Journal on Mathematics Education
Volume 14, No. 3, 2023, pp. 449-468

Prospective elementary teachers' informal mathematical proof using GeoGebra: The case of 3D shapes

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Received: 15 February 2023 | Revised: 17 April 2023 | Accepted: 9 June 2023 | Published Online: 16 June 2023

Abstract

Mathematical proofs play a paramount role in developing 21st-century skills, and the use of technology in mathematics learning has widely paved the way in the instruction of mathematical proofs. In mathematics education, GeoGebra has a significant role as a dynamic mathematics software in supporting students' learning process. This study aims to use GeoGebra in supporting prospective elementary teachers' mathematical proofs of the volume of 3-D shapes. This research used a case study method with 23 first-year prospective elementary teachers as participants from a public university in Riau, Indonesia. The data were gathered by means of students' work recordings in the GeoGebra classroom and video recordings from their interactions in the course of small group and classroom discussions. The videos were transcribed using verbatim, and then the mathematical proofs were analyzed using praxeological analysis. The findings show that prospective elementary teachers still had challenges to connect the construction of the volume of 3-D shapes using GeoGebra to its informal mathematical proofs. However, GeoGebra provides an opportunity to learn informal mathematical proofs for prospective elementary teachers.

Keywords: 3D-Shapes, GeoGebra, Mathematical Proofs, Prospective Elementary Teachers

How to Cite: Putra, Z.H., Afrillia, Y.M., Dahnilsyah, & Tjoe, H. (2023). Prospective elementary teachers' informal mathematical proof using GeoGebra: The case of 3D shapes. *Journal on Mathematics Education, 14*(3), 449-468. http://doi.org/10.22342/jme.v14i3.pp449-468

Proof is an important aspect of mathematics and helps to develop mathematical knowledge in student learning (Zengin, 2017). Mathematical proofs are without a doubt the foundation of mathematics (Balacheff & Boy de la Tour, 2019). However, several studies have shown that many students struggled with proof construction (Baker & Campbell, 2004; Moore, 1994; Weber, 2001). Therefore, mathematics learning at the tertiary level emphasizes on students construct mathematical proofs to support their development of mathematical thinking and reasoning (Syamsuri et al., 2018).

In recent years, the application of technology into the instructional process has received much attention (Bulut et al., 2015; Magana, 2014; Radović et al., 2020), but so far few studies if any focus on mathematical proof. Jankvist and Misfeldt (2019), for instance, investigated the didactical impact of CAS-aided proofs in secondary mathematics textbooks. They discovered that employing CAS as an integrated element of deductive mathematical proofs in textbooks resulted in an undesirable generation of proof schemes among students, as well as the difficulty in understanding these problems using the frameworks of epistemic and pragmatic mediations. Furthermore, a study conducted by Nyaumwe and Buzuzi (2007) focused on teachers' attitudes towards proofs in the secondary school mathematics curriculum, and the findings demonstrated that teachers expressed neutral attitudes toward technology as a method of proof. Conversely, proof using technology captures certain mathematical details that can aid in understanding some more sophisticated notions (Guerrero, 2018), but there are still limited studies focused on the use of technology in supporting students' and teachers' understanding of mathematical proofs.

Among many technological tools to support learners to learn mathematics, GeoGebra has been a prominent provider for learning and teaching mathematical proofs, such as employing the dynamic mathematics software of GeoGebra to teach geometrical proofs (Balacheff & Boy de la Tour, 2019). GeoGebra has proven to be a useful technique for improving prospective teachers' attitudes toward evidence and proving (Zengin, 2017). Putra et al. (2021) have found that GeoGebra can be used as a pedagogical tool in improving prospective elementary teachers' mathematical and didactic knowledge. It becomes interactive mathematical software that can be used to demonstrate or visualize mathematical concepts and as a tool to construct mathematical concepts (Tamam & Dasari, 2021).

GeoGebra integration in teacher education has supported prospective elementary teachers in constructing 3-D shapes (Putra et al., 2021). Prospective elementary teachers could improve their comprehension of mathematical concepts such as how to create the formula for the volume of spheres by utilizing GeoGebra. The prior study, however, did not focus on how prospective elementary teachers develop their mathematical proofs of any 3-D shape formula. As a result, the current study aims to evaluate how prospective elementary teachers can generate informal mathematical proofs of 3-D shape formulas using GeoGebra. More specifically, the present study will address the following research questions:

1. What are the aspects of informal proofs of 3-D shape formula prospective elementary teachers develop in supporting learning instruction using GeoGebra?
2. How do prospective elementary teachers use their informal proofs of a 3-D shape formula to support informal mathematical proofs for other formulas using GeoGebra?

Mathematical Proof

The proof is a series of logical statements, one suggesting the next, that explains why a particular proposition is true (Stefanowicz et al., 2014). As a foundation of mathematical knowledge, proofs and proving are important aspects of mathematics education (Syamsuri et al., 2018). However, there are some difficulties in mathematical proofs. Moore (1994) stated that there are seven difficulties in mathematical proving: the students (a) did not know the definitions; (b) had little intuitive understanding of the concepts; (c) had insufficient concept images for doing the proofs; (d) were unable, or unwilling, to generate and use their examples; (e) were unable to understand and use mathematical language and notation; (f) did not know how to use definitions to obtain the overall structure of proofs; and (g) did not understand how to start proofs. One of the primary goals of mathematical proofs is to communicate mathematical ideas rather than formal specifics in reasoning (Sjögren, 2010).

The degrees of informality in mathematical proofs are considerably varied; for example, proofs in abstract algebra or analysis real might be highly comprehensive, but proofs in more advanced literature typically skip details (Sjögren, 2010). Although mathematical truths can be understood without strong proof, such knowledge fails to meet the standards we should expect of mathematical knowledge. However, by developing rigorous proofs (i.e., formalizing ordinary informal proofs), learners may increase their understanding of these facts by learning mathematical proofs for specific purposes (Marfori, 2010).

In the present study, we use informal mathematical proofs. In common mathematical practice, informal mathematical proofs are incomplete proof sketches that might be formalized by an expert in the relevant topic (Marfori, 2010). There is some formal higher-order language and a deductive system where a corresponding formal proof exists for each informal mathematical proof treated as (implicitly) higher-order, and hence there is some first-order system where a corresponding formal proof exists as well (Azzouni, 2009). In theory, every informal proof should be turned into a formal derivation in a suitable formal system (Marfori, 2010). However, in elementary school, mathematical proofs may refer to informal mathematical proofs where details of the step for proving are not present. For example, the volume of a cuboid as height, width, and length may be proven by examining how many unit cubes cover the volume of a cuboid.

Mathematical Proof using Technology

On the use of technology, learners may be encouraged to consider existing software for use in evidentiary learning and teaching (Sümmermann et al., 2021). The primary role of proof is to verify the truth of mathematical propositions (De Villiers, 2004). When proof is considered to be highly essential in mathematics education, proof practices performed with GeoGebra software make a significant contribution to students and teachers (Zengin, 2017). Teachers have only given proofs in the geometry classroom as a technique of achieving certainty; that is, to try to raise concerns in their pupils' minds about the validity of their empirical findings, and so to drive a need for deductive proof. It has also dominated most mathematics teacher education programs (De Villiers, 2004).

GeoGebra is a great tool for improving the quality of learning, especially for exploring, visualizing, and constructing mathematical proofs. It improves students' mathematical abilities such as mathematical proof reasoning and problem-solving abilities. GeoGebra is beneficial to both students and teachers, and it is simple to use and accessible from anywhere and at any time (Hohenwarter & Fuchs, 2005; Tamam & Dasari, 2021). Therefore, it is a potential for using GeoGebra to understand mathematical proofs, especially geometry.

Teachers' Knowledge of Mathematical Proofs

A study conducted by Sumardyono (2018), discovered that most teachers did not possess adequate competence in compiling mathematical proofs. Most teachers did not understand the mathematical proofs, even most of them could not remember correctly. Prospective teachers had difficulty in learning geometry including difficulty in responding to the goals and objectives of the learning and determining the beginning of a proof, difficulty in finding ideas, difficulty in applying definitions, characteristics, and theorems in constructing proofs, and difficulty in determining the correct steps of proof (Reflina, 2020).

Teachers relate to the GeoGebra program, which influences prospective teachers' views about evidence (Zengin, 2017). Following the proof activities using GeoGebra, prospective teachers highlighted the importance of proof. They stated that the program provided an engaging, intriguing, amusing, and visually stimulating learning environment (Zengin, 2017). Furthermore, they indicated that this learning environment promoted genuine learning rather than rote learning. It was discovered that the program concretized abstract proofs, allowing proofs to be comprehended better and simpler. As a result, prospective teachers discovered that the software made proofs more persistent. In addition to the software's beneficial benefits, it was discovered that prospective teachers liked proofs more than the others as they had less anxiety about proofs, and were more confident in themselves while proving a theorem (Zengin, 2017).

Proofs, for example, may be made more entertaining and enjoyable for students by using GeoGebra software (Zengin, 2017). GeoGebra would allow students to widely construct their idea and expand their knowledge (Faris, 2018). Integrating GeoGebra software into all levels of education, from primary to higher education, can help teachers improve teaching and learning at all levels (Zengin, 2017).

Praxeological Analysis as a Theoretical Framework for Mathematical Proof

The theoretical framework for mathematical proof used in the present study is based on the anthropological theory of the didactic (ATD, Chevallard, 1992). ATD is a mathematics education theory to model a human action using a basic unit known as praxeology (Bosch & Gascón, 2006; Chevallard, 1992, 2006). According to Chevallard (2006)

"A praxeology is, in some way, the basic unit into which one can analyse human action at large. [ ] What exactly is a praxeology? We can rely on etymology to guide us here – one can analyse any human doing into two main, interrelated components: praxis, i.e., the practical part, on the one hand, and logos, on the other hand. "Logos" is a Greek word that, from pre-Socratic times, has been used steadily to refer to human thinking and reasoning – particularly about the cosmos. Let me represent the "praxis" or practical part by P, and the "logos" or noetic or intellectual part by L, so that praxeology can be represented by [P/L]. How are P and L interrelated within the praxeology [P/L], and how do they affect one another? The answer draws on one fundamental principle of ATD – the anthropological theory of the didactic –, according to which no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast."

A praxeology consists of praxis and logos. The praxis is made of a type of task (T) and techniques (τ) to solve the given task. While logos are also made of two components, technology, and theory. Technology is used for explaining the technique, and then theory works to explain a set of technologies (Putra, 2019). The praxeology of an institution related to the teaching and learning process of mathematical praxeology is modelled in didactic praxeology (Putra, 2019; Rasmussen, 2016). For example, the praxeological analysis in the didactical study of phenomena related to the teaching and learning of abstract algebra (Hausberger, 2018). Many tasks in abstract algebra involve proofs, thus logistic praxeology (Jovignot et al., 2017). In particular, it allows identifying the steps at which mathematical arguments are silenced, supporting that mathematics and logic are closely related in proofs (Jovignot et al., 2017).

"Mathematical praxeology", i.e. the scientific description and analysis of what we humans do and what happens when we "do mathematics" (Chevallard, 2006). If teachers struggle alone with incomplete praxeology, they will find it difficult to support students' mathematical praxeology. The need for changes in students' (mathematical) praxeology also has consequences for teachers' didactic praxeology, because they have to face didactic tasks to support students' praxeology (Putra, 2019). When we focus here on the praxeology of mathematics taught in universities, it is clear that praxis (e.g. calculating the learned Fourier series of a given function) is closely related to various forms of logos - from ad hoc explanations of standard techniques to theories involving general definitions, theorems and evidence (Kondratieva & Winsløw, 2017).

Praxeological application in mathematical proofs is by examining the structure of objects, and the possibility of generalizing statements and proofs. From the insertion of these generalizations into an axiomatically organized theory, abstract axiomatic structures can be applied as a viewpoint of simplifying conceptual generalizations to demonstrate the properties of mathematical objects (Hausberger, 2018).

METHODS

Research Approach

This study applied a case study method to deeply analyze and evaluate what prospective elementary teachers had learned from the integrated GeoGebra in mathematical proofs in the initial elementary teacher training. The case study approach enables a thorough comprehension of the phenomena (Heale & Twycross, 2018). Gerring (2007) indicated that a case study may be defined as an in-depth examination of a particular instance to shed light on class situations in greater extent.

The present study focused on a case study on the use of GeoGebra to prove the volume of some 3-D shapes by first-year prospective elementary teachers. This work was part of the researchers' major study for developing a model to support teachers' mathematical, didactic, and technological knowledge by integrating GeoGebra into mathematics instruction.

Participants

Participants were twenty-three first-year prospective elementary teachers (3 men and 20 women) from an elementary teacher education study program, at the public university in Riau, Indonesia. All participants were in the even semester of the 2021/2022 academic year. They were taking a course on Geometry and Measurement and had already completed a fundamental mathematics course about Numbers and Algebra. The topics covered in that course were basic geometry figures, perimeter and area of 2-D shapes, volume and surface areas of 3-D shapes, correlations between angles and measures, congruent triangles, characteristics of parallelograms, and similar triangles. The primary textbook for this course was written by Bittinger and Beecher (2012) and all instructional resources were made available on Google Classroom, which was integrated into the University learning management system.

Design of the Tasks

The designed tasks in this study pertained to informal mathematical proofs about the volume of 3-D shapes which were based on the didactical research of the use of GeoGebra to support prospective elementary teachers' knowledge. The first task was about mathematical proof of the volume of cuboid. This task was adapted from Brzezinski (2022) under the title of "Volume: intuitive introduction". The researchers translated the tasks into Indonesian and then added some questions to support participants to come to informal mathematical proofs about the volume of the cuboid. A similar idea was used to design the tasks of the volume of a pyramid (Dhakal, 2022), tube, and cone (Ramsay, 2022). Before the tasks were given to the participants, the research team had a focus group discussion to evaluate the quality of the tasks. Besides, three fourth-year prospective elementary teachers were asked to solve the tasks and asked about their understanding of the instructions in the tasks. After considering no such challenges and ambiguities in those tasks, the researchers considered the tasks sufficient to be used to collect the data.

Research Procedures

This study was conducted in the course of Geometry and Measurement class within 2 months (3 hours of lectures in a week). The content about mathematical proofs of some 3-D shapes was presented in the last meeting before the final exam. It means that prospective elementary teachers had almost completed those course contents. In that meeting, the researchers prepared 3 types of tasks integrated into the GeoGebra classroom: 1) cuboid, 2) pyramid, and 3) tube and cone. After that, there reflected those task types. In solving those tasks, the participants were first asked to work individually, discussed in a small group, and then in the classroom discussion.

Data Analysis

The collected data were prospective elementary teachers' work recordings in the GeoGebra classroom, and video recordings from their interactions during small group and classroom discussions. The videos were transcribed verbatim, and then the mathematical proofs were analyzed. The analysis of informal mathematical proofs was meant to investigate the logos part of mathematical praxeologies answered and discussed by prospective elementary teachers. The logos part was inferred from the explanation of techniques presented to solve a mathematical task by prospective elementary teachers. We classified the parts of the logo into 5 to 6 criteria; no answer, given an answer that does not relate to the task, present formula without any justification, and 2 to 3 informal mathematical proofs from simple to more complex proofs. To maintain the credibility of the data analysis, the first and second authors analyzed the logos of prospective elementary teachers' answers separately. For questionable answers, the first two researchers conducted a discussion with the others to have an agreement.

RESULTS AND DISCUSSION

Results of mathematical proofs were presented based on the analysis of mathematical praxeologies discussed by prospective elementary teachers. We began the presentation of prospective elementary teachers' proof of the volume of the cuboid, followed by pyramid, cube and cone.

Mathematical Proofs of Volume of a Cuboid

The first type of task given to prospective elementary teachers was to discover the volume of a cuboid (T_c). This task type is aimed at triggering prospective elementary teachers to prove that the volume of a cuboid is based on the number of unit cubes building it. The volume of the cuboid is mostly written as length times width and height. Through this task type, we expected prospective elementary teachers could realize how to derive the mathematical formula of the volume of a cuboid.

To reach the formula of the volume of cuboid, prospective elementary teachers were guided by five tasks presented in the GeoGebra applet. The first task (t_{c1}) was to let prospective elementary teachers interact with the activity of constructing a cuboid based on the number of layers, length, and width of the cuboid (Figure 1). Then, the activity was followed by the following mathematical tasks:

t_{c2}: In the given GeoGebra applet, create a cuboid with 1 layer measuring 3 units long and 5 units wide. How many unit cubes are needed to build the cuboid?

t_{c3}: Then, create a cuboid with 3 layers high, 3 units long and 4 units wide. How many unit cubes are needed to build the cuboid?

t_{c4}: What can you conclude about the volume of a cuboid?

t_{c5}: Give your opinion regarding teaching volume of cuboid using GeoGebra.

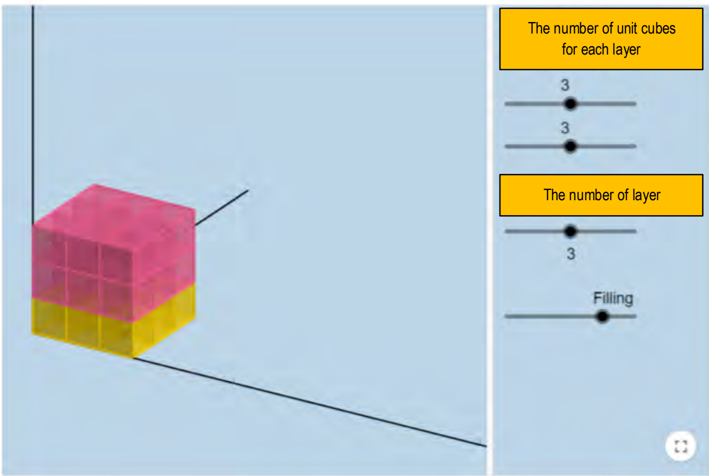

Figure 1. An activity of constructing a cuboid using a GeoGebra applet

Regarding the mathematical task t_{c2} all prospective elementary teachers gave correct answers, and only one prospective elementary teacher gave an incorrect answer in reverse, in mathematical task t_{c3} all prospective elementary teachers gave correct answers. They did not find any difficulties in finding how many unit cubes were used to build a cuboid. These first two tasks were important to help them answer the mathematical task t_{c4}.

Mathematical task t_{c4} aims to lead prospective elementary teachers to prove the volume of a cuboid as length time width and height. Out of 23 prospective elementary teachers, only 5 prospective elementary teachers could explain the link between the concept of the volume of a cuboid to the formula (Table 1). We call this informal mathematical proof 3 and it is the highest level of informal proof that prospective elementary teachers could explain.

Table 1. Mathematical logos for the mathematical task t_{c4}

No	Logos	Number of answers
1.	No answer	1
2.	Given an answer that does not relate to the task	5
3.	Present the formula without any justification	3
4.	IMF 1: The volume of a cuboid is the total number of unit cubes contained in the cuboid	3
5.	IMF 2: The volume of a cuboid is equal to the number of unit cubes that build it	6
6.	IMF 3: The volume of a cuboid is equal to the number of unit cubes that build it; therefore, it comes to the formula of length time width and height	5

Concerning IMF3, here is an example of prospective elementary teachers' written answers.

"My conclusion is that in finding the volume of a cuboid, we can count all the number of unit cubes in it. This is what makes the formula of cuboid as v= p × l × t." (PET19).

PET19 understood that the volume of a cuboid is based on the number of unit cubes that construct the cuboid. The number of unit cubes can be calculated by looking at the length, width, and height of the cuboid. This mathematical proof was confirmed by some prospective elementary teachers during the classroom discussion. It is presented as an excerpt follows:

R : What about task 4? What did you answer in number 4?
PET2 : Do you mean our opinion about the volume of a cuboid?
R : Yes.
PET2 : In my opinion, a cuboid is a 3-D shape that the volume that can be filled by some objects.
R : Okay. The 3-D shape is filled by some objects. Is there anything here that relates to the first two previous tasks to answer task 4, please?
PET19 : So, the volume of a cuboid is the total number of unit cubes contained in the cuboid. Then, from the activities we have done, we use unit cubes to make a cuboid so we can calculate the volume of a cuboid by counting the total number of cubes contained in the cuboid. So, to determine the volume of a cuboid, we can count the number of unit cubes that we have arranged that make up the cuboid.
R : Okay, so later you will come to the mathematical equation, won't you? From what you described.
PET19 : So, the mathematical equation is as explained by my previous friends, namely that the number of layers is the height, then the width and the length, so we multiply the length, width, and height.

PET2 could justify the volume of a cuboid based on the objects that filled the cuboid. This answer was considered IMF 1 because PET2 only provides a general idea about the meaning of the volume of the cuboid. She could not be able to link this idea to the mathematical equation of the volume of a cuboid. Meanwhile, PET19 elaborates more on the mathematical proof of the volume of the cuboid. She could link the informal definition of a volume of a cuboid to the mathematical equation of it after the researcher triggered her on the discussion.

The mathematical task of finding the formula of the cuboid becomes a starting point for prospective elementary teachers to further explore an informal mathematical proof of its volume. More than half of them could provide informal mathematical proofs, but only less than a quarter connect their mathematical ideas to the formula of the volume. This could happen because mathematical proofs in simple geometrical concepts vary to a great extent compared to abstract algebra or analysis real (Sjögren, 2010).

Mathematical task t_{c5} aims to give an opinion regarding teaching the volume of a cuboid using GeoGebra. Out of 23 prospective elementary teachers, only 8 prospective elementary teachers present didactical praxeology regarding how to teach students to find the volume of a cuboid. For instance, PET8 wrote.

"In my opinion, learning volume by using this app can make it easier to understand how to find the volume of cuboid. Because can directly experiment with the applet." (PET8).

PET8 realizes that the GeoGebra applet could help students to find the volume of the cuboid. The representation of unit cubes can be used to come to the formula of the cuboid.

Mathematical Proofs of Volume of a Pyramid

The second type of task given to prospective elementary teachers was to find the volume of a pyramid (T_p). This type of task aims to trigger prospective elementary teachers to prove that the volume of a cube is based on the volume of three pyramids. Therefore, the volume of a pyramid is mostly written as 1/3 times the base area and height. Through this task type, we expect prospective elementary teachers could realize how to derive the mathematical formula of the volume of a pyramid.

To come to the formula of the volume of a pyramid, prospective elementary teachers were guided by three tasks presented in the GeoGebra classroom. They also must look at the volume using the GeoGebra applet. Then, the activity was followed by the following mathematical tasks.

t_{p6}: What can you explain the relationship between the volume of the pyramid and the volume of the cube?

t_{p7}: Suppose the cube below (Figure 2) has a length of 5 cm, a width of 4 cm, and a height of 6 cm. What is the volume of the green pyramid?

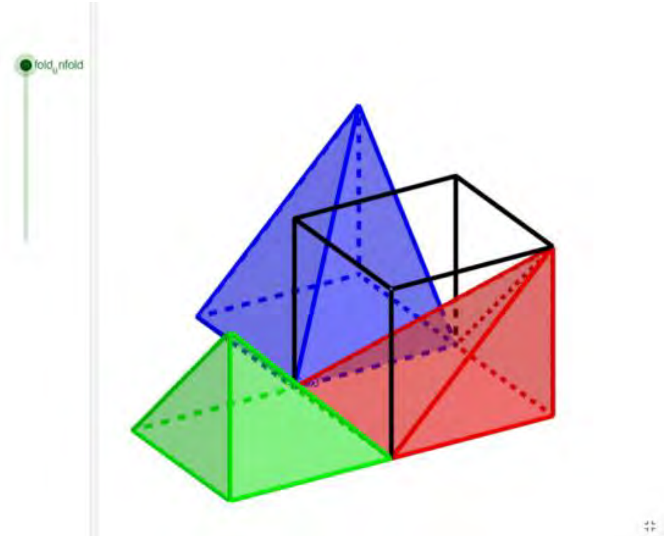

Figure 2. An activity of constructing a pyramid using a GeoGebra applet

t_{p8}: Do you think the volumes of the three pyramids are the same? Prove your answer!

Regarding the mathematical tasks $t_{p6,}$ it aims to lead prospective elementary teachers to come to an informal proof or reasoning regarding the volume of a pyramid as 1/3 times the base area and height. Out of 23 prospective elementary teachers, only 3 prospective elementary teachers could explain the relationship between the volume of the pyramid and the volume of the cube. The following is an example of prospective elementary teachers' written answers from PET3.

"The relationship between the volume of the pyramid and the volume of a cube is that if three pyramids are combined, it will form a cube." (PET3)

PET3 realized that the volume of a cube is the total number of three pyramids contained in the cube. Another example comes from prospective elementary teachers' written answers from PET13.

“The volume of pyramid = 1/3 volume of the cube, which means if three pyramids are put together it will form a cube.” (PET13)

PET13 recognized that the volume of the cube is equal to the number of three pyramids that build it. The following is an example of prospective elementary teachers' written answers from PET7.

“The volume of the cube is 3 times the volume of the pyramid. In a cube, we can form 3 pyramids, where the sides of the pyramid are the same as the base of a pyramid. Then the equation can be made that the volume of the cube is equal to the volume of the pyramid. Then the formula is 1/3 x p x l x t or 1/3 x base area x-height.” (PET7)

PET7 considered that the volume of a cube is equal to the number of three pyramids that build it, therefore it comes to the formula of 1/3 times base area and height. Thus, PET7 used inductive reasoning to account for the formula of the volume of a pyramid. This is a fundamental role in the study of mathematics and problem-solving situations (Christou & Papageorgiou, 2007).

Regarding the mathematical task t_{p7}, from 23 prospective elementary teachers,' only 4 prospective elementary teachers gave correct answers. While 16 of them gave incorrect answers and 3 of them gave no answers. They did find it difficult to answer the question of the volume of the green pyramid.

Mathematical tasks t_{p8} aims to lead prospective elementary teachers to come to prove the volume of a pyramid (Table 2). We call this informal mathematical proof 3 and it is the highest level of informal proof that prospective elementary teachers could do, and only a prospective elementary teacher could reach this level. Most of them still had difficulties in connecting the simulation of 3D shapes on the GeoGebra applet to the mathematical representation of the formula of 3D shapes. These difficulties have led to the both less intuitive understanding of the concepts and inadequate concept images for doing the mathematical proofs (Moore, 1994).

Table 2. Mathematical logos for the mathematical task t_{p8}

No	Logos	Number of answers
1.	No answer	2
2.	Disagree the volume of the three pyramids is not the same	3
3.	Agree on the volume of the three pyramids is the same without any justification	2
4.	IMF 1: The volume of a pyramid is the same as 1/3 of a cube	5
5.	IMF 2: The volumes of the three pyramids are the same because the cube is divided through its diagonals	10
6.	IMF 3: The volume of the three pyramids is the same because the cube is divided through its diagonals, therefore it comes to the formula of the volume of a pyramid as 1/3 times the area of the base and height	1

Concerning IMF3, here is an example of prospective elementary teachers' written answers.

"[It is the] same because the cube is divided into 3 parts based on the diagonal, we get 3 pyramids of the same shape and size. So, the volume of the pyramid is one-third of the total volume of the cube. … The formula for the surface area of a pyramid depends on the shape of its base, as well as its volume.
The volume of the pyramid = 1/3 x area of the base x height.

To get the number 1/3 of the equation, if the cube is divided into three identical parts based on its diagonals three congruent pyramids will be obtained. So, the volume of the pyramid is one-third of the total volume of the cube." (PET5).

PET5 could prove that the volumes of the three pyramids are the same. The volume can be calculated by looking at the 1/3 x area of the base x height of the pyramid. This mathematical proof was confirmed by some prospective elementary teachers during the classroom discussion. It is presented as an excerpt follows:

PET16 : I think it's the same, because when it is opened... then the cube will form 3 pyramids, when 3 pyramids are seen from the same side they will look like the same pyramid shape, and after completing the given task, it can be seen that the volume of the green pyramid is 50 cm^3, so it can be said that each pyramid has the same volume as the green one, which is 50 cm^3.

R : Anyone disagree? Your friend said that the three volumes of pyramids are the same, if you look at it, some are high, and some are low, how about that? Do you agree with your friend's answer that the three volumes are the same? Come on, who wants to convey, you want to convey? Please.

PET8 : If the volume is the same, sir, but for the height of the pyramid or the area of the third base, it can be different, but the volume is the same.

R : why is it like that?

PET2 : Because, right, there are length, width, and height, so later on, when the base is on, the side of the base of the pyramid hits the height of the cube, so it will be different from the side of the pyramid which only affects the length and width.

R : Is there anything different? Guess why the 1/3 formula appears, right? You have memorized the formula, and right from that, you should be able to understand why there can be a number 1/3 like that, let's try.

PET20 : because this pyramid requires an approach of building a cube, so to make the cube it takes 3 pyramids, so to find the volume it becomes 1/3 of a cube because it takes 3 pyramids to build a cube.

R : okay, but there are conditions right, not just any pyramid, right? Can it be any pyramid? Yes, there are rules too, yes, where there is a connection between whether the height or the upright sides are connected, so the size, okay, anyone want to add? Do you want to add?

PET5 : to get 1/3, right, the volume of the pyramid is 1/3 times the area of the base times the height, to get 1/3, for example, the cube is divided into 3 equal parts, based on the diagonal, then we get, we get 3 pyramids that have the same shape, so the volume is 1/3 of the total volume of the cube.

Through the discussion, prospective elementary teachers could elaborate their ideas on how to connect the volume of a cube or cuboid to the volume of a pyramid. Prospective elementary teachers improved their mathematical reasoning in the discussion through a trigger from the researcher (Mata-Pereira & da Ponte, 2017). For instance, PET16 shows the volume of the three pyramids is the same through a concrete example from the volume of the green pyramid. Then, it is further elaborated by the other prospective elementary teachers to show the connection between the volume of a cuboid to the volume of three pyramids.

Mathematical Proofs of Volume of a Tube

The third type of task given to prospective elementary teachers was to find the volume of a tube (T_t). This task type is aimed to trigger prospective elementary teachers to prove that the volume of a tube is base area x height or $\pi \times r^2 \times t$. Through this task type, we expect prospective elementary teachers could realize how to derive the mathematical formula of the volume of a tube.

To come to the formula of the volume of a tube, prospective elementary teachers were guided by two tasks presented in the GeoGebra applet. Then, the activity was followed by the following mathematical tasks.

t_{t9}: The volume of a tube can be analogous to the volume of a rectangular n prism. Since the volume of a rectangular n prism is the base area x-height, the volume of the tube is also the same, namely…
t_{t10}: Because the base of the tube is a circle, the volume of the tube can be written mathematically as...

Mathematical task t_{t9} aims to lead prospective elementary teachers to realize the connection between the volume of a prism to the volume of a tube. Of the 23 prospective elementary teachers, only 9 of them could explain that the volume of a tube can be derived from the volume of a rectangular n prism. The following is an example of prospective elementary teachers' written answers from PET19.

"The volume of the tube is also equal to the volume of an n-sided prism where r^2 = area of the base on an n-sided prism. Therefore, the volume of the tube v= $\pi \times r^2 \times t$." (PET19).

PET19 understood that the volume of a tube can be analogous to the volume of a rectangular n prism, therefore it comes to the formula of v= $\pi \times r^2 \times t$.

Mathematical task t_{t10} aims to lead prospective elementary teachers to come to prove the volume of a tube as $\pi \times r^2 \times t$. Out of 23 prospective elementary teachers, only 5 of them could prove the formula of the volume of the tube (Table 3). We call this informal mathematical proof 2 and it is the highest level of informal proof that prospective elementary teachers could explain.

Table 3. Mathematical logos for the mathematical task t_{t10}

No	Logos	Number of answers
1.	No answer	2
2.	Given an answer that does not relate to the task	1
3.	Present the formula without any justification	10
4.	IMF 1: The volume of a tube as base times height, since the base is a circle, so the volume is $\pi \times r^2 \times t$	5
5.	IMF 2: The volume of a tube is analogous to the volume of an n-prism since the volume of an n-prism is base area times height, therefore the volume of the tube is $\pi \times r^2 \times t$	5

Concerning IMF2, here is an example of prospective elementary teachers' argument during the discussion.

PET4 : well, I will explain the volume of tubes and cones, for task 9 the volume of the tube can

> be analogized to the volume of an n-sided prism because the volume of an n-sided prism is the area of the base times the height, then the volume of the tube is also the same, namely for the volume of the tube which is the same as the volume of a prism or can be said to be a square key times t, because in the tube the base is the same as the base of the circle.

PET3 elaborates more on the mathematical proof of the volume of the tube. She could link the informal definition because the base of the tube is a circle, which can be written mathematically as the volume of the tube after the researcher triggered her on the discussion. Thus, it is obvious that through constructing rigorous proofs prospective elementary teachers could improve their knowledge of these truths (Marfori, 2010). Besides, the use of representation on the GeoGebra applet allows prospective teachers to engage in mathematical tasks regarding mathematical proofs (Zengin, 2017).

Mathematical Proofs of Volume of a Cone

The fourth type of task given to prospective elementary teachers was to find the volume of a cone (T_c). This task type is aimed to trigger prospective elementary teachers to prove that the volume of a cone could be derived from the volume of a cube and from their previous knowledge about the volume of a pyramid. The volume of a cone is mostly written as 1/3 x π x r² x height.

To come to the formula of the volume of a cone, prospective elementary teachers were guided by three tasks presented in the GeoGebra applet. Then, the activity was followed by the following mathematical tasks.

t_{c11}: Try the activity on the following GeoGebra applet (Figure 3). Look at the relationship between the volume of the tube and the volume of the cone. When the radius is reduced/ increased, the ratio of the volume of the tube to the volume of the cone becomes...

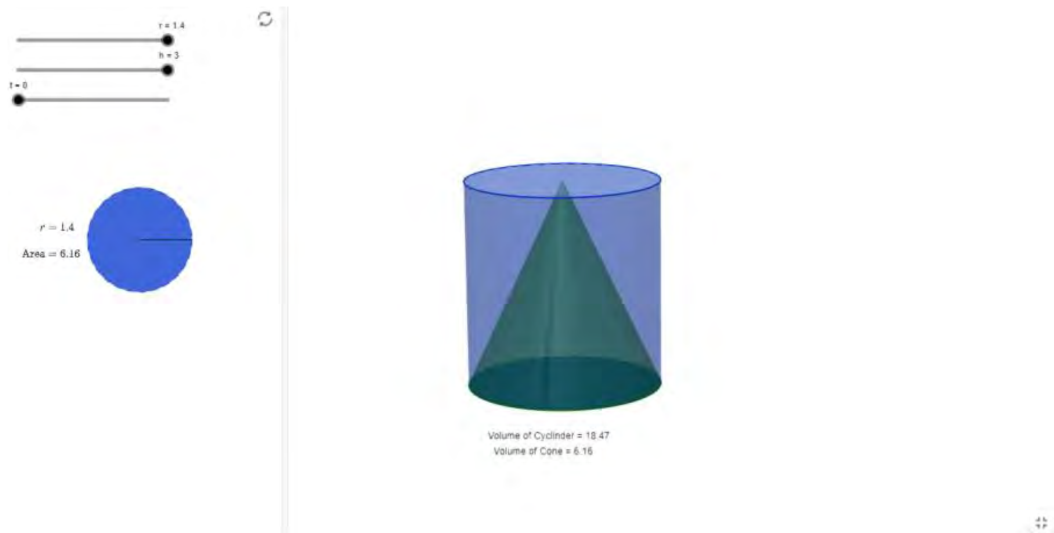

Figure 3. An activity of constructing tube and cone using a GeoGebra applet

t_{c12}: Compare the volumes of two different heights (record the results) and two different radii (record the results). What do you notice about the difference in volume with each change?

t_{c13}: So, what can you conclude about the volume of a cone?

Mathematical tasks t_{c11} to t_{c13} aim to lead prospective elementary teachers to figure out the relationship between the volume of the cone to the volume of the tube. Out of 23 prospective elementary teachers, only 2 prospective elementary teachers could explain the relationship between the volume of the tube to the volume of the cone when the radius is reduced/increased (t_{c11}).

The following is an example of prospective elementary teachers' written answers from PET16.

"If the radius of a cone or tube is reduced, the ratio of its volume decreases and if the radius of a cone or tube is increased, its volume also increases." (PET16)

PET16 understood the volume of a tube with the volume of a cone will change to small if the radius is reduced and will change to large if it is enlarged.

The mathematical task t_{c12} aims to lead prospective elementary teachers to realize the relationship between the volume of a cone and a tube. Of the 23 prospective elementary teachers, only 3 prospective elementary teachers can explain the difference in the ratio of the volumes of tube and cone with different heights and radius. The following is an example of prospective elementary teachers' written answers from PET2.

"The volume will change when height and radius are also changed." (PET2).

PET2 understood the volume of a tube with the volume of a cone will change if its height and radius are decreased/increased.

Another example comes from prospective elementary teachers' written answers of PET20.

"If the ratio is reduced by 0.3 then the volume of the tube = 0.85 and cone = 0.28, If enlarged by 1.1, then the volume of the cylinder = 11.4, cone = 3.8. The results of the two are different because the height and radius of the cylinder and cone are different. Judging from the tube is higher than the cone. Then the volume of the cylinder is greater than the cone." (PET20)

PET20 understood the volume of a tube with the volume of a cone will change to small if the height and radius are reduced and will change to large if it is enlarged, therefore it comes to the formula of 1/3 x π x r² x height.

Table 4. Mathematical logos for the mathematical task t_{c15}

No	Logos	Number of answers
1.	No answer	3
2.	Given an answer that does not relate to the task	5
3.	Present the formula without any justification	8
4.	IMF 1: The volume of a cone is analogies to the volume of a prism	1
3.	IMF 2: The volume of a cone is analogies to the volume of a prism, since the base is a circle, the volume becomes 1/3 x π x r² x height	6

Mathematical tasks t_{c13} aims to lead prospective elementary teachers to come to the volume of a cone as 1/3 x π x r² x height. Of the 23 prospective elementary school teachers, only 6 prospective elementary teachers can conclude the volume of a cone is a third of the volume of a tube (Table 4). We call this informal mathematical proof 2 and it is the highest level of informal proof that prospective elementary teachers could explain.

Concerning IMF2, here is an example of prospective elementary teachers' written answers.

“A cone is a pyramid because it has a vertex so that the volume of the cone is equal to the volume of the pyramid, which is the area of the base x height. Because the base of the cone is a circle, the formula is equal to the area of the circle, thus, the volume of the cone is 1/3 of the volume of the cylinder with the formula 1/3 × π × r × r × t.” (PET20)

PET20 made an analogy between the volume of a cone to the volume of a pyramid. Pólya (1990) argued that analogy has a role in inductive reasoning. So, PET20 could come to the formula of volume of the cone as 1/3 x π x r^2 x height. This mathematical proof was confirmed by some prospective elementary teachers during the classroom discussion. It is presented as an excerpt follows:

PET4 : So, what can you conclude about the volume of the cone? So, we can conclude that the the cone is a 3-dimensional building that has a shape like a special pyramid, so the radius and the height of the cone is twice or more, so the volume of the cone will be larger.

PET4 elaborates more on the mathematical proof of the volume of a cone. He could link the informal definition conclusion as the volume of the cone after the researcher triggered her on the discussion.

The general findings of this study have shown that prospective elementary teachers still struggle to present the mathematical proofs of the formula of the four 3D shapes. Their main difficulty is to relate the simulation of 3D shapes on the GeoGebra applet to the mathematical representation of the formula of 3D shapes. These difficulties might be attributed to their lack of intuitive knowledge of the topics as well as insufficient concept pictures for performing mathematical proofs (Moore, 1994). In addition, prospective elementary teachers in most universities in Indonesia, or around the world, do not require advanced mathematics courses that present mathematical proofs (Siswono et al., 2020). Therefore, most of them do not know how to present sufficient mathematical proofs.

CONCLUSION

Mathematical proofs are very challenging as pointed out in many previous studies. Through this study, researchers discovered the abilities of prospective elementary teachers and found some evidence. Prospective elementary teachers’ mathematical proofs are still insufficient. Only a few prospective elementary teachers can provide mathematical proof, while the ability to prove mathematics is needed for a teacher to teach their students. Thus, it can be said that supporting mathematical proofs for prospective elementary teachers is very necessary as a provision for teaching prospective teachers in the future.

This study fills in the gaps from previous studies, especially related to the use of GeoGebra, as a trigger in supporting prospective elementary teachers' competencies in constructing informal mathematical proofs. Using praxeological analysis from the anthropological theory of the didactic (Chevallard, 2006), the researchers revealed the extent to which prospective elementary teachers can carry out informal mathematical proofs. Although proofs using GeoGebra have the potential to support prospective elementary teachers' mathematical proofs, the limitations of the intervention given to them unable to develop their competencies in mathematical proofs. Future studies need to develop more mathematical proof instructions integrated with GeoGebra and tested them with more prospective elementary teachers from different teacher institutions. Besides, the instructions are also needed for tests with in-service elementary teachers because informal mathematical proofs are part of the mathematics curriculum in elementary school (Kemendikbudristek, 2022). Inservice teachers are required to be able to develop students' mathematical proofs because it is an important skill in the 21st century and the current industrial revolution.

Acknowledgements

The study was conducted with prospective elementary teachers from Elementary School Teacher Education Study Program, University of Riau in the academic year of 2021/2022, and the writing was done during the first author's Fulbright visiting scholar in the Pennsylvania State University, Berks Campus, with fourth author.

Declarations

Author Contribution	:	ZHP: Conceptualization, Collecting the Data, Writing – Original Draft, Methodology, and Visualization. YMA: Writing and Analyzing the Data. D: Review and Editing. HT: Validation, Supervision, and Review of the Manuscript.
Funding Statement	:	This research was funded by Fulbright Indonesia under the grant of 2022 Fulbright Visiting Scholar Program with IIE ID number PS00332630.
Conflict of Interest	:	The authors declare no conflict of interest.
Additional Information	:	Additional information is available for this paper.

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