Student *Commognitive* Analysis in Solving Algebraic Problems

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Abstract: This study aims to describe students' commognitive in solving algebraic problems. This research is a qualitative research with descriptive approach. The subjects of this research were fifth semester students of the mathematics education study program at the College of Taman Siswa Teacher Training and Education. The research subjects were 9 students and 3 people were selected each as representatives of subjects who answered questions without solving Polya problems (S1), subjects answered with Polya problem solving but were incomplete (S2) and subjects who used Polya problem solving and were correct. (S3). The research method consists of four steps, namely: preparation, research subjects and locations, data collection, and data analysis. The research instruments were algebraic test questions and interviews. The results of the Research showed that the S1 subject only raises word use in solving problems. For the S2 subject, besides raising the word use, it also uses an exploratory routine in solving questions, namely using the necessary but wrong procedure. S2 also experienced an error when determining the time from East Indonesia Time (WIT) to West Indonesia Time (WIB) where S2 added up instead of subtracting 2 hours so the result was wrong. Then for the S3 subject, bring up word used, symbolic mediators, exploratory and ritualistic routines and endorsed narratives in solving problems. The research findings showed that of the 9 research subjects, only 1 subject has all four commognitive components with Polya problem solving, the other 8 are still incomplete. Thus, for further research it is recommended that to see students commognitive it is necessary to use Polya's problem solving with positioning in group discussions.

Keywords: commognitive, students, solving algebraic problems, word use, visual mediators, routines, endorsed narratives

INTRODUCTION

Communication is an important part of mathematics in general and mathematics education. Through communication, ideas become objects of reflection, improvement, discussion, and change. The communication process also helps to build the understanding. When students are challenged to think and make reasons about mathematics and communicate the results of their thoughts to others either orally or in writing, they learn to explain and convince (NCTM, 2000). Brodie agrees that an understanding of mathematics is assumed through mathematical communication (Brodie et al., 2010). It can be said that communication is an inseparable part of the understanding. Brodie further stated that because communication is an important part of understanding, communication is used by students to discuss their understanding with others (Brodie et al., 2010).

The Ontario Ministry of Education explains that communication is the process of expressing mathematical ideas and understanding orally, visually and in writing, using numbers, symbols, pictures, graphs, diagrams, and words (Education, 2005). Students communicate for a variety of different purposes and opponents, such as communicating with teachers, peers, a group of students, or an entire class. Communication is an important process in learning mathematics. Through communication students can contemplate and reflect on their ideas, their understanding of mathematical relationships and their mathematical arguments.

Mathematical communication is an important process in learning mathematics (Cohen et al., 2015; Daher, 2012; Kieran, 2001; Kosko, 2014; Lestari et al., 2019; Thinwiangthong et al., 2012; Umar, 2012). Sfard further stated that mathematical communication is a process of conveying mathematical ideas both in writing and orally by each individual (Sfard, 2001, 2008, 2015). Communication is an important skill in mathematics because it is used to express mathematical ideas to oneself or others either in writing in the form of diagrams, symbols or orally. Mathematical communication is a process of conveying messages, ideas, ideas or opinions in mathematical terms both in writing and orally.

When coming up with an idea to solve the problem at hand, there is information processing that occurs. Information processing is a mental process known as cognitive process (Campos et al., 2013; Iglesias-Sarmiento and Deaño, 2011; Sánchez et al., 2013). Cognitive processes are mental processes in individuals (Montague et al., 2014). Cognitive processes that occur within one's self includes: 1) the process of obtaining new information, 2) the process of transformation information received, 3) the process of testing or evaluating the relevance and accuracy knowledge (Sutarto, 2017). Cognitive processes can be understood as a process of getting new information in memory to be digested and understood into a knowledge.

Communication and cognition are known as *commognitive* (Caspi and Sfard, 2012; Kim et al., 2017; Sfard, 2001, 2006, 2008, 2015; Sriraman, 2009; Viirman, 2015). *Commognitive* can be interpreted as a mental process and the delivery of information to oneself or others that is carried out verbally or non-verbally. *Commognitive* consists of four main components, namely *word use, visual mediators, endorsed narratives* and *routines. Word use* is the use of words in learning mathematics. *Visual mediator* is the media used in learning mathematics. *Visual mediators* can also be in the form of graphs, diagrams and symbols, as well as physical objects used as media/ props. *Routine* is a process of rules, steps that describe a pattern in learning mathematics. The steps in learning are defined as defining, estimating, proving, and generalizing. *Narrative* is a mindset used in learning mathematics about definitions, theorems, principles and facts (Viirman, 2015).

Commognitive research on students' as prospective teachers in teaching and learning has been carried out before, Berger (2013); Heyd-Metzuyanim & Tabach (2018); Ho et al. (2019); Nardi et al. (2014); Tababaru (2016); Tuset (2018); Viirman (2015); Zayyadi et al. (2019, 2020). Berger (2013) used *commognitive* theory to examine the activities of a pair of mathematics teachers in South Africa in giving mathematics assignments using *Geogebra*. Nardi, et al. (2014) investigated the effective communication through analysis of the use of words and visual mediators in the context of problem solving in small groups, analyzing variations in defining routines and *commognitive conflicts* in the transition from school to university. Viirman (2015) conducted research on explanations, motivations and asking questions in teaching from a *commognitive perspective*. In his research, Viirman investigated the learning practices carried out by seven mathematics teachers which were presented in three categories namely, giving explanations, motivation and asking questions. Explanatory routines including known mathematical facts, summaries and repetitions, different representations, everyday language; motivational routines including use of references, mathematical traits, humor; and routines in asking questions including questions about facts understood by students, controlling questions and rhetorical questions.

Research conducted by Heyd-Metzuyanim & Tabach (2018) explains the implications of the *commognitive theoretical framework* in four areas of practice: pre-service teacher preparation, in-service professional development, introduction to mathematics texts for secondary school students, and diagnosis of learning difficulties in mathematics, and ends with discussions about affordability and the challenges of linking *commognitive* with practice. One of the results of research conducted by Heyd-Metzuyanim & Tabach (2018) is that *commognitive* can be used as a tool for sharing learning experiences owned by a teacher to be given to students in the learning process. Research conducted by Tasara (2017) investigated mathematics teachers teaching basic differentials. In his research, Tasara revealed that the inconsistent use of the word "gradient" in material "gradient" can make it difficult for students to understand when "derivative" is used to mean gradient. This shows that the differences in the use of words in learning must really be considered. In addition, the *commognitive framework* provides a powerful conceptual lens to examine how teachers teach mathematics at the micro level. In this case, the research conducted by Tasara places more emphasis on the components of the use of words and *narratives* in investigating the teaching of the mathematics teacher.

Commognitive frameworks that can provide pre-service teacher teaching information in achieving mathematics learning goals was conducted by Tuset (2018). Tuset showed that a *commognitive framework* can provide an analysis of prospective teachers in the learning process. First, to describe in detail the components of the geometric discourse carried out by prospective teachers to achieve learning objectives. Second, to identify and explain the use of learning tools by prospective teachers provided in educational programs.

Commognitive research involving solving IDEAL problems (I- *Identify problems and opportunities*, D- *Define goals*, E- *Explore possible strategies*, A- *Anticipate outcomes and act*, and L- *Look back and learn***)** conducted by Zayyadi, et al. (2019) shows that students are still more focused on the end result than on the IDEAL problem solving process and many students do not look back at the IDEAL stage in doing their work. From the *commognitive framework*, the subject tends to use mathematical words and visual mediators at the stage of understanding the problem, and narrative and routine at the stage of exploring and implementing strategies. This research provided initial insight into how students' describe mathematic problems from a *cognitive perspective*. Then Zayyadi, et al. (2020) conducted research with the aim of describing the content and pedagogic knowledge skills of prospective teachers in learning mathematics from a *commognitive perspective*. In this reserach, there are fundamental differences in the *commognitive components* of content knowledge and pedagogical knowledge of prospective teachers. The findings of this study related to the *commognitive pedagogical ability of prospective teachers can be referred to as commognitive* pedagogical. Student *Commognitive* can be measured by solving math problems.

Problem solving is the essence of learning mathematics (Subanji, 2013). Problem solving is a type of learning to think at higher level, so mathematics often requires problem solving skills in cultivating students' creative minds (Chong & Shahrill, 2016; Yassin & Shahrill, 2016). One of the frameworks for thinking about problem solving was proposed by (Polya, 1973) where the strategy is recognized by many researchers as the steps used in solving mathematical problems. Polya suggests four stages for problem solving, namely: 1) Understanding *the* problem; 2) Planning the problem (*Devising a plan*); 3) Carry out problem solving (*Carrying out the plan*); and 4) Looking back *(*Lederman, 2009; Lee, 2017; Okafor, 2019; Simpol, et al., 2018; Tohir, et al., 2020) The stages of solving the Polya problem can be presented in Table 1 below.

Problem Solving Stages	Description
Understanding the problem	It should be clear what the question means, what you are looking for the answer. Need to first realize the key point and context of the problem, then be able to find the answer
Devising a plan	Clearly knowing the relationship between problem points, choosing the appropriate approach and drawing up a plan to solve the problem, which is the most important task in the problem solving process
Carrying out the plan	Follow Steps 1 and 2, and practically calculate alone / in groups and find the answer
Looking back	Looking back at the entire troubleshooting process; check calculations and answers; discuss the meaning of the problem

Table 1: Polya's Problem Solving Stages

Polya's stages are closely related to open problem solving. Open problem solving is one way to reveal students' *cognitive* components, namely by giving questions that are non-routine in nature so that they can dig in depth and not rely on just one answer. An example of a non-routine question is an *open ended* question. Subanji (2013) states that an *open ended question* is a question that has a non-single answer or way of solving it. Furthermore (Yee, 2009) also argues that *open ended* questions are questions that have more than one way to solve them, or have various possible correct answers. *Open ended* questions can give freedom to students in expressing answers, encouraging students to generate various kinds of different thoughts according to their abilities as Cifarelli & Cai (2005) stated that problems in *open ended* are directed to guide students in understanding problems that can be solved with a different and correct point of view.

Commognitive research related to solving algebraic problems with *open ended* questions has never been done by other researchers. In this research, it is necessary to conduct a study of student *commognitive* from the point of view of *open-ended algebraic problem solving*. Therefore, this study aims to describe student *commognitive based on open ended algebraic problem solving*.

METHOD

This research aims to describe students' *commognitive* when solving algebraic problems. The research is qualitative, with a descriptive approach. The four important steps in this research are: (1) preparation, (2) research subjects and locations, (3) data collection, and (4) data analysis.

Preparation

In the preparation stage, the researcher developed test and interview instruments that enabled students to be involved in the process of solving algebraic problems. The test instrument involves math questions in Figure 1, which provide opportunities for students to demonstrate the use of *commognitive components* (*words use, visual mediators, routines* and *endorsed narratives*) in solving algebraic questions. In table 2. Furthermore, interviews are designed to articulate students' thought processes when they solve algebraic problems. The following are algebra problem solving test instruments:

Two tourists departed on different airplanes from Jakarta to Jayapura. The first plane took off from Jakarta airport at 20.30 West Indonesia Time (WIB) local time and the second plane one hour later, the first plane landed at Jayapura airport at 08.30 East Indonesia Time (WIT) local time and the difference in time for the second plane arrived 1.5 hours afterwards. If during the flight the first plane stopped at Surabaya and Makassar airports for 30 minutes each, and the second plane stopped at Surabaya, Makassar and Timika airports for 30 minutes each, except for Timika with a 30 minute delay. a. Write down everything that is known from the problem above! b. How many hours does each tourist travel from Jakarta to Jayapura without stopping? c. Is there a difference in the travel time of the two tourists without stopping? d. What can you conclude?

Figure 1: Algebraic Problem-Solving Problems

***Commognitive* Component**		**Indicator**
Visual Mediators	*Symbolic Mediators*	Presenting mathematical information with symbols or algebra in solving problems
	Iconic Mediators	Make representations in the form of graphs, tables, diagrams and pictures in solving problems
	Concrete Mediators	Using real objects as media in solving problems
Word Use	*Mathematical Terms*	Using keywords or mathematical terms contained in the problem in solving the problem
	Mathematical and Non-Mathematical Terms	Using keywords or mathematical terms and not mathematical in solving problems
Routines	*Ritualised*	Can use the necessary procedures in solving problems
	Exploratory	Can provide explanations or reasons for how to solve the problem at hand and can convey when the selection procedure is used
	Applicability	Judging from how to solve the given problem, such as using symbols, making pictures, or directly calculating
	Corrigbility	Examining explanations or narratives of the reasons for using certain procedures or ways of solving problems by making conclusions and checking again
	Flexibility	The use of more than one way to solve a given problem, the general formula or form used and can be seen from the visual media used
Narratives	*Remember and explain*	Can explain reasons and relate objects, relationships with previous material and processes, such as definitions, theorems and proofs in solving problems.

(Adapted from Sfard, 2008; Mpofu & Pournara, 2018)

Table 2: Indicators *Commognitive* Components Used in Solving Algebraic Problems

Research Subjects and Setting.

The research was conducted at Teacher, Training and Education of Taman Siswa Bima NTB Indonesia with research subjects in semester V. it is consisted of 20 members, they were chosen because they had taken school mathematics course. Research subjects were selected using a *purposive sampling technique,* so that out of 20 people, 9 people were selected as research subjects.

Data Collection

The data collection process began with giving math questions based on algebraic problem solving to 9 research subjects to be solved individually. Of the 9 subjects, 1 subject answered using Polya's problem solving and the answer was complete and correct, 4 subjects answered using Polya's problem solving but the answer was incomplete, and 4 subjects answered not using Polya's problem solving and the answer was wrong. After that, the researcher conducted data analysis on 3 subjects, each of whom represented the three categories that the researcher mentioned above.

Data Analysis

In the problem-solving process, students are asked to express their thoughts out loud. Students are given the opportunity to explore, record and express all their thoughts and ideas. Researcher observed and recorded all behaviors including students' verbal thoughts while they were solving open problems. After the students completed the questions given, the researcher then transcribed the data, and the students who were the research subjects were interviewed individually to find out and explore their *cognitive components in solving open problems.* After that, the researcher carried out data reduction by removing elements that were considered unimportant from all data (observations, interviews and field notes) to be examined in data analysis.

RESULTS AND DISCUSSION

In this study, 9 subjects were given questions about algebra and asked to answer by writing all the steps according to the question accompanied by more detailed reasons. Based on the data obtained, there are different strategies used by students in solving the problems. There are students who start with examples so that the solution to the end is clear, while there are others who go straight without examples. Some of the strategies used by students include Polya's problem solving steps. Based on the data collected, there were 9 research subjects as presented in Table 3, consisting of 1 subject who answered correctly with the Polya problem solving steps, 4 subjects who answered with Polya problem solving but were not perfect, and 4 other subjects without using problem solving Polya and wrong. After that, the researcher analyzed the data by looking at the tendency of the answers made by the subject , namely the *commognitive* component based on the stages of solving the Polya problem used .

Using Polya Troubleshooter	No Troubleshooting Polya (*)
5 subjects	4 Subjects who answered did not use Polya's problem solving strategy and their answers were incorrect
1 who answered with Polya's problem solving strategy and the answer was correct	
4 which answered with Polya's problem-solving strategy but was incomplete	

Table 3: Research Subjects

Information:

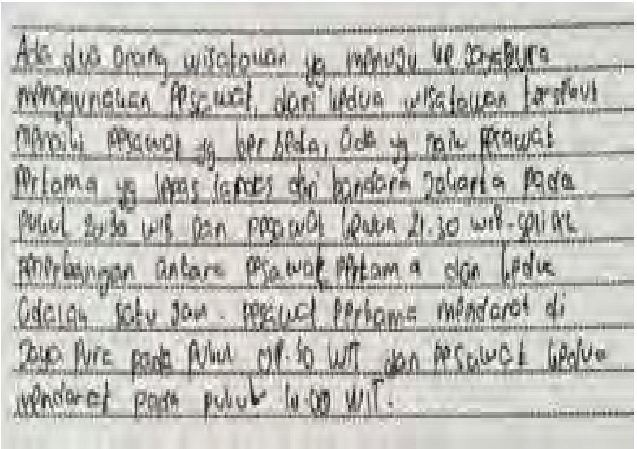
S1 : The subject answered the question without solving the problem and the answer was wrong

S2 : The subject answered the question by solving the Polya problem but it was not complete

S3 : The subject answers the questions by solving the Polya problem and is correct

Subjects without Using Polya Problem Solving (S1)

One of the 4 subjects (*) who answered the question without solving Polya's problem, without paying close attention to the problem and not following the questions asked, so the answer was wrong. For examples of answers from this subject can be seen in Figure 2 below:


Ada dua orang wisatawan yg menuju ke Jayapura menggunakan pesawat, dari kedua wisatawan tersebut memiliki pesawat yg berbeda, ada yg naik pesawat pertama yg lepas landas dari bandara Jakarta pada pukul 20.30 WIB dan pesawat kedua 21.30 WIB sehingga perbedaan antara pesawat pertama dan kedua adalah satu jam. pesawat pertama mendarat di Jayapura pada pukul 08.30 WIT dan pesawat kedua mendarat pada pukul 10.00 WIT.

Tehere were two tourist who went to jayapura by plane, they used different planes, firsts used the first plane which take off from Jakarta at 20.30 WIB. Andwhile the second one take off at 21.30 WIB so the different time of the two planes is 08:30 so the the first plane landed at Hayapura at 08:30 while the second landed at 10 WIT

Figure 2: Example of one answer from 4 subjects (S1)

Information:
WIB : Western Indonesian Time
WITA : Central Indonesian Time
WIT : East Indonesia Time

Based on the data above, S1 does not use Polya's problem solving stages in finding solutions to the problems given. The first subject (S1) immediately rewrote the existing questions without paying attention to the intent of the questions. S1 writes the second plane's arrival time at 10.00 WIT because the time difference between the first and second planes is 1.5 hours. S1 did not pay attention to the time changes from WIB to WITA and from WITA to WIT, so that S1 came to the wrong conclusion. In this case the S1 *commognitive component* that appears is only *word use*, so the researcher tries to explore it by interviewing. The following is a transcript of the results of the researcher's interview with S1:

R : Do you understand the questions given?
S1 : Hmmm, I understand sir (a bit doubtful about the answer)
R : How do you write what is known from the questions above?
S1 : By writing everything in the question sir
R : Oh, I see... How many hours did each tourist travel I and second traveler from Jakarta to Jayapura?
S1 : Tourist I 12 hours of travel because it starts at 20.30 WIB and arrives at 08.30

WIT, for tourists II because it starts at 21.30 WIB and arrives at 10.00 WIT, then the time needed is 12 hours 30 minutes sir.

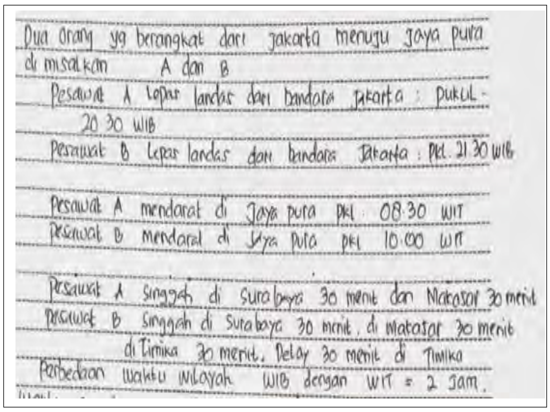
R : Have you not noticed the time difference between WIB, WITA and WIT?

S1 : Oh yes sir, I didn't pay attention to that. (smile)

Based on the researcher's interview with the S1, it can be said that the S1 did not understand the questions well even though the S1 should have understood the differences between WIB, WITA and WIT because this material had previously been obtained at school. It appears that S1 does not yet have sufficient mathematical communication as Uptegrove (2015) states that the influencing factor for effective communication to occur is the students' understanding. When S1 writes everything that is known from the problem, S1 writes everything without choosing which one is important to write. Because S1 also writes the time without paying attention to the time difference from WIB to WITA and from WITA to WIT, S1 assumes that the time at all destinations is the same so that S1 immediately concludes that the travel time for tourists I is 12 hours and the travel time for tourists II is 12 hours 30 minutes, even though the difference between WIB and WIT should have been 2 hours. S1 does not explore strategies that might provide solutions and does not re-check. From this it can be understood that S1 only uses *word use in the commognitive* component in solving questions, namely using the terms difference, star, until, arrive. The terms used by S1 are not only used in mathematics, but these terms are also used in everyday life, namely the use of words (math and non-mathematics) as Yenmez and Özpinar (2017) reveal that solving mathematical problems does not rule out the possibility of using a combination of terms. mathematics and terms in everyday life.

Subject Using Polya Problem Solving but the Answer is Incomplete (S2)

In addition to the types of answers written by the 4 subjects above, there were 5 subjects who provided answers by solving the Polya problem, 4 of which were subjects whose answers were incomplete. The following presents the answers from the Subject (S2):



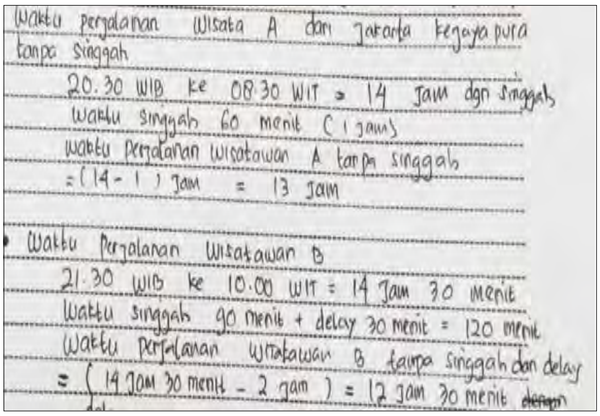
two people departing from Jakarta for Jayapura are A and B
Plane A takes off from Jakarta airport: 20.30 WIB
Plane B takes off from Jakarta airport: 21.30 WIB
Plane A landed in Jayapura at 08.30 WIT
Plane B landed in Jayapura at 10.00 WIT
Plane A stops in Surabaya for 30 minutes and Makassar for 30 minutesPlane B stops in Surabaya for 30 minutes, in Makassar 30 minutes and Timika 30, delays 30 minutes in Timika
The time difference between WIB and WIT = 2 hours

Figure 3: Stages of Understanding & Problem-Solving Plans for S2

At this stage, S2 tries to understand the problem and applies *symbolic visual mediators*, namely for example tourists I and airplane I with symbol A and tourists II and airplane II with symbols B. Then S2 writes down everything that is known by sorting it starting from tourists I and II star from Jakarta until arriving in Jayapura. To dig deeper about *symbolic visual mediators*, the researcher interviewed S2 according to the results of the following transcription:

R : Do you understand the questions given?
S2 : Yes sir, I understand that
R : Why use symbols A and B to compare tourists and aircraft?
S2 : Oh that sir…? To make it easier for me to complete the next question Sir
R : Why is the example not separated between tourists and planes?
S2 : Oh he, it could also be like that sir, but I didn't separate it
R : Are tourists the same as planes?
S2 : Oh that's right, it's different sir

Based on the results of the interview, it is illustrated that S2 does not differentiate between tourists and planes, so the symbols used are the same, namely tourists I with plane I and tourists II with plane II, although this makes it easier for S2 to solve the next problem as (Sfard, 2008) states that *visual mediators* are important in establishing effective communication as they help create a general focal point. Then, the problem-solving stage is shown in Figure 4 below:


Waktu perjalanan wisata A dari Jakarta ke Jayapura tanpa singgah
20.30 WIB ke 08.30 WIT = 14 jam dgn singgah
Waktu singgah 60 menit (1 jam)
Waktu perjalanan wisatawan A tanpa singgah
= (14 - 1) jam = 13 jam
Waktu perjalanan wisatawan B
21.30 WIB ke 10.00 WIT = 14 jam 30 menit
Waktu singgah 90 menit + delay 30 menit = 120 menit
Waktu perjalanan wisatawan B tanpa singgah dan delay
= (14 jam 30 menit - 2 jam) = 12 jam 30 menit dengan

Travel time for traveler A from Jakarta to Jayapura without stopping:20.30 WIB to 08.30 WIT = 14 hours with a stopover, a layover time of 60 minutes (1 hour) Traveler A's travel time without a stopover = (14 – 1) hours = 13 hours
Tourist travel time B 21.30 WIB to 10.00 WIT = 14 hours 30 minutes Lay time 90 minutes + delay 30 minutes = 120 minutes Travel time for tourist B without stops and delays = (14 hours 30 minutes – 2 hours) = 12 hours 30 minutes

Figure 4: S2 Problem Solving Stages

At the problem solving stage, even though S2 used *the exploratory routine* , namely using the necessary procedures to solve the problem (Mpofu and Pournara, 2018), S2 was wrong in determining the travel times of tourists I and II. S2 already knows that the difference between WIB and WIT is 2 hours, but S2 mistakenly places it so that the narration is wrong. The following are excerpts from the results of interviews with S2 researchers:

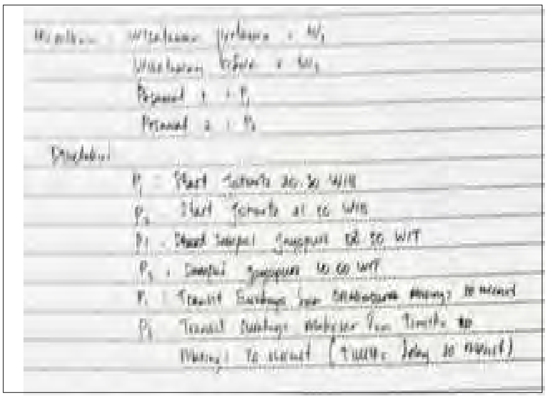
R : Do you know the difference between WIB and WIT?
S2 : Yes sir, I know
R : How much is the difference?
S2 : 2 hours sir
R : Which time is faster, WIB or WIT?
S2 : WIT Sir
R : Pay close attention to your answers, if you say that WIB is 2 hours faster than WIT, but why is your answer like that?
S2 : Which one sir?
R : This (while pointing to S2's answer which wrote 14 hours and 14 hours 30 minutes), try to count first, start at 20.30 WIB and arrive at 08.30 WIT, how many hours before changing to WIB?
S2 : 12 hours sir
R : Well, because the stars from WIB and WIB are 2 hours faster than WIT, plus or reduced?
S2 : Oh yes sir, I was wrong, it should have been reduced, not added. Means not 14 hours sir but 10 hours minus 1 hour layover time to 9 hours. So are for tourists II sir, it must be reduced by 2 hours not added by 2 hours. So 10 hours 30 minutes reduced again by 2 hours layover and delay, so 8 hours 30 minutes
R : Got it now
S2 : Yes Sir…thank you for reminding me

From the excerpts of the interview results above, it can be seen that S2 mistakenly determined the time difference from WIB to WIT, where S2 added 2 hours instead of subtracting 2 hours, so that the travel time for tourist I from Jakarta to Jayapura was 9 hours without a stopover, calculated by S2 13 hours, as well as for tourist II, the journey time should have been 8 hours 30 minutes because the stop and delay time of 120 minutes or 2 hours was calculated by S2 12 hours 30 minutes so that S2's answer was automatically wrong. This is as the expert stated that *routine* describes a person's activity patterns such as calculating, proving and abstracting (Tasara, 2017). Because S2 only knows that the difference between WIB and WIT is 2 hours, and does not know which time comes first, S2 has difficulty synthesizing or integrating his knowledge about real life with solving his mathematical problems. S2 experienced differences *in word use* between mathematics and non-mathematics, but S2 could not correctly determine how this led to a ritualized solution-activity so that S2's explanation was incomplete.

After conducting the interviews, S2 proposes a good narrative that includes all the information required by the researcher, but S2 reverses the procedure when synthesizing real life into a mathematical problem. After the researcher kept digging with questions, S2 realized that he was using real life knowledge incorrectly so he knew what the correct answer looked like. Therefore, S2 internalizes or adapts new information and adapts flexibility although at a lower level than presented. In contrast to S1 who ignores real life knowledge because it is too difficult for him to process and integrate it into problem solving so S1 ignores it. Based on this opinion, S2's activity pattern is wrong, because at the problem-solving stage it is wrong, then automatically S2's conclusion is wrong.

Subject Using Polya Problem Solving and Answer Correct (S3)

Of the 5 subjects who answered with Polya's problem solving steps, only 1 subject gave the correct answer and all four *commognitive components* appeared, namely *visual mediators, word use, routines* and *narratives*. The subject is named as S3. At the stage of understanding the problem, S3 tries to exemplify tourists I and II with symbols W1 and W2, then planes I and II with symbols P1 and P2. This indicates that S3 understands the question well, shown in Figure 4 below:



First travelers: W1
Second traveler: W2
Plane 1: P1
Plane 2: P2
Is known:
P1: start Jakarta 20.30 WIB
P2: start Jakarta 21.30 WIB
P1: until Jayapura 08.30 WIT
P2: until Jayapura 10.00 WIT
P1: transit Surabaya and Makassar 30 minutes each
P2: transit Surabaya, Makassar and Timika 30 minutes each (Timika delay 30 minutes)

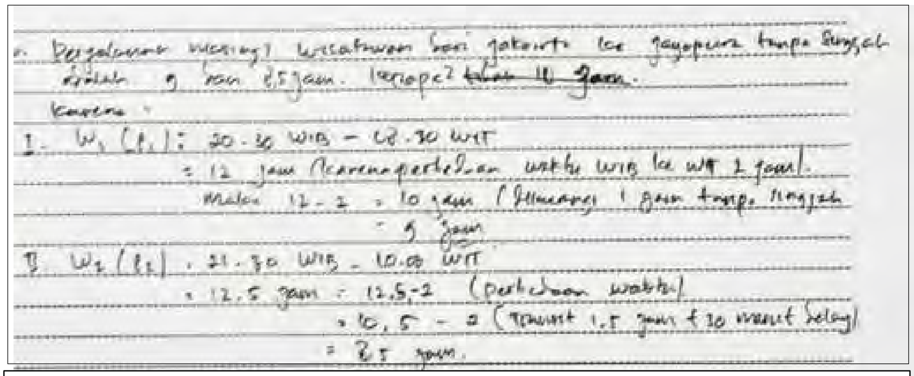
Figure 5: Stages of Understanding & S3 Problem Solving Plan

At the stage of understanding the problem and planning the problem, it appears that S3 uses *word use* well, where S3 after assuming tourists with symbols W1 and W2 and airplanes with symbols P1 and P2. Then S3 determines the starting time for P1 and P2 from Jakarta and arriving at Jayapura, including the transit time. After that, at the problem-solving planning stage, *visual mediator symbolic* S3 shows how to present mathematical information with symbols or algebra in solving problems. The following is a transcript of the results of the researcher's interview with S3:

R : Do you understand the questions given?
S3 : Yes sir, I really understand it
R : How do you understand it?
S3 : First of all we assume the first tourists with W1 and the second tourist with W2, as well as the plane, the first plane with P1 and the second plane with P2
R : What is your goal for example like that?
S3 : To make it easier for me to complete the following questions Sir, because if it's not like that, then I will be in trouble finish it.

Based on the interview above, it is illustrated that S3 performs *visual mediators* with *symbolics* with the aim of making the subject more effective in solving the problems given, as Ryve et al.

(2013) revealed that one of the factors that causes effective communication is the use of *visual mediators*. Then the next step, S3 solves the problem properly and correctly as shown in Figure 5 below.



Perjalanan masing2 wisatawan dari jakarta ke jayapura tanpa singgah adalah 9 dan 8,5 jam. kenapa?
karena :
I. W1 (P1) : 20.30 WIB – 18.30 WIT
= 12 jam (karena perbedaan waktu WIB ke WIT 2 jam).
Maka 12 – 2 = 10 jam (dikurangi 1 jam tanpa singgah
= 9 jam
II. W2 (P2) : 21.30 WIB – 10.00 WIT
= 12,5 jam = 12,5 – 2 (perbedaan waktu)
= 10,5 – 2 (transit 1,5 jam + 30 menit delay)
= 8,5 jam.

The journey of each tourist from Jakarta to Jayapura without stopping is 9 and 8.5 hours. Why?
Because:
I. W1 (P1) = 20.30 WIB – 18.30 WIT
= 12 hours (because the time difference between WIB and WIT is 2 hours)
Then 12 – 2 = 10 hours (minus 1 hour without stopping)
= 9 hours
II. W2 (P2) = 21.30 WIB – 10.00 WIT
= 12.5 hours = 12.5 – 2 (time difference)
= 10.5 – 2 (1.5 hours transit + 30 minutes delay)
= 8.5 hours

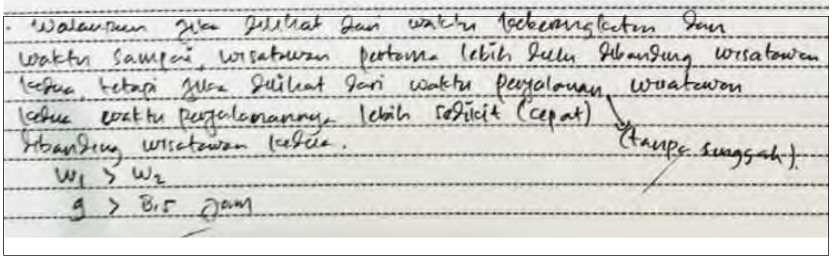
Figure 6: S3 Troubleshooting Stages

Observing the solution to the problems carried out by S3 above, S3 uses *exploratory routines* and *ritualized routines.* Following are the results of the researcher's interview with S3.

R : Are you sure that your work is correct? (while pointing at sheet answer S3)
S3 : Very sure sir (answer enthusiastically)
R : Why are you so sure?
S3 : Because I determine the length of the first tourist's trip with the second traveler, where W1 travels for 12 hours because the difference between WIB and WIT is 2 hours so the time becomes 10 hours, then reduced the layover time by 1 hour, so 9 hours. Then for W2, In the same way, the travel time is calculated first, we find the number 12.5 hours or 12 hours 30 minutes minus 2 hours to 10 hours 30 minutes, then subtracting again the 2 hour layover and delay time, so the result is 8 hours 30 minutes.

Based on the results of the doctoral work reinforced by the interviews, it can be understood that the doctoral in solving questions uses an *exploratory routine* where the doctoral student provides an explanation of how to solve the problem at hand and can convey when the selection procedure is used. The explanation given by S3 is correct, namely S3 determines in advance the time used from Jakarta to Jayapura, then S3 explains the change in time from WIB to WIT, then reduces it with layovers and delays. By exploring this, S3 demonstrates its ability to solve a given problem. At the same time, S3 also performs *ritualized routines*, namely using the necessary procedures to solve problems. So in this regard, Thoma and Nardi (2016) explain that *routine* can be said to be a description of the subject's pattern of activity when solving a given problem.

The final stage of solving Polya's problem is looking back. At this stage S3 checks again to ascertain whether the work is correct or needs to be repaired again, after which S3 gives a conclusion. The following is the conclusion as the last step of S3 solving the questions given, as shown in Figure 7 below:



Although when viewed from the time of departure and arrival time, the first tourist is ahead of the second tourist, but when viewed from the travel time (without a stopover) the second tourist travels less (faster) than the second tourist.
W1 > W2
9 > 8.5 hours

Figure 7: Stages of Looking Back at S3

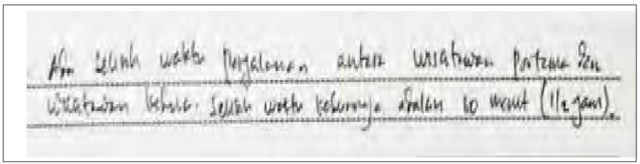
The following is a transcript of the results of the researcher's interview with S3:

R : Why do you conclude that?

S3 : Because based on my description from above Sir, it was found that the time used by W1 is more than the time used by W2, although W1 starts earlier and arrives faster

R : Why is it like that?

S3 : Because W2 has more layover time and delay than W1, where W1 the layover time is only 1 hour while the W2 has a 1.5 hour layover time Plus the delay is 1 hour so it's 2 hours. So there is a time difference between the two of them 30 minutes W1 is faster, like this sir (S3 shows the difference in arrival time both of them).



There is a difference in travel time between the first tourists and the second tourists. The time difference between the two is 30 minutes (1/2 hour)

R : Oh I see...

S3 : Yes sir.

From the results of the interview above, S3 seems to use *endorsing narratives* to conclude his work, where S3 provides the argument that the second traveler's travel time is less than the time used by the first traveler even though the first traveler is 1 hour earlier than the second traveler. Then S3 mentions the difference in the travel time of the two. Thus, S3 analyzes the questions well so that they arrive at the correct conclusion, namely linking the answers written previously with the rules so that the conclusion of S3 is W1 > W2 because 9 hours is more than 8 hours 30 minutes. In line with (Sfard, 2007, 2008) which states that *endorsed narratives* are descriptions or descriptions so that they can be judged as true or false.

CONCLUSION

This Research reveals the *commognitive* of students in solving Polya problems, namely mental processes and conveying information to themselves or others in the process of expressing ideas to solve open problems which are carried out verbally and non-verbally. Students' *cognitive* in this study consisted of four components, namely *word use*, *visual mediators*, *routines* and *endorsed narratives.* The research results show that:

Subject 1 (S1), at the lowest level, it is not seem to use *symbolic visual mediators* effectively, he also fails to recognize the importance of non-mathematical terms in problem situations, namely *word use* and the first stage of Polya, the stage of understanding the problem. So, S1 performed math routines in the right way but the problem solving planning stage is flawed because he fails to realize the need to integrate routines that involve non-mathematical problem information. As such, the narrative is not complete, and while S1 understands that it is important when pointed out by the researcher the verification step of solving Polya's problem, S1 cannot easily adapted the narrative due to lack of flexibility. The researcher concluded that weaker problem solvers have difficulty understanding the importance of non-mathematical terms and failed to formulate correct problem-solving strategies. S1 also experience flexibility in adapting or accommodating information presented by researchers who show errors, a slow internalization process.

Subject 2 (S2), the problem solver is showing good use of *visual mediators symbolic* in solving the Polya stage 1 problem. S2 formulates the right strategy and gives a good narrative. However, the terms synthesis of real life and unreal life are wrong so the routine is wrong. With the assistance of researchers, S2 seemed to be aware of his mistakes like S1 but also adapted and fixed them using the verification step in Polya's troubleshooting. Subject 3 (S3) is a subject who is able to complete all stages of Polya's problem solving correctly, using effective mediators, relevant and accurate routines and presenting good narratives for their activities. The researcher speculates that peer interaction will help students easily recognize their mistakes when adapting. Due to their flexibility, low level students may need more assistance, otherwise they may find it very difficult to work in groups. We suggest this for future research using positioning in group discussions.

Acknowledgements

We do thanks to State University of Malang Indonesia for costing this research of disertation under the supporting of contract number 5.4.652/UN32.20.1/LT/2023, and all stakeholders involved include College of Teacher Training and Education (STKIP) Taman Siswa Bima Indonesia

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