The Pirie Kieren dynamic model of the growth of mathematical understanding: The critical concept of folding back

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ABSTRACT

The Pirie-Kieren Model (PKM) was a paradigm shift in theories of learning by presenting a coherent, consistent theory compatible with complexity theory. PKM recognized that learning is non-linear, recursive, iterative, and emergent. PKM was one of the first theories to depart from the linear models of learning that dominated theories of learning prior to PKM.

One of the critical concepts in PKM is folding back. This paper considers implications from PKM: Implications for practice and implications for theory. For practice, PKM has implications for both instruction and for assessment. Teachers need to be given theories of teaching that are congruent with PKM's non-linear and recursive model of learning. Teachers need such theories of teaching to recognize the elements of PKM such as non-linearity, recursiveness, iterations, and emergence. For assessment, since one of the implications of PKM is that each student will typically be at different points in their learning at any particular time, mass norm-referenced or criterion-based assessments are clearly inappropriate, and other assessment models, such as ipsative assessment, need to be implemented for appropriate assessment of learning.

Keywords: Pirie-Kieren model, mathematics, understanding, taxonomies, theories of learning, theories of teaching

INTRODUCTION

Thirty-five years ago, Pirie and Kieren (1989) postulated a theory of the growth of mathematical understanding that was revolutionary in that it deviated from the linear models of learning that dominated our understanding of how students learn and understand. The theory was congruent with complexity theory in that the model reflected that learning is non-linear, recursive, iterative, and emergent. PKM is illustrated in Figure 1.

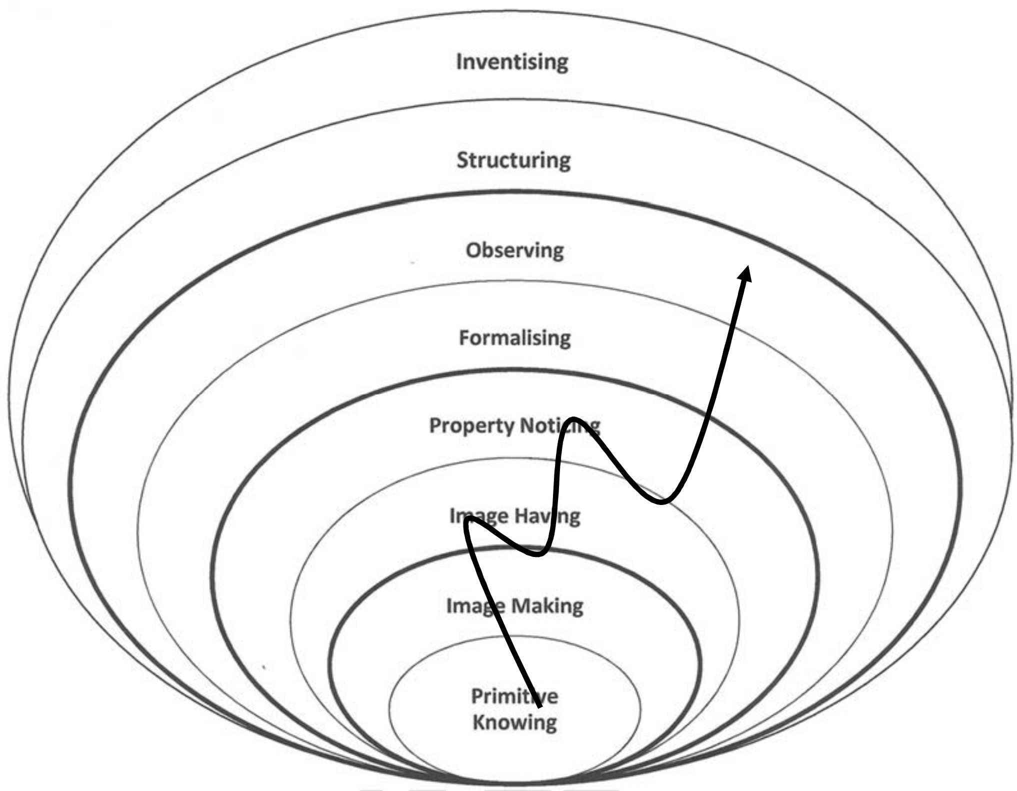

Figure 1. The Pirie Kieren Model. Reproduced with permission from Irvine (2017b).

Primitive Knowing is the initial level of knowledge about the topic, before study begins. . *Image Making* is all beginning representations of understanding. These may be verbal, pictoral, visual or mental. According to constructivist theory, learning involves students building onto their existing schema (Fosnote. 2005). At the *Image Having* level, the learner has acquired sufficient understanding that actual images are no longer necessary.

At *Property Noticing* stage, the student constructs examples that identify essential attributes of concepts. *Formalising* requires learners to abstract the learner abstracts a common attribute or method from the property common attributes based on what was noticed at the previous stage. At *Observing* the learner formulates theorems. At the *Structuring* level, the learner accumulates theorems to form a theory. *Inventising* allows students to demonstrate the depth of their understanding by proposing new or extended questions about the topic.

It is important to note that these levels are not targets for students, i.e., not all students will reach the outer levels of the model, and student understanding of some levels will be imperfect or contain knowledge gaps. However, all students will grow their understanding beyond primitive knowing.

A critical feature of the model is *Folding Back*. Folding back is a dynamic, recursive process that recognizes students' understanding does not proceed in a linear manner. Deepening understanding may mean returning to previous levels to develop better understanding of the concept and address problems that have arisen. Folding back is a major attribute of understanding. Students who fold back retain all their previous levels with a newer lens. This richer lens is called *Thickening*. The student's knowledge is richer or "thicker". This is an important feature of PKM, and is at the heart of enriching learners' understanding. Through folding back learners journey to understanding is not linear but rather a nonlinear path that May include multiple returns to previous levels to deepen understanding, illustrated by a "wandering" line on Figure 1.

PKM AND TAXONOMIES OF EDUCATION

There are a large number of taxonomies of education. Anderson et al., for its revision of the well-known Bloom's Taxonomy, consulted over nineteen taxonomies (Anderson et. al., 2001). The best-known taxonomies ae Revised Bloom's Taxonomy (RBT; Anderson et al., 2001) shown in Figure 2, and Marzano's New Taxonomy (MNT; Marzano & Kendall, 2007) shown in Figure 3.

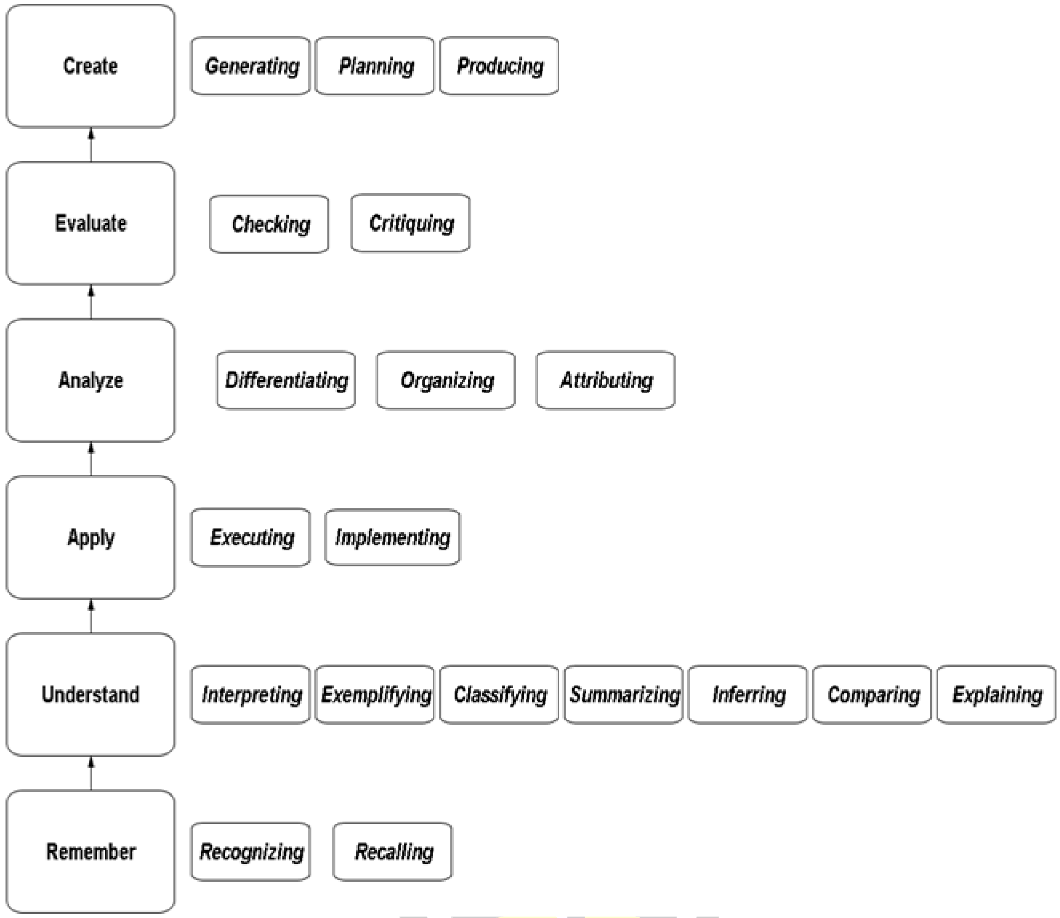

Figure 2. Revised Bloom's Taxonomy. Reproduced with permission from Irvine, 2021.

The levels of PKM can be related to levels of the cognitive domains of RBT and MNT. Image Making corresponds to the RBT level of Understanding-Interpreting, which involves clarifying, translating, and representing. Image Having is related to RBT's Understanding-Exemplifying, which involves illustrating information. The PKM level Property Noticing is analogous to RBT's Understanding-Classifying level, which involves categorizing information based on similarities. Next, PKM's Formalising matches RBT's Understanding-Summarizing level, recognizing and abstracting general themes. The Observing level of PKM is analogous to RBT's Understanding-Comparing level, recognizing correspondences among similar sets of items and formulating theorems.

There is a significant jump to the next level of PKM, namely, Structuring. This level matches RBT's Analysis-Organizing level, finding coherence and integrating information to identify common structures. Finally, the Inventising level of PKM fits the RBT level Create-Generating level, generating hypotheses and postulating new knowledge related to the topic. The final two levels of PKM, Structuring and Inventising, are higher-order thinking Skills (HOTS) that correspond to levels identified as HOTS in RBT.

A similar analysis relating PKM to another taxonomy, MNT, can be found in Irvine, 2017b. Both RBT and MNT are linear and do not recognize PKM's major concept of folding back. Folding back distinguishes PKM as being coherent with complexity theory. For PKM learning is non-linear, recursive, iterative, and emergent.

Figure 3. Marzano's New Taxonomy. Reproduced with permission from Irvine, 2021.

PRIMITIVE KNOWLEDGE

As pointed out earlier, Primitive Knowledge is simply the pre-exiting knowledge related to the topic under study. Each student's primitive knowledge of a topic will typically be different than any other student. Each student begins their study of a topic with their own base of knowledge, and their understanding will grow from that base. Pirie and Kieren (1994) emphasize that mathematic concepts are interconnected and do not stand alone. This interconnectedness of topic can be referred to as chaining. Pirie and Kieren (1994) provide the example of a student's (unique) understanding of fractions forms their primitive knowledge for understanding decimals. The accumulated knowledge of decimals in turn becomes primitive knowledge for the study of percent.

Figure 4 illustrates an example of chaining, based on Pirie and Kieren's fraction example. Each student's level of understanding of any topic will vary. Thus, a student's primitive knowledge when beginning the study of linear relations will perforce be different than any other student's primitive knowledge, at least to some degree. This is a significant challenge for a teacher who structures lessons on linear relations since they will need to meet the needs of each student while moving the understanding of all students forward.

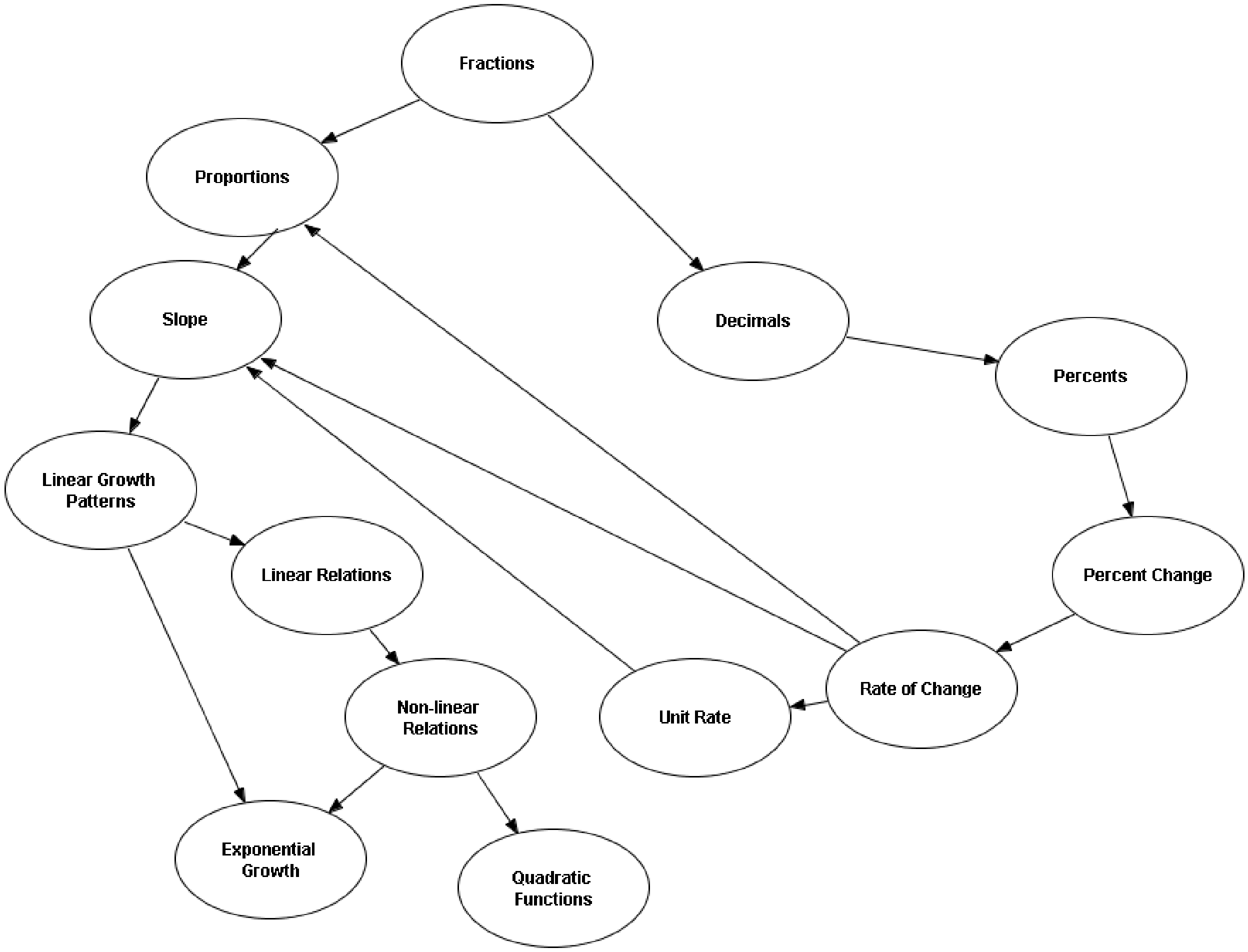

Figure 4. An example of chaining

RESEARCH EVIDENCE SUPPORTING PKM

PKM has been investigated thoroughly and supporting evidence is extensive (Martin, 2008; Pirie & Kieren, 1992; Pirie & Martin, 2000). An interesting modification was by Towers (1998) who, rather than using the nested concentric nested circles changed the representation to parallel lines. Gokalp and Bulut (2018) argue that this new representation allows illustration of more complex learning trajectories. However, the parallel line representation loses the nested feature of the original representation, and does not clearly illustrate that outer levels of PKM are built upon the earlier levels. This is a major loss in understanding of the PKM theoretical stance. PKM is an explicitly constructivist theory (Pirie & Kieren1992, 2008).

Constructivism brings with it the problem that each student will have somewhat differing perspectives on any topic as well as differing levels of understanding of the topic. This can cause difficulties for the teacher who attempts to plan a learning trajectory and initiating event(s) for their class (Simon, 1995). However, this is a problem related to a constructivist lens and not a problem particular to PKM.

Warner (2008) discusses the issue of creating tasks that are low floor, high ceiling, wide walls for a group of diverse learners, each of whom has different conceptual starting points, finishing points, and learning paths. This also is a problem for constructivism and not particular to PKM. There is abundant research evidence both supporting PKM and applying the model to various situations, grade levels, and mathematical topics.

At the elementary school level, PKM has been applied to topics such as fractions (Pirie & Kieren, 1989); multiplication problems (Pirie & Kieren, 1993); problem solving (Borgen & Manu, 2002); sequences (Guner & Uygun, 2020); number sense (Thom & Pirie, 2006); multiple representations (Gokalp & Bulut, 2018). High school level studies include frequencies and proportions (Wright, 2014) and geometric transformations Gülkılık et al.,2 015)

At the university level, PKM has been applied to the understanding of dynamic geometry problems (Yao, 2020a, 2020b; Yao & Manouchehri, 2020); mathematical proofs (Hakim & Murtafiah, 2022); infinite series (Codes et al., 2013).

PKM can reflect not only individual learning but also pairs or group learning (Martin et al., , 2002); with classroom teachers (Droujkova et al., 2005); with workplace training (Martin & LaCroix, 2008); with teacher candidates (Slaten, 2007). Towers and Davis' (2002) study produced folding back diagrams for two individual learners working as a pair. In response to teacher prompts, the two students' thinking grows nonlinearly, sometimes converging and sometimes diverging, as illustrated by folding back diagrams superimposed on each other. Thus, the flexibility of PKM to model numerous different learning situations is a strength of the model.

THE CRITICAL CONCEPT OF FOLDING BACK

As mentioned previously, the concept of folding back, together with its associated concept of thicker knowledge, is a key distinguishing facet of PKM. Folding back recognizes that growth of understanding is seldom linear, and that to deepen understanding requires revisiting previous knowledge while retaining all new understanding that has occurred. It is impossible to unlearn what has been learned, and so when folding back, learners bring new perspectives and new understandings, which deepen their understanding of previously learned concepts and broaden their overall knowledge base.

Most researchers identify that folding back occurs when a student encounters a problem or situation at an outer level that does not immediately have a specified solution (Martin, 2008; Warner, 2008). By returning to an earlier level, the student can accumulate knowledge that may help address the new problem, while consolidating their knowledge of the earlier level. Pirie and Martin (2000) identify this process by the term collecting. Collecting involves retrieving knowledge from a previous level for a specific purpose, and then transferring that knowledge to address a new situation. Because the learner returns to the lower level with a thicker understanding, the result may be a reorganization of the knowledge at the lower level, resulting in new or different understanding of the concepts.

Gokalp and Bulut (2018) apply the concepts of PKM, especially folding back, to create maps of student learning in a variety of mathematical situations. They examine some of the reasons why students consider mathematics difficult and how these learning maps may identify and address some of the difficulties.

A particularly significant application of PKM by Hill et al. (2020) looks at the relationship between folding back and mathematical wellbeing. This research delves more into the "why" students fold back and considers the impetus of folding back from the perspective of student engagement and affect. Hill et al. identify the reasons for folding back as not only to retrieve information from a lower level but also to reduce anxiety and increase feelings of wellbeing. By returning to a more comfortable level, where students feel that their knowledge base is more developed, students may become more engaged in their current problem and proceed more comfortably to attempt a solution. Thus, Hill et al. also identify engagement and motivation as the impetus to both proceed to outer levels of PKM and also to use folding back to gain greater mathematical wellbeing.

IMPLICATIONS FOR PRACTICE

PKM has implications for teachers' knowledge of learners, for theories of teaching and for assessment of students' understanding. Two significant implications of PKM to teaching practice are in the areas of instruction and assessment. Constructing a theory of teaching based on PKM faces all the difficulties of using any constructivist theory of learning as a basis for a theory of teaching. In the same way, since PKM predicts that each student will begin at and arrive at different points in their learning trajectory, assessment methods need to reflect these differences; thus, a one size fits all assessment technique is inappropriate if we accept PKM as a foundational learning theory.

Instruction

PKM enables teachers to develop better knowledge of their pupils, how they learn and what supports might best fit with their students' level of understanding at any given time. This requires teachers to be active observers of their students, with observation broadly construed to include visual observation, listening to students' discourse with peers and teachers, examining student artifacts, and using probing questions to elicit students' explanations and reasoning.

Using the PKM model diagram, teachers are able to create a visual record of each student's growth in understanding for a given topic (Pirie & Kieren, 1989, 1992; Towers & Davis, 2002). This allows teachers a more fulsome picture of each student's support needs and directs teachers towards more appropriate next steps. PKM recognizes the uniqueness of each student's learning trajectory and allows teachers to better meet the needs of their students.

PKM is an explicitly constructivist theory of learning (Pirie & Kieren, 1989). There has been significant confusion about how teachers need to support constructivist learning.
The different levels of PKM and the different rates at which students progress to their ultimate final levels of understanding of a topic require the teacher to differentiate instruction and feedback to meet the needs of each student. Figure 5 illustrates one possible strategy to help teachers meet the needs of each student and support their students' learning.

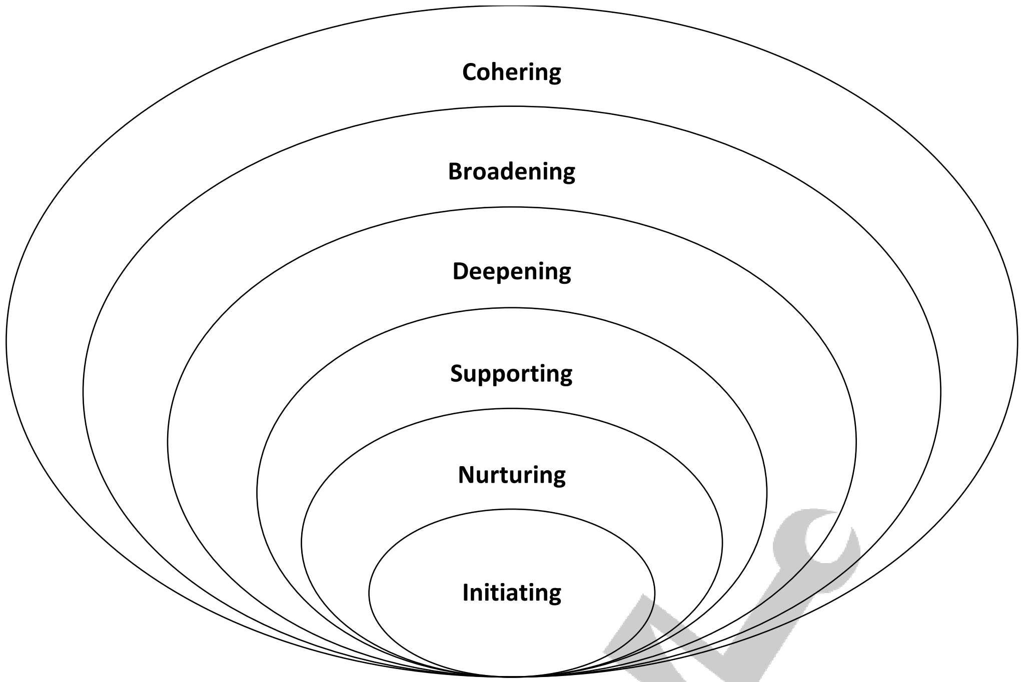

Figure 5. A possible strategy for teachers, based on PKM. Reproduced with permission from Irvine, (2017b).

Initiating is the initial teacher task, the problem or inquiry topic that the teacher has identified as a starting point for their students. This task should generate student interest and spur students' on to engage fully in the inquiry. The initiating task needs to be low floor, high ceiling, wide walls; every student needs to engage with the task at their own level. For more about what the teacher needs to prepare in creating an appropriate initiating task, see Irvine (2017b).

Once students reach the Image Making level, the teacher's role involves *Nurturing*. Nurturing involves teachers prompting, questioning, asking for clarification, and addressing misconceptions. The teacher's role is to assist and support students in image making by scaffolding learning and suggesting alternatives to support students in making multiple images to grow their understanding.

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Transfer has been identified as "integral to our expectations and aspirations for education" (Perkins & Salomon, 1988, p. 22). Barnett and Ceci (2002) found that transfer is more common when involving near transfer, and this will typically be the case for activities in Broadening.

Finally, at the HOTS levels of Structuring and Inventising, the teacher will move to the *Cohering* stage for the student, where students are encouraged to conjecture, formulate hypotheses, and identify methods for verifying those hypotheses.

With every instance of folding back, students thicken their understanding. This not only returns them to an earlier level with greater understanding, it also allows learners to deepen their understanding of earlier levels.

Pirie and Kieren state that it is not possible to identify specific teacher behaviours that support a constructivist theory of learning such as PKM. However, this paper argues that broad classes of teacher behaviours that support students' growth of understanding can be and in fact must be identified in order to clarify for teachers what kinds of behaviours and responses are necessary to assist students in growing their understanding of mathematics. Teachers must recognize that student motivation, including interest, attitudes and engagement, are critical in moving students from lower levels of PKM to outer levels.

Assessment

One major implication of PKM for practice is in assessment. Students will begin their study of a topic with a relatively unique set of primitive knowledge. Their understanding will proceed in a (probably unique) non-linear fashion to their ultimate level of understanding, which typically will have some elements in common with other students but will feature a lens unique to each student. Therefore, common assessments for all students that assume common levels of understanding are inappropriate.

While criterion-referenced assessments may be used, each student's level of understanding will be somewhat unique. A more appropriate assessment model is ipsative assessment (Hughes, 2014; Hughes et al., 2014). Ipsative assessment is designed to measure the growth in a student's understanding rather than assume a specific required level of attainment. Ipsative assessment is common in athletics ("Personal best") and is becoming more common in academic pursuits such as doctoral studies (Hughes, 2014; Newton & Martin, 2013). As does PKM, ipsative assessment focuses on the process of understanding rather than the product.

Based on the assumption of PKM, it is also appropriate to broaden student assessments to include interviews, presentations, assignments, projects and model construction. In this way students can demonstrate their level of understanding without reference to a norm or absolute. Students' higher-order thinking skills (HOTS) can also be illustrated through problem posing, investigations, and hypothesizing (Irvine et al., 2016). PKM recognizes that each student's learning is unique, and therefore assessing each student's level of understanding must also recognize that uniqueness.

An added bonus of using ipsative assessment is that it has been found to increase student motivation and engagement (Hughes et al., 2014). However, maximizing the impact of ipsative assessment requires teachers to provide high-quality individualized feedback in order to guide students' learning growth (Hattie & Timperley, 2007; Nicol & Macfarlane-Dick, 2006).

IMPLICATIONS FOR THEORY

PKM has implications for both taxonomies of education and for theories of learning.

Taxonomies of Education

Taxonomy is defined as classification into ordered categories. Usually, taxonomies consist of non-overlapping categories organized across one or more dimensions. We have already seen how PKM can have a one-to-one correspondence with well-known taxonomies such as RBT and MNT. However, PKM can be considered a stand alone taxonomy of education in its own right. The levels of PKM identify both levels of understanding and also the actions that correspond to those levels. For example, Image Making identifies initial understandings as well as what learners do at that level of understanding, i.e., making visual, mental, pictorial, algebraic or verbal. The Property Noticing level of PKM involves learners classifying common attributes by identifying similarities and differences of specific cases to conjecture common properties of classes. This is a significant strength of considering PKM as a taxonomy of education. The action-oriented stance of PKM emphasizes that understanding is not a passive activity. Rather, understanding involves demonstrating that understanding through action.

Most well-known taxonomies of learning, such as RBT, have an implied linearity. While taxonomies may be simply systems of classification, in education most taxonomies, either explicitly or implicitly, imply an underlying theory of learning.

Theories of Learning

There are hundreds of theories of learning (Davis, 1996). Some, like PKM, have substantial research evidence to support them. PKM is especially significant because it posits a non-linear, recursive and iterative learning trajectory through the critical concept of folding back. This differs substantially from most other theories of learning, which are typically linear and often assume, especially in mathematics, that learning is cumulative and that prior knowledge, once obtained, is retained with limited need to revisit or refresh understanding. Many taxonomies of education have an associated theory of learning.

MNT presents an explicit, linear theory of learning (Figure 6). MNT's major contribution is to designate the self system (motivation, engagement, attitude) as primary to learning (Irvine, 2017a). However, 1the remainder of MNT is assumed to be linear, with the top two levels, (self system, metacognition) top-down linear, and the bottom level (cognitive system) linear, bottom-up) A major exclusion in MNT is feedback loops, which are implied but not explicitly included in the model (Irvine, 2017a).

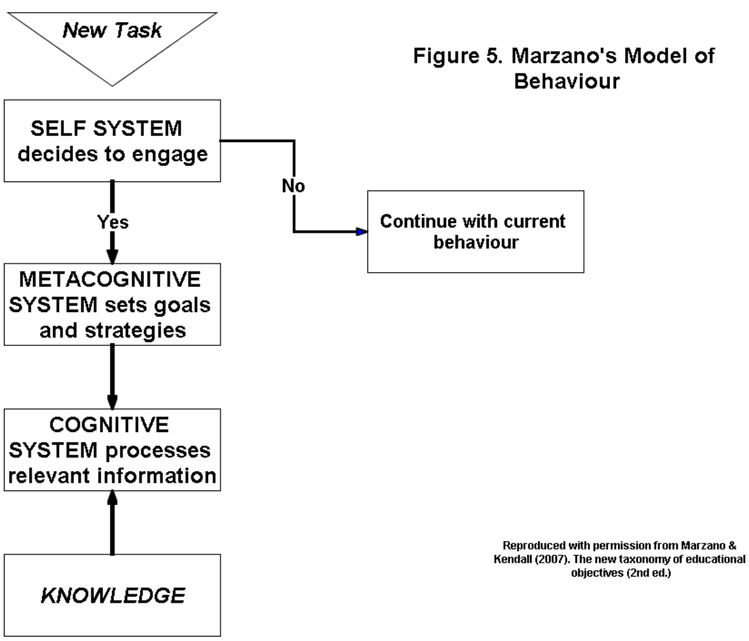

Figure 6. Marzano's theory of learning. Reproduced with permission from Irvine (2017a).

Fink's taxonomy of significant learning (FTSL: Fink, 2013) also explicitly details the underlying theories of learning. FTSL is often represented by a circular wheel, with segments consisting of foundational knowledge, application, integration, human dimension, caring and learning how to learn (Figure 7). Fink explicitly links his taxonomy to constructivism, and promotes active learning as key (Irvine, 2021). While Fink implies with his circular representation that his theory of learning is nonlinear, an important dimension is the revisiting of segments as student understanding grows. However, Fink's theory of learning does not include the concept of folding back, but rather continual advancement of understanding, albeit with "thicker" knowledge for each subsequent rotation.

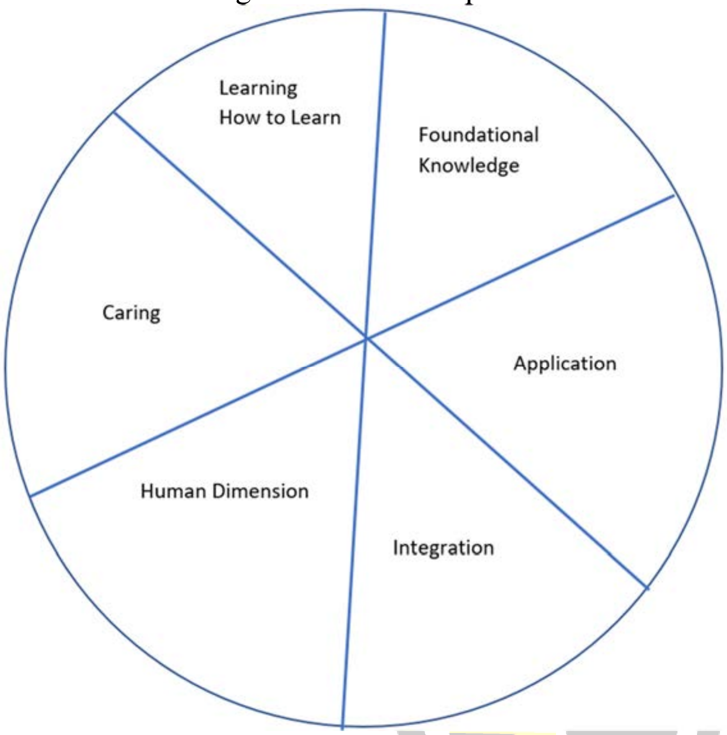

Figure 7. Fink's Taxonomy of Significant Learning. Reproduced with permission from Irvine (2021).

Other taxonomies, such as original Bloom's taxonomy (OBT: Bloom et al., 1956) and revised Bloom's taxonomy (RBT; Anderson et al., 2001) may not explicitly describe the associated theory of learning, but clearly assume linearity. The authors of RBT state that their taxonomy is for learning, teaching, and assessing, but do not explicitly outline the theory of learning. Since RBT is a linear taxonomy, it must be assumed that the associated theory of learning is also linear.

Shulman's table of learning (STOL; Shulman, 2004) is often presented in two formats: A linear, top-down taxonomy that emphasizes the role of engagement and motivation, similar to MNT (Figure 8); and a circular format meant to identify Shulman's concept of reciprocal relationships (Figure 9), commitment with engagement, understanding with judgment, and action with reflection (Irvine, 2021). These reciprocal relationships illustrate a break with linearity as well as delineation of concepts such as understanding linked to judgement and action with reflection. Shulman thus begins moving from a linear theory of learning to a more nonlinear, but not recursive and not truly nonlinear in the sense of PKM's folding back.

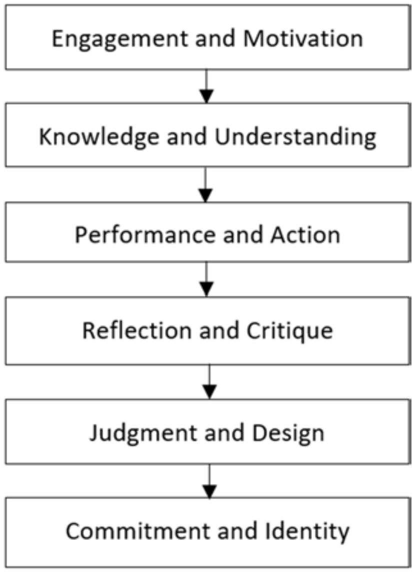

Figure 8. Shulman's table of learning, from Shulman (2004).

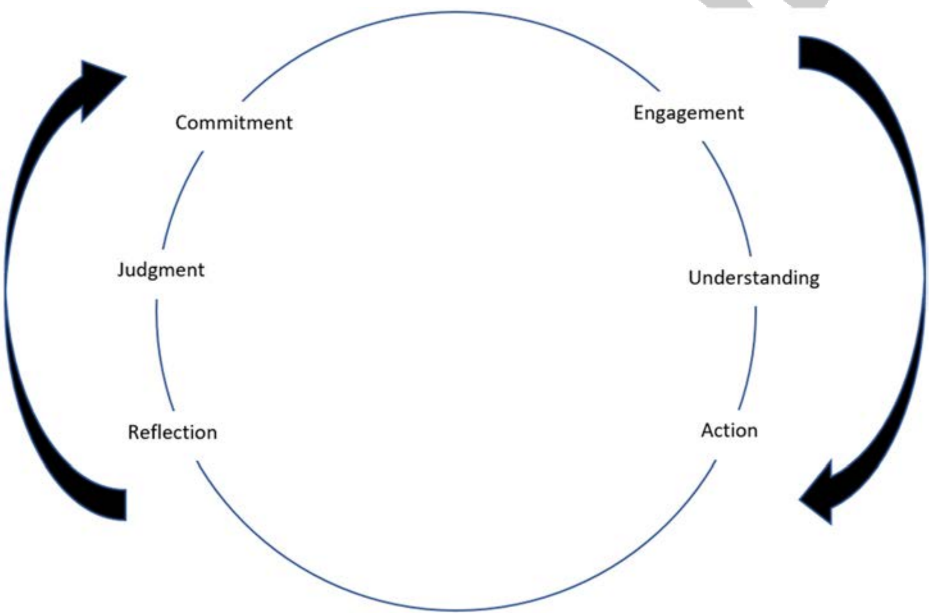

Figure 9. Shulman's conception of continuous learning; from Shulman (2004, p. 75)

Much current research in complexity theory indicates that learning is anything but linear (Davis, 1996; Davis et al., 2008). Learning is most frequently nonlinear, recursive, and emergent in nature. The concept of folding back, together with the concept of thickening, aptly illustrates a theory of learning that is congruent with complexity theory. PKM's folding back is a significant advancement in thinking about theories of learning. The break with traditional linear theories of learning is a major contribution of PKM, and a foundation for thinking about how students learn in the future.

CONCLUSION

PKM is a well-developed theory of learning that is supported by a significant volume of research evidence. When first proposed over thirty years ago it was a massive paradigm shift away from the linear theories of learning that dominated the thinking of the time. Now supported by research evidence across multiple grade levels and multiple subjects and topics within mathematics, PKM is a foundational theory within mathematics learning. The revolutionary concepts of folding back and thickening fundamentally change the lenses that can be applied to student learning, and thus also change perspectives on how students can be supported and assessed by teachers in order to optimize student understanding in mathematics.

Further, PKM has made contributions in theory. Linear taxonomies of education can no longer be assumed to validly represent students' path to understanding, and linear theories of learning can no longer be supported, based on the abundant research evidence supporting PKM. It is somewhat surprising that PKM had not so far occupied a more prominent place as a foundational notion of students' growth of mathematical understanding. As a theory of learning congruent with complexity theory,

PKM occupies a significant place in both the theory and practice of students' mathematical understanding and should therefore feature prominently in teacher education programs as well as teacher professional development.

Finally, it is informative to examine the potential impact of PKM on mathematical well-being (MWB; Clarkson, 2013; Clarkson et al., 2010), the development of which presents a five-level model of MWB but says nothing about the potential for folding back to lower levels, as well as the possibility that MWB is not a state but rather trait phenomenon and may present at different levels for different levels of understanding under PKM.

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