicative relationships between homologous terms of both ratios.
- Incorrect strategies: using only part of the data, e.g., comparing only the numerators of each ratio, making additive comparisons, subtracting elements of each ratio, or using an arbitrary unit of comparison.

Development of Proportional Reasoning

Inhelder and Piaget (1958) were pioneers in studying proportional reasoning development. The authors suggested that pre-proportionality involves the coordination of functions, while proportionality supposes the coordination of operations. The first strategies for comparing fractions are additive, as children believe ratios' equivalence implies the equality of difference between their terms. A stage of logical operations comes after children understand the relationship between the four terms of a ratio (a/b=c/d), which implies that if a increase, d must decrease when the other terms are constant.

Noelting (1980a, 1980b) posed the problem of mixing water and orange juice and expressed each mixture as an ordered pair (a,b), where the first term corresponds to the number of glasses of orange juice (a) and the second to the number of glasses of water (b). After his work, the author considered the following levels of reasoning in comparing two mixtures of compositions (a_1,b_1) and (a_2,b_2), which we will take into account in our research:

- Level 0 - Symbolic: comprising problems that are solvable by differentiating between the two elements in each ratio.
- Level IA - Lower intuitive: with structure {$a_1 < a_2$; $b_1 > b_2$}. The usual strategy is the comparison of the first terms (a_1 and a_2) without considering the second terms.
- Level IB - Medium intuitive: with structure {$a_1 = a_2$; $b_1 > b_2$}; when the first components are equal, children compare the second terms and perceive them as reciprocals of the first terms.
- Level IC - Higher intuitive: with structure {$a_1 > a_2$; $b_1 < b_2$} or {$a_1 > b_1$; $a_2 < b_2$}. Children perceive either the ratio as an entity and compare both internal relationships as complementary or compare the inequality of the terms in each ratio.
- Level IIA - Lower concrete operational: with composition {$a_1 = b_1$; $a_2 = b_2$} and is characterized by the unit equivalence class. Children must acknowledge the four relations between terms and need to use multiplication.
- Level IIB - Higher concrete operational or fraction equivalence class: with structure {$a_1/ b_1= a_2/ b_2$}. The difference between IIA and IIB is the independence of the first and second terms within the equivalence class.
- Level IIIA - Lower formal operational: with the form {$m\cdot b_1 = b_2$; $m\cdot a_1 < a_2$} or {$m\cdot a_1 = a_2$; $m b_1 > b_2$}. A new strategy emerges involving the combination of multiplication and addition. The child first constructs an equivalent ratio to a ratio and afterward compares both ratios through an additive operation.
- Level IIIB - Higher formal operational: where children compare any fractions. There is no multiplicative relationship between terms; therefore, the strategy involves a double process of "co-multiplication", generating equivalent classes for each ratio with an additional additive comparison of the first following terms. The addition of ratios and fractions algorithm is the final stage of the period.

Research on Understanding Fair Games

Watson and Collis (1994) analyzed the strategies used by 16 Australian children aged 8-10 years to decide if a game consisting in rolling a die was fair. The authors identified different beliefs held by the children, for example, that some outcomes of rolling a die are more likely than others, that the dice can have a will of their own and favor one player, or that the physical characteristics of the dice (e.g., its color) influence the likelihood of the different outcomes. Later, Watson and Moritz (2003) described similar findings in a longitudinal study with students from Grades 3 to 9, which was repeated with a part of the sample two years later. The authors reported progress for many students who initially had erroneous beliefs. However, some students maintained that there are numbers that come out more often or that the dice were always fair.

Schlottmann and Anderson (1994) analysed 5-10-year-olds children's intuitions about mathematical expectation, indicating that they considered both the probability of winning and the value of the prize to decide whether a game is fair. However, they could have done better when calculating probabilities using additive strategies. Other studies examined the influence of out-of-school experiences on developing the idea of fairness and its relationship with probability (Lidster et al., 1996).

Vidakovic et al. (1998) examined Grade 8 students' ideas about fair games. They informed that children's ideas came from their everyday lives and not from instruction. Students considered a game fair when everyone had the same chance of winning. Some students argued that some numbers would be more likely to occur. Paparistodemou et al. (2002) studied the reasoning of 6 to 8-year-old children when playing with an electronic game where they had to match the teams' chances of winning. The children could vary the number, size, and position of the teams' balls, but not a randomly bouncing white ball. The authors reported that the children associated a fair organization of the game with a lack of pattern, complex movement, symmetry in the positioning of the objects, and equal size of the objects.

More closely associated with our study is the research of Cañizares et al. (2004), who related 10 to 14-year-olds' understanding of the idea of fair game to their probabilistic beliefs. Some children showed the following ideas a) introducing external factors such as "cheating" into their arguments; b) equalising payoffs in games where participants were not equally likely; c) requiring a high number of rounds to make the game fair by ignoring independence of trials; and d) considering all games to be fair if there was no cheating. Many children could establish the prizes corresponding to two players to turn an unfair game into a fair one. However, some felt that, even with equal expected payoffs, the game would still be unfair if the prizes differed in a single trial. The authors concluded that most students believed that fair game was synonymous with equiprobable results and, therefore, their difficulties were not in judging whether the game was fair or not but in determining whether the outcomes were equiprobable. Other students might incorrectly assume all random events to be equiprobable, generalising Laplace's rule (equiprobability bias, Lecoutre, 1992). The results improved with age and the student's mathematical ability. We found similar results in an exploratory study with a small sample of Costa Rican 6th grade primary school students (Hernández-Solís et al., 2021).

Some research on teachers' professional knowledge of probability (Azcárate, 1995; Mohamed & Ortiz, 2012; Ortiz et al., 2012) has found that trainee teachers fair games. Participants based their arguments on the equiprobability of outcomes, arithmetic rules or incorrect combinatorial argumentation. Batista et al. (2022) obtained similar findings when analysing children's and adults' reasoning in evaluating and justifying the fairness of a game. They indicated that participants relied on their ideas of randomness, analysis of the sample space, comparison of probabilities and understanding of independence. Their results demonstrated the task difficulty, where many participants considered equal probabilities a prerequisite for accepting the game fairness. Their arguments were influenced by their beliefs and personal experiences.

We also consider Green's (1982) study, which replicated in a paper-and-pencil version some experiments by Piaget. He determined the probabilistic reasoning level of participants in his sample, thereby obtaining a parallel with the developmental stages of probabilistic reasoning described by Piaget and Inhelder (1951). In a replication of this study, Cañizares (1997) found beliefs in the participants' arguments, such as not recognising independence in lottery contexts and looking for causal explanations that affect the probability of the events (e.g., a child's favourite colour or age). She also identified arguments associated with physical considerations, such as the arrangement of the balls in the picture.

METHODS

Sample

The sample comprised 292 Costa Rican students in Grades 6-9 (Basic General Education) and 10 (Diversified). We present the sample composition in Table 1, together with the type of questionnaire (A or B) assigned to each student. We describe the questionnaires in the following section. When the questionnaires were applied, and considering the educational planning established by MEP (2012), students in Grades 6 to 8 only had studied probability in primary school (in Grades 1 to 5). Students in Grade 9 studied probability in the previous grade, and students in Grade 10 studied probability in Grades 8 and 9.

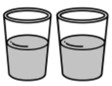
Table 1. Sample composition by grade and questionnaire.

Grade	Age (years)	Questionnaire A	Questionnaire B	Total
6	11-12	35	33	68
7	12-13	26	26	52
8	13-14	31	33	64
8	14-15	26	26	52
10	15-16	27	29	56
Total		145	147	292

Questionnaires

Students were given one of two questionnaires, A and B, each with three ratio comparison items of increasing complexity and similar to those used by Boyer and Levine (2015), Noelting (1980a, 1980b), Karplus et al. (1983), Pérez-Echeverría et al. (1986) and Tourniaire and Pulos (1985) (see Figure 1). Together, between both questionnaires, we included problems of six different reasoning levels of Noelting (1980a, 1980b). We gave each questionnaire to half the students from each school year to have approximately the same number of students with the same characteristics answering each item (Table 1). The questionnaire was divided into two parts to avoid repeating the answer when advancing the resolution of many similar tasks.

Elena and Juan made some lemonade. Elena mixed 2 glasses of lemon juice with 3 glasses of water. Juan combined 1 glass of lemon juice with 3 glasses of water. All glasses contained the same amount of liquid. Look at the picture.

Elena		Juan	
Lemon juice	Water	Lemon juice	Water
			

Which of the lemonades tasted more like lemon?

( ) Elena's.

( ) Juan's.

( ) Both were identical.

( ) I don't know.

Explain why you gave that answer.

Figure 1. Item 1

In Table 2, we specify the characteristic of each item. The composition of the item (a_1,b_1) vs (a_2,b_2) describes the number of glasses of juice and water in each mixture to be compared. For example, in Item 1 (from questionnaire A), the students should compare a mixture with 2 glasses of juice and 3 glasses of water with another mixture containing 1 glass of juice and 3 glasses of water.

Table 2. Classification of items, according to questionnaire, proportional reasoning level (Noelting, 1980a; 1980b) and difficulty level (Pérez-Echeverría et al., 1986).

Item (questionnaire)	Composition (a_1,b_1) vs (a_2,b_2)	Noelting proportional reasoning level	Age (years-months)	P. Echeverría difficulty level
1 (A)	(2,3) vs (1,3)	IA. Lower intuitive	(3-6)	1
2 (B)	(5,1) vs (5,4)	IB. Medium Intuitive	(6-4)	1
3 (A)	(2,2) vs (4,4)	IIA. Lower concrete operational	(8-1)	2
4 (B)	(3,1) vs (6,2)	IIB. Upper concrete operational	(10-5)	2
5 (A)	(3,1) vs (4,2)	IIIA. Lower formal operational	(12-2)	3
6 (B)	(3,2) vs (4,3)	IIIB. Upper formal operational	(15-1)	4

In Item 7 (Figure 2), we assess the personal meanings that the students assigned to a fair game. The task has two steps. First, the student must realise that María is ahead in the game (she has a higher probability of winning than Esteban). Second, the student must quantify the number of chocolate bars Esteban should get by winning so that the game is fair. To do this, the prizes should be inversely proportional to each player's probability of winning. Therefore, with only one item, we can check if the students understand that each player has a different probability of winning and if they can adjust the prize to turn the game into a fair game. The computation needed in the item is easy enough for the students in the sample, as it involves only determining the favourable cases for each player to later apply the Laplace's rule.

As regards the ratio comparison, we need six different items because we intend to study how the different proportional reasoning level of students is related to their solution of the fair game item.

María and Esteban play a game of rolling a die with 6 sides numbered 1 to 6. María wins 1 chocolate bar if the die comes up 2, 3, 4 or 5 or 6. If the result comes up 1, Esteban wins a certain number of chocolate bars. How many chocolate bars must Esteban win when the die is 1 for the game to be fair?

Explain why you gave that answer.

Figure 2. Item 7

Coding of Strategies and Analysis

Once the questionnaires were collected, we performed a content analysis of each student' response to the different items. According to Krippendorff (2013), content analysis permits the establishment of categories that emerge objectively because of the systematic study of written information. We first classified the students' responses according to their correctness, and secondly, we analysed the students' strategies, starting from those described in previous research. With an inductive and cyclic process, including successive revision and discussion of cases, we refined the classification, and one author coded the responses. Another author coded the responses in a subsample of 20 students to compute the inter-coder reliability. We obtained a value for the reliability coefficient Cohen's Kappa = .97, used to evaluate similitude of coding in different coders, which is very high.

In the following, we describe in detail the categories of strategies in both types of items (proportional and fair game items), including examples of students' responses in each category. We supplement this analysis in the results section with numerical information, shown as percentages of students in the different categories considered.

We classified the strategies that students could use to solve the items, starting from those identified in previous works (Noelting, 1980a; 1980b; Pérez-Echeverría et al., 1986; Tourniaire & Pulos, 1985) and completed them by content analysis. The strategies are described below, adding an example of a student's response to each. The codes (a_1,b_1) and (a_2,b_2) describe the number of glasses of juice (a_1 and a_2) and water (b_1 and b_2) in the first and second mixture to be compared, respectively.

Ratio Comparison Strategies

- Comparing totals in each ratio, that is, the number of glasses (lemon juice and water) in each mixture; for example, when $a_1+ b_1 > a_2+ b_2$ then $(a_1, b_1) > (a_2, b_2)$. Noelting (1980a, 1980b) did not report this strategy, which is usually incorrect, although it provides correct responses in Item 1. It involves a semiotic conflict since the student uses a non-normative procedure to compare two ratios.
  S6: Elena's mixture tastes stronger because she used more amount (Item 1: (2,3) vs (1,3)).

- Comparing the first terms "a" of the ratios (comparing numerators). This procedure provides correct results only when the second terms are the same ($b_1 = b_2$). Students already at Noelting's lower intuitive level (IA) can use it. The student must differentiate both terms of the ratio. In this study, it provides correct answers to items 1 and 7. In case the denominators are different, the student shows a semiotic conflict in assuming it is possible to compare ratios using only the numerators.
  S13: Mixture 1. Because it contains more lemon (Item 1: (2,3) vs (1,3)).

- Comparing the second terms "b" of ratios (comparing denominators), which provides correct answers if the first terms are identical ($a_1 = a_2$). The student must understand that with the same numerator the ratio is larger if the denominator is smaller. This requires understanding that "b" is the reciprocal of "a". This strategy corresponds to Noelting's intermediate intuitive level (IB). It includes the previous procedure, since the students must first compare the first terms and realise that they are equal. The strategy gives correct answers to the Item 2 and in the remaining items involves a semiotic conflict.
  S172: Mixture 2. Because Elena has less amount of water (Item 2: (5,1) vs (5,4)).

- Comparing the differences between the terms of each ratio. Linking the four terms in the problem with additive comparisons by analysing the difference between these terms. According to Noelting (1980a, 1980b), this conduct indicates that the student constructs the ratio by analysing the internal relationships between its terms. Noelting described this procedure as an "intro" strategy; that is, the student perceives the difference between terms in each ratio to be different, and then, the student chooses the one with a higher difference. The strategy is typical for Noelting's higher intuitive level (IC) and is valid in items of lower levels; for example, in items 1 and 2. In the remaining items suppose a semiotic conflict, in assuming a ratio is higher than another if the difference of its terms is higher.

S39: Mixture 1. Because Elena has one more glass of water than juice and Juan has two more glasses of water than juice (Item 1: (2,3) vs (1,3)).

- Ratio of equivalence to unity. The student compares one ratio (a_1/ b_1) with the other (a_2/b_2). A multiplicative operation is applied to the terms of both ratios, discovering an equivalence to unity. While the previous strategy combines the terms in the same ratio, in this case, the terms of both ratios should be considered. It provides correct answers when each ratio terms are identical. This strategy corresponds to Noelting's lower concrete operational level (IIA). In our study, it gives appropriate answers to Item 3.
  S69: Both are identical, because both lemonades have the same amount of juice and water in different mixtures (Item 3: (2,2) vs (4,4)).

- Equivalence relation between ratios. The student compares the ratios by a multiplicative operation, finding that they are equivalent. This procedure leads to correct answers only when the ratios belong to the same equivalence class of fractions. The procedure corresponds to Noelting's highest concrete operational level (IIB) and gives correct answers in Item 4.
  S156: Both are identical. Because Elena's amount of lemonade is smaller and so is the water, Juan's amount is as if they were multiplied by 2 (Item 4: (3,1) vs (6,2)).

- Correspondence between the ratio terms. The student establishes a proportionality criterion between one ratio terms (a_1/ b_1) to determine whether the relationship in the other ratio (a_2/ b_2) is smaller or larger. This strategy gives correct answers when two of the four terms are multiples, which will allow to establish a relationship of the first ratio with the second. It corresponds to Noelting's lower formal operational level (IIIA) and solves Item 5.
  S109: Mixture 1. It is concentrated in less water. For each water glass there are 3 lemon glasses so that Juan should use 6 glasses of lemon to be as acidic as that from Elena (Item 5: (3,1) vs (4,2)).

- Proportionality. The children reduce the ratios to common denominator fractions and compared. With this strategy, they can compare any ratios and gives correct answers for any ratio comparison task. It corresponds to Noelting's higher formal operational level (IIIB) and solves all the items of the questionnaire.
  S80: Mixture 1. Because the percentage of lemon juice in Elena's lemonade (60%) is higher than that of Juan's (57%) (Item 6: (3,2) vs (4,3)).

Strategies for Determining the Fairness of the Game

We deduced the strategies from Cañizares et al. (2004) and Hernández-Solís et al. (2021). The strategy is correct if the student correctly computes the price. We found the following correct strategies:

- C1. Correct estimation of the prize, computing the probability for each player and using mathematical expectation. Let "a" be the favourable cases of the event Esteban wins, "b" the possible outcomes when rolling a die, and k_1, k_2 the number of chocolates each player would win. The student equals the mathematical expectation of each player (Esteban: $k_1(a/b)$ and María: $k_2((b-a)/b)$) looking for a value k_1 such that $k_1(a/b) = k_2((b-a)/b)$, where $k_1(1/6) = 1(5/6)$ so

that $k_1 = 5$.
S71: 5, María gets a probability 5/6 to win 1 chocolate bar while Esteban gets 1/6 probability: hence, it is only fair when he obtains 5 chocolate bars.

- C2. Correct estimation of the prize, applying inverse proportionality. Let "a" be the number of favourable cases and "b" the number of chocolates to win in the game, then $(a_1, b_1)=(5,1)$ represents that María has 5 chances to win and wins one chocolate, and $(a_2, b_2)=(1,x)$ represents that Esteban has one favourable case to win and "x" number of chocolates: $x = \frac{5 \cdot 1}{1} = 5$. Therefore, students use inverse proportionality, because when comparing the ratios a_1/b_1 and a_2/b_2 they find that $a_1 \cdot a_2 = b_2 \cdot b_1$ (Ben-Chaim et al., 2012).
  S218: Five chocolate bars. If María wins 1 chocolate bar each time she gets one of the 5 faces, Esteban should win 5 bars each time he gets 1 face.

- C3. Correct estimation of the prize, with a confuse argument. We interpreted that the student solved the problem correctly but had no linguistic fluency to explain the method he followed.
  S233: Five, because María has more probability.

In partially correct responses, the student computes correctly the probability, but provides an incorrect estimation of the prize or do not estimate the prize. The student realises Maria's advantage, i.e., that the game is not fair; but needs to perform a correct assignment of chocolate bars to Esteban. This response is linked to additive strategies of intuition that involve no mathematical procedure but only an estimation of the value; answers such as 2, 4, and 6 chocolates may appear, which suggest a lack of competence in inverse proportionality by the student. These responses involved a semiotic conflict that consists on not assuming that a fair game implies the same mathematical expectation for both players. We differentiate the following partially correct responses:

- PC1. The student correctly computes the probability, but misestimates the prize, as S35.
  S35: 6 bars, because his probability is $\frac{1}{6}$ and María probability is $\frac{5}{6}$

- PC2: the student realises that the probabilities are different, but do not compute the probabilities or the prize as S4.
  S4: More chocolate bars, since María has many probabilities to win and Esteban only 1 probability.

Finally, in incorrect responses, both the prize and the probability need to be corrected. An example is:

S13: Two, because María has more probability.

Other incorrect responses were due to not mathematical arguments, as in the case of S192, who only suggest that both players should reach an agreement to divide the prize. Other students, such as S179, reason according to the outcome approach (Konold, 1989) as they misunderstood the question and, instead of computing a probability, intend to guess the result of the game. The student presents a semiotic conflict (Godino et al. 2007; 2019) since he or she is assigned to the mathematical object “fairness of the game” a meaning different from that in the teaching institution. Finally, S274 suggests that any chance game is fair, a semiotic conflict described by Cañizares et al. (2004). This student assigns one chocolate to Esteban, considering that both players have the same probability of winning. This strategy is related to the equiprobability bias (Lecoutre, 1992), where the person conceives the equiprobability of all events associated with any random experiment, even if there are more favourable cases for one event than another. This is another semiotic conflict since the student assigns equiprobability to all the events of a random experiment.

S192: Maria and Esteban should reach an agreement.

S179: We cannot know since you can win many times.

S274: 1 because Maria also gets a chocolate bar if it results in a 6.

RESULTS AND DISCUSSION

We analysed each student's response to each item to determine the student's proportional reasoning, understanding of fair game and the relationship between these two variables.

Proportional Reasoning

In Table 3 we show the percentage of correct strategies in the ratio comparison items per grade and item in the six tasks that evaluate the students' proportional reasoning. We include the Noelting (1980a, 1980b), and Pérez-Echeverría et al. (1986) levels of the item to improve the interpretation of these results. We considered than a strategy is correct in an item if it provides a mathematically valid solution in that item.

Table 3. Percentage of correct strategies in the ratio comparison items.

			Grade				
Item	**Noelting level**	**P. Echeverría level**	**6**	**7**	**8**	**9**	**10**
1	IA	1	77.2	80.7	90.4	76.8	88.9
2	IB	1	87.9	92.3	78.8	88.4	79.3
3	IIA	2	68.6	69.2	64.5	69.2	70.4
4	IIB	2	27.2	38.4	39.4	42.3	51.7
5	IIIA	3	11.4	15.3	19.4	30.7	37.0
6	IIIB	4			3.0	7.7	10.3

We can see in the table that, as the level of Noelting in the item increases, the percentages of correct strategies diminish, especially from level IIA onwards that require correspondence or proportional strategies. This result is reasonable since the valid strategies in these items require a higher level of proportional reasoning. The percentage of correct strategies per grade is very high and similar for items 1 and 2. This can be solved correctly by comparing the terms' absolute value or the difference in the ratios. These are the level 1 items in the classification by Pérez-Echeverría et al. (1986).

The percentages of correct strategies in the other items increase (or remain similar) with grade. Until level IIB, over half the students per grade perform correct strategies; however, from this level onwards, some grades do not reach 50% of correct strategies. Although in Pérez-Echeverría et al. (1986), the classification of both items has the same difficulty, in our study, the difficulty of Item 4 is higher than that of Item 3. Our students quickly identified fractions equivalents to unity but had difficulty comparing fractions equivalents in general. Then, the classification of difficulty in Pérez-Echeverría et al. (1986) does not apply in our study for items in their level 2, although it remains for the other items.

In Table 4, we present the percentages of students according to the level of proportional reasoning achieved in comparing ratios by grade. We assign a student to a given level if he/she has correctly solved the item associated with that level (providing a correct answer and a correct argument). In this way, level 0 corresponds to students who, having completed the items, did not solve any of them correctly either because they failed in the strategy or the answer.

To visualize the data patterns, we display these results in Figure 3 when we reproduce the cumulative number of students who achieved at least a given level in each problem. From this table and figure, we observe a notable percentage of students in level 0 in all grades, which indicates the difficulties these students still have with proportional reasoning, even in Grade 10. Even if these students may have found the correct solution to one item, they did so with incorrect strategies.

Table 4. Percentage of students according to proportional reasoning level achieved

	Grade				
Level	**6**	**7**	**8**	**9**	**10**
0	11.8	11.5	14.1	13.5	10.7
IA	11.8	11.5	12.5	5.8	10.7
IB	30.9	26.9	21.9	23.1	17.9
IIA	29.4	25.0	21.9	23.1	19.6
IIB	13.2	17.3	18.8	17.3	21.4
IIIA	2.9	5.8	9.4	13.5	14.3
IIIB		1.9	1.6	3.8	5.4

That is, in our study, 10% of students in all grades failed to solve any problem correctly, and the higher levels (IIIA and IIIB) were obtained by a too small percentage of students in all grades. However, from Grade 8 onwards, more than half of the students were at least in one of the levels IIA or superior (Grade 8: 51.7%, Grade 9: 57.7%, and Grade 10: 60.7%). Over half of the Grades 6 and 7 students (60.3% and 51.9%, respectively) attain levels IB and IIA. Consequently, our results cast doubt on the validity of Noelting's age range for reaching the higher levels of proportional reasoning in Costa Rican students. Thus, we also alert teachers to reinforce their students' proportional reasoning.

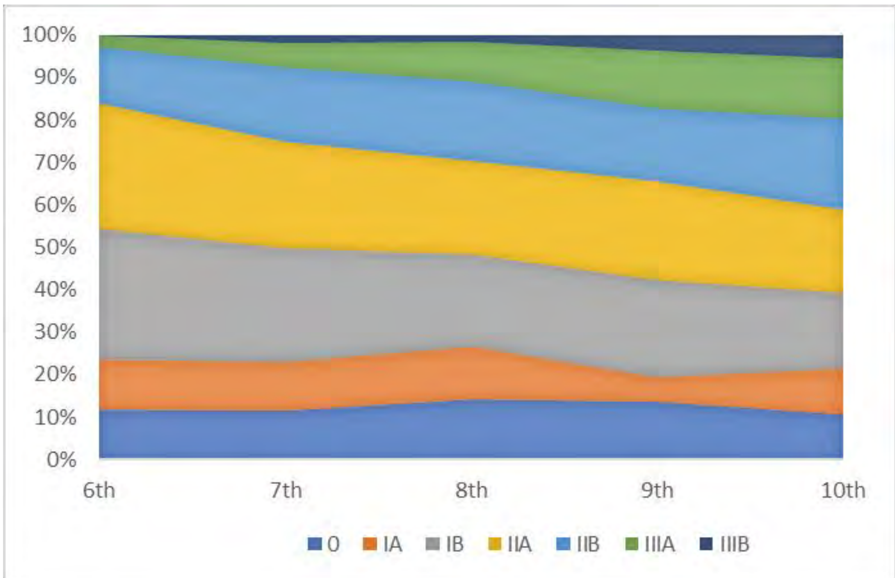


Figure 3. Percentage of students reaching at least each different Noelting's proportional level.

The proportional reasoning level increases with the school grade. However, the age at which students reach these levels in our work is higher than that assumed by Noelting (1980a, 1980b), at all levels: In Noelting (1980a, 1980b), 82.4% of 12-year-olds were classified between levels IIA and IIIA while in our study, that percentage is 45.5%. At 13 years of age, in Noelting's study 83.8% students were between levels IIB and IIIA: in our case, only 23.1%. Among 14-year-olds, 80% were between IIIA and IIIB in the Noelting study; in our research 11%. Between 15 and 16 years of age, 94.7% reach at least level IIB; in our study, 34.6% of 15-year-olds and 41.1% of 16-year-olds had acquired this level or superior.

Understanding Fair Games

In Table 5, we present the percentage of student who employed the different strategies previously described in the analysis of categories of responses to solve Item 7. The proportion of correct solutions accounted for 45.6%, 42.4%, 51.5%, 75.1% and 53.6% responses in Grades 6 to 10, respectively. Cañizares et al. (2004) obtained 45.1%, 62.1% and 46.6% correct solutions in Grades 6 to 8. So, the author obtained higher proportion of correct solutions in Grade 7 as compared with our sample. This was contrary to our expectation, since in the study of Cañizares et al. (2004) the students had no previous study of probability.

Table 5. Percentage of students according to strategy used in Item 7 by grade.

Response		Grade				
		6	7	8	9	10
Correct estimation of the prize	Mathematical expectation	2.9	7.7	9.4	13.5	7.1
	Inverse proportionality	33.8	30.9	34.4	50.0	35.8
	Correct estimation, confuse or not argument	8.9	3.8	7.7	11.6	10.7
Correct computation of probability with no estimation of the prize	Incorrect estimation of the prize and correct probability		3.8	6.3		3.6
	Correct probability, do not estimate the prize		1.9	1.6		
	No computation, but observes the different probabilities	2.9	3.8	3.1		3.6
Incorrect response	Incorrect probability and prize	27.9	28.9	18.7	11.5	19.6
	Non-mathematical argument			1.6		
	Outcome approach			1.6		
	Equiprobability	11.8	17.3	6.3	5.8	7.1
	Incorrect response, no argument or confuse argument	4.4	1.9	1.6	3.8	1.8
No response		7.4		7.7	3.8	10.7

In Grade 9, the students had recently studied probability at the end of the previous school year and were revising the topic when the questionnaire was applied, and that explains the better results in this group. The results in Grade 6 were better than those in our exploratory study (Hernández-Solís et al., 2021) where children obtained only 32,7% correct responses. There were, in general, scarce use of non-mathematical arguments or semiotic conflicts such as not assuming that a fair game implies the same mathematical expectation for both players, considering the game is fair only if both players have identical probabilities to win, or believing that any game is fair (Cañizares, 1997). We also interpret as semiotic conflicts the equiprobability bias (Lecoutre, 1992) and the outcome approach (Konold, 1989). However, the proportion of those who provided incorrect probabilities and prize varied between 11% and 28.9% in the different grades. This support our conjecture that the understanding of fairness is linked to the students' proportional reasoning. We will study this conjecture in the following sections.

To better visualize how students' understanding of fair games increases with the grade, in Table 6 we summarise the different strategies. We added a score to each student with the following criteria that takes into account the probabilistic reasoning involved in the student's response:

0. When the student did not answer the question or provided an incorrect response. Here, we included the different semiotic conflicts described, non-mathematical strategies, and no argument or confuse argument (some examples are S192, S179 and S274 in the section describing the strategies in determining the fairness of the game).
1. When the student did not solve the problem but perceived the different probabilities for the two players in the game. In general, these students used additive reasoning to solve the problem by comparing only the favourable cases for each player but did not find the proportion of favourable to possible cases in the problem (see S4 in the mentioned section). Therefore, they revealed a semiotic conflict in using a non-normative procedure to compare two ratios.
2. When the student correctly computed each player's probability, although he did not estimate the price to obtain a fair game. Here, we included those students who did not estimate the price and those who provided an incorrect estimate. These students compared the proportion of favourable to possible cases for each player, that is, applied the Laplace' rule (see response by S35) and revealed a correct proportional reasoning in case of direct proportion.
3. When the student correctly computed the probability of the two players and correctly estimated the price to turn the game fair. However, their explanations were incorrect or confuse. These students possess adequate proportional reasoning for direct proportion and also correctly apply inverse proportionality (see response by S233).
4. Those students who provided both the probabilities and the price and used a mathematical correct argument to justify their response. Here, we included those who used mathematical expectation and others who used inverse proportionality. In both cases, in addition to correct proportional (direct and inverse) reasoning, they were able to correctly justify their procedures (see responses by S71 and S218).

Table 6. Summary of strategies in Item 7 by grade

Response	Score	6	7	8	9	10
Correct estimation and argument	4	36.7	38.5	43.8	63.5	42.9
Correct estimation	3	8.9	3.8	7.7	11.6	10.7
Correct probability	2		5.8	7.9		3.6
Different probability	1	2.9	3.8	3.1		3.6
Other responses	0	51.5	48.1	37.5	24.9	39.2

The results in Table 6 suggest that less than half the students in each grade reached the maximum level of understanding the fair game, providing a correct estimation of the prize with an adequate argument. However, the percentage increases by grade, except for Grade 9 in which the majority of students achieved the maximum score. On the contrary, the lower score (0) of students who did not reply or provided incorrect responses showed different semiotic conflicts diminished with grade. The average score per grade was 1.8 (Grade 6), 1.8 (Grade 7), 2.2 (Grade 8), 2.9 (Grade 9) and 2.1 (Grade 10).

Relating Understanding of Fair Game and Proportional Reasoning

To further study the correspondence between proportional reasoning level and understanding of fair game, in Table 7 we present the percentage of student who employed the different strategies described to solve Item 7 for each of the different reasoning levels of Noelting (1980a, 1980b).

We observe that the use of the correct strategies to determine the prize in the fair game and provided a correct argument (mathematical expectation and inverse proportionality) increases with the Noelting's reasoning level and more than half the students in levels IIIA and IIIB used these strategies. Less consistently, the proportion of those who correctly computed the probabilities for both players also raised. The average score by level is 1.9 (Level 0), 1.4 (IA), 1.8 (IB), 2.2 (IIA), 2.6 (IIB), 2.8 (IIIA) and 3.4 (IIIB). On the contrary, the proportion of those with incorrect strategies or semiotic conflicts diminish with increasing the proportional level.

Table 7. Percentage of students according to the strategy in fair game task and proportional reasoning level achieved

			Proportional reasoning level						
Response	**Score**		0	IA	IB	IIA	IIB	IIIA	IIIB
Correct estimation of the prize	4	Mathematical expectation	2.6	6.5	2.8	6.0	11.8	23.1	28.6
	4	Inverse proportionality	25.6	25.8	39.4	38.8	45.1	34.6	42.9
	3	Correct estimation, confuse or not argument	20.5	3.2	2.8	11.9	5.9	11.6	0.0
Correct probability with no estimation of the prize	2	Correct probability, incorrect estimation	5.1			1.5	3.9	3.8	28.5
	2	Correct probability, no estimation			1.4		2.0		
	1	Observes the different probabilities	2.6	3.2	4.2	1.5	2.0	3.8	
Incorrect response	0	Incorrect probability and prize	10.3	32.3	25.5	29.8	15.6	11.6	
	0	Non-mathematical argument					2.0		
	0	Outcome approach	2.6						
	0	Equiprobability	12.8	16.1	11.3	9.0	7.8		
	0	Incorrect response, no argument, not understanding the problem, or confuse argument		6.4	7.0			7.6	
No response	0		17.9	6.5	5.6	1.5	3.9	7.7	

Theses tendencies are more visible in Table 8, where we summarise, the scores received in Item 7 (fair game) in each reasoning level. Hence, we found only 28.2% students attained 4 points in level 0 while 71.5% of students achieved the maximum score in level IIIB. On the contrary, while 43.6% of students with 0 score stayed at level 0 in Noeltings' classification, no student in level IIIB and only 23.1% in level IIIA remained in that level 0.

Table 8. Percentage of students with different scores in understanding fair games in each proportional reasoning level.

Score	0	IA	IB	IIA	IIB	IIIA	IIIB
4	28.2	32.3	42.2	44.8	56.9	57.7	71.5
3	20.5	3.2	2.8	11.9	5.9	11.6	
2	5.1		1.4	1.5	5.9	3.8	28.5
1	2.6	3.2	4.2	1.5	2.0	3.8	
0	43.6	61.3	49.4	40.3	29.3	23.1	

All the above results are confirmed by the study of correlations between the different variables considered (Table 9). We note the higher correlation coefficient between the proportional reasoning level and understanding of fair game scores, as compared with the correlation of proportional reasoning level with school grade, and the correlation between understanding of fair game scores with school grade. These correlation values are small but statistically significant; in particular, between the fair game understanding and proportional reasoning level is highly significant.

Discussion

In this study, we jointly evaluated the understanding of fair games and the proportional reasoning level in a sample of Costa Rican students from Grades 6 to 10. The findings showed an increase in the level of proportional reasoning with the grade, although the age at which the higher levels are reached was higher than that assumed by Noelting (1980a, 1980b). The difficulty levels described by Pérez-Echeverría et al. (1986) do not apply in our study for items in their level 2, although it remains for the other items.

Table 9. Pearson's correlation coefficients between proportional reasoning, understanding fair game score and grade

	Fair game understanding score	p-value	Grade	p-value
Proportional reasoning level	.210	.000	.124	.034
Grade	.137	.019		

Like other studies (e.g., Cañizares et al., 2004; Hernández-Solís et al., 2021; and Schlottmann & Anderson, 1994), many students used additive strategies to compute probabilities. The percentage of students who applied correct strategies in the fair game problem also increased with grades, similar to students in the study of Schlottmann and Anderson (1994) to decide the game's fairness.

A difference with Cañizares (1997) is that in her research, many students considered that only games where the two players had the same probability of winning were fair. Similar beliefs were held by children and adults in Batista et al. (2022) research. Neither did we find in our sample the erroneous beliefs described by Watson and Collis (1994), such as that the dice could favour a player or that physical features could affect the fairness of the game, which is reasonable given the higher ages of students in our study. None of our students argued that some numbers were more likely to happen, contradicting Grade 8 students in Vidakovic et al. (1998). Consequently, in our study, the students showed better proportional and probabilistic reasoning, which we attribute to the teaching of probability they had received in the previous years. In contrast, in the research described in section 2.4, children did not study probability at school.

Along the study we identified different semiotic conflicts related to understanding a fair game: a) not assuming that a fair game implies the same mathematical expectation for both players; b) assuming the game is fair only if both players have identical probabilities of winning; and c) assuming any game is fair: We also interpreted in terms of semiotic conflict the outcome approach (Konold, 1989) and the equiprobability bias (Lecoutre, 1992) and found other semiotic conflicts related to incorrect procedures to compare ratios in the proportionality items.

Finally, we obtained a relationship between the understanding of the fairness of a game and the level of proportional reasoning. Students in the higher levels of proportional reasoning more frequently applied correct strategies and arguments in their study of the game's fairness. This fact was confirmed by a significant correlation between a score given to the correctness in determining the fair game and proportional reasoning level.

CONCLUSIONS

Proportional reasoning is one primary competence of students because it influences their understanding of probability, algebra, measure, and other mathematical contents (Burgos & Godino, 2020; Lamon, 2007). However, in our research, few students in Grade 10 achieved the highest level IIIB in Noelting's classification (1980a, 1980b). This fact may impede these students' progress in their learning of other mathematical topics. Consequently, there is a need to pay attention to the students' proportional reasoning level when organizing instruction.

Games of chance play a primary role in probability teaching (Batanero et al., 2005) and are the primary context where children become aware of randomness and perform probabilistic estimations, even before instruction (Batanero et al. 2019). However, in agreement with previous research (e.g., Cañizares et al., 2004), we found that it is not easy for all the students to recognize a game's fairness and estimate the price in a fair game when two players have different probabilities. Teachers should reflect on this fact, and we encourage them to use problems related to fair games in their probability teaching. Due to the relationship between proportional reasoning and understanding fair games, working with one of these situations will help students with the other, and in particular, fair game tasks will contribute to increasing the students' proportional reasoning.

Given the study's limitations, where the students come from only one country, and we only used a task related to understanding fair games, we encourage other researchers to continue this investigation to analyze whether our results stand in other settings or with different tasks.

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