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Primary School Teachers' Noticing Skills Regarding Students' Thinking: The Case of Whole Number Subtraction

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ABSTRACT

This qualitative case study investigates how, and to what extent, primary school teachers notice students' mathematical thinking in the context of whole number subtraction. A task involving a student's invented strategy was used to collect data. Three noticing questions connected to the task were asked to 45 teachers. Their written answers were analyzed to reveal teachers' noticing skills based on attending, interpreting, and deciding how to respond. The study results revealed that most participants provided limited evidence of attending and interpreting skills, and some responses showed a lack of evidence. More specifically, they could not identify the relationship between digits in ones and tens places. Further, their interpretations did not directly focus on students' solutions and consisted of general statements and misconceptions.

Keywords: teachers' noticing, professional noticing of children's mathematical thinking, students' invented strategies, whole number subtraction, primary school teachers

Introduction

The classroom is a complex environment where many situations are experienced simultaneously; however, teachers cannot focus on all situations in this environment (Sherin et al., 2011). Teachers' knowledge and beliefs about the students, the content of the lesson, the curriculum, and teaching and learning affect everything that needs to be considered during teaching (van Es, 2011). Among these constructs, teachers' knowledge and beliefs about students is a keystone while building instruction, and it has an essential role in students' learning (Darling-Hammond & Ducommun, 2010; Hoth et al., 2018). Based on the knowledge of students, the teachers plan the activities, the problems, and the teaching strategies before the lesson. However, they should be aware of all the situations in the classroom and emphasize if any situations will support students' learning during the lesson (Stockero et al.,2017). In order to be aware of the instant events that emerge during the lesson and to be able to make in-the-moment instructional decisions, the teachers need to have noticing skills which are the essential components of teaching expertise (Jacobs et al., 2011; Sherin et al., 2011; Sherin & Star, 2011).

A better understanding of noticing allows further development in mathematics teaching and learning (Amador et al., 2021a; Jacobs et al., 2011). Therefore, the need to understand teacher noticing has emerged, and the interest of researchers in teacher noticing has increased abruptly in recent years (Stahnke et al., 2016). In order to meet this need partially, in this research study, we aimed to explore primary school teachers' noticing skills of students' mathematical thinking in the context of whole number subtraction.

Teacher Noticing

As a professional vision, teacher noticing is one of the critical components of teaching. However, it can be challenging because it requires one to perceive multiple aspects of the classroom environment (van Es & Sherin, 2021). Generally, teacher noticing includes attending and making sense of the specific events in a classroom and transforming the teachers' attention and interpretation in a teaching setting (Jacobs & Spangler, 2017). More specifically, teachers need to choose the events they will focus on and determine their focus duration to manage "a blooming, buzzing confusion of classroom events" (Sherin & Star, 2011, p. 73). Then, they need to interpret what they see and make connections between the observed events and related issues of the events (Sherin et al., 2011). Processing all these issues during mathematics teaching is not easy for teachers (Estapa et al., 2017).

Although researchers have different ideas about the components of teacher noticing and how to measure and develop it, they agree that identifying the noteworthy events and making sense of them are two essential features of teacher noticing (Sherin et al., 2011). For instance, van Es and Sherin (2008) claimed that mathematical thinking, pedagogy, classroom environment, and classroom management are the events that teachers should notice. Based on their claims, van Es and Sherin developed a framework, *Learning to Notice*, which has two main dimensions, each with four levels (baseline, mixed, focused, extended): what teachers notice and how teachers notice. The *what teachers notice* dimension of the framework includes noticing the classroom environment, students' behaviors and learning, teacher pedagogy, particular students' mathematical thinking, and the relationship between particular teaching strategies and resultant students' mathematical thinking. On the other hand, the *how teachers notice* dimension consists of providing comments and making connections between events and principles of teaching and learning.

In their subsequent work, van Es and Sherin (2021) introduced a new framework based on noticing as an active process. This considers the active interaction with the environment by enabling more observation and interpretation. These authors expanded the dimensions of attending and interpreting, which they put forward on their original Learning to Notice framework (van Es & Sherin, 2002) and included one dimension called shaping. Van Es and Sherin (2021) grounded shaping on the interaction between teacher and students instantly within a classroom environment. This interaction aims to get additional information related to students thinking, which serves for attending and interpreting, and curriculum materials.

Consequently, van Es and Sherin (2021) based their revised Learning to Notice framework on three dimensions, attending, interpreting, and shaping. Although another most cited framework for noticing, called Professional Noticing of Children's Mathematical Thinking, involves attending and interpreting dimensions (Jacobs et al., 2010), both frameworks discussed these dimensions from different points of view. More specifically, van Es and Sherin (2002; 2021) focused on attending and interpreting the noteworthy events that occurred in classroom settings, but Jacobs et al. took students' understanding into consideration while defining attending and interpreting dimensions. In addition, as a third dimension, Jacobs et al. (2010) presented deciding how to respond, meaning teachers' next instructional move to extend and support a student's thinking. To put it differently, in their framework called Professional Noticing of Children's Mathematical Thinking, Jacobs et al. focused more on students' thinking and defined teacher noticing as the ability to attend to the mathematical details in students' strategies, interpret students' mathematical understanding of the particular subject reflected in their strategies, and make decisions to support and improve student's learning based on their understandings. Since this study aims to investigate the extent to which primary school teachers notice students' mathematical thinking in the context of whole number subtraction, the Professional Noticing of Children's Mathematical Thinking framework served as a theoretical framework for the study.

Professional Noticing of Children's Mathematical Thinking

Within the scope of Professional Noticing of Children's Mathematical Thinking theory, Jacobs et al. (2010) identified a particular focus among the levels of the Learning to Notice framework, which is an extended level, and selected a particular slice of teaching which is teachers' in-the-moment decisions while they are responding to students' strategies. Jacobs et al. emphasized that the reasons for selecting a particular focus for noticing are attending more to how and to what extent teachers notice students' mathematical thinking rather than attending to the variety of what teachers notice. From this point of view, these authors attached particular importance to a group of teachers' expertise, with a specialized type of noticing, on which the theoretical framework of this study is based. It was built as a set of three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond based on children's understandings (Jacobs et al., 2010).

The first dimension describes the teacher's explanations related to how a student approaches the mathematical situation/activity/ problem, how he solves it, what materials and strategies he uses, and what the details of his strategies are (Jacobs et al., 2010). The second dimension is defined as a teacher's reasoning consistent with the mathematical details specific to a particular student's strategy and the research on students' mathematical development rather than revealing a holistic picture of students' mathematical understanding (Jacobs et al., 2010). Therefore, it is the ability of the teacher to interpret mathematically how a student understands the subject and how consistent this knowledge is with the knowledge of students. In addition, Jacobs et al. distinguished this interpretation from superficial evaluations, as many researchers did (van Es & Sherin, 2008). The last dimension deals with to what extent teachers use the knowledge they have learned from students' understanding in a given situation while deciding what to do with their next moves. Jacobs et al. (2010) stated that to decide how to respond based on students' understanding, the teacher has also attended to students' strategies and interpreted their understanding. This means that these three skills are interrelated with each other.

Students' Invented Strategies in Whole Number Subtraction

The National Council of Teachers of Mathematics (NCTM; 2014) emphasized that analyzing students' thinking is an important tool for teachers to make instructional decisions for improving students' learning. However, understanding students' thinking is challenging for teachers, and they need to spend substantial time and effort analyzing and interpreting students' thoughts (NCTM, 2014; Son, 2016). Regarding this, Franke et al. (2007) stated that one of the ways of understanding students' thinking and reaching their minds is to understand their invented strategies. Invented strategies are defined as different from standard algorithms and do not require using materials (Carpenter et al., 1998). Students start school with a great deal of knowledge about the concepts, and they can construct strategies for solving mathematics problems, especially for adding and subtracting (Carpenter & Fennema, 1992). These strategies play a prominent role in helping students develop number sense and learn multi-digit operations (Carpenter et al., 1994). Furthermore, it is claimed that inventing strategies requires making sense of mathematics (Carpenter et al., 1998) since these are flexible methods that change according to numbers and circumstances (Van de Walle et al., 2013). In order to connect students' strategies and standard algorithms, teachers need to attend to and interpret students' invented strategies (Carpenter et al., 1998).

Number-based invented strategies are categorized under three labels: decomposition, sequential, and varying strategies (Verschaffel et al., 2007). In decomposition strategies or partial differences strategies (Son, 2016) that involve decomposing the minuend and the subtrahend based on the place value of each number, the decomposed numbers (tens and ones) are subtracted separately. For instance, for the question, 57 – 28 = ?; the solution is 50 – 20 = 30, 7 – 8 = -1, so the answer is 30 + (-1) = 29.

The current study uses a subtraction operation involving a decomposition strategy to analyze primary school teachers' noticing skills of children's mathematical thinking. In the literature, this strategy is the most common strategy invented by students in subtraction problems (Carpenter et al., 1998). The decomposition strategy was also used in studies, which aim to understand pre-service teachers' interpretations (Son, 2016), or to compare pre-service teachers' and practicing teachers' content knowledge (Philipp et al., 2008). Furthermore, this strategy finds the answer by subtracting tens and ones separately without regrouping them. This requires students to conceptualize negative numbers or debt that will be subtracted later, which highlights conceptual understanding. Thus, in this study, by giving a solution involving the decomposition strategy to teachers, we had an opportunity to investigate the teachers' deeper mathematical understanding regarding subtraction with whole numbers through various connections within number relationships.

Rationale of the Study

In recent years, the construct of teacher noticing has gained significant importance in mathematics education research areas. Studies on noticing show variations in the dimensions of noticing (Amador et al., 2016; Blömeke et al., 2015; Jacobs et al., 2010; van Es & Sherin, 2021), the interventions used to measure or analyze teacher noticing skills (Choy, 2013; Sherin & van Es, 2009; Stockero, 2014), and the critical issues that are significant to notice (Jacobs et al., 2010; Star & Strickland, 2008; van Es & Sherin, 2008; 2021). Some of those studies analyzed teacher noticing using video clubs (Amador et al., 2021b; Girit-Yildiz et al., 2023; González &Vargas, 2020; Ivars et al., 2020; Ulusoy & Cakiroglu, 2021; Warshauer et al., 2021), others used students' written work or verbal responses (Dogan-Coskun et al., 2021; Jacobs et al.,2010; Roller, 2016; Sánchez-Matamoros et al., 2019; Tekin-Sitrava et al., 2021). In addition, some studies focused on teachers noticing the events in classroom environments (Stahnke & Blömeke, 2021; van Driel et al., 2021); however, others attended to teachers' noticing students' mathematical thinking (Teuscher et al., 2017).

When we turn our attention to participants, the studies conducted with pre-service teachers showed that pre-service teachers generally focused on the general aspects of the classroom, such as teacher actions (Santagata et al., 2007), management and student-teacher interaction (Star & Strickland, 2008), rather than focusing on students' thinking (Jacobs et al., 2010). This is not surprising since they do not have teaching experience, which is one of the variables that influence teacher noticing (Sherin et al., 2011) and provides support for attending to and interpreting children's understandings (Jacobs et al., 2010; Schoenfeld, 2011). Therefore, it would be hard to get an in-depth exploration of pre-service teachers' knowledge and interpretation regarding students' thinking. On the other hand, in-service teachers have more knowledge and skills to make students understand the concepts within the complex classroom environment in which student learning occurs (Miller, 2011). From this point of view, the participants, who can make instant decisions to reveal students' understanding by observing the complex and dynamic structure of the classroom environment, would provide more comprehensive and in-depth information about teachers' ability to notice. Additionally, working with experienced teachers may allow us to enhance the theoretical framework by articulating the extent to which in-service teachers attend to the details of students' strategies, how they interpret students' understanding as students reflected in their strategies, and how to respond to the basis of students' understanding. Therefore, it would be significant to study with in-service teachers to get more in-depth data on teacher noticing skills and a more categorized framework.

The focus of this study, whole number subtraction, is one of the important topics in primary school mathematics curriculum (Ministry of National Education [MoNE], 2018), and it has a crucial role in teaching and learning mathematics conceptually (Van de Walle et al., 2013). Therefore, teachers need to have the ability to attend to, interpret, and respond to students' understanding to empower them mathematically (Thanheiser, 2009). Although various studies aim to investigate in-service and pre-service teacher knowledge of whole number subtraction operation and students' strategies for whole number subtraction, there is a gap in the literature on teacher noticing of students' strategies for subtracting whole numbers. In other words, although teachers' and students' understanding of whole number subtraction was examined in the previous studies (e.g., Roy, 2014; Thanheiser, 2009), there are limited studies related to teachers' attending to students' strategies for whole number subtraction, their interpreting of students' understanding based on students' strategies, and the decisions on how to respond based on students' understandings (e.g., Son, 2016; Yeo & Webel, 2019). However, most of the studies focused on examining the noticing skills of prospective teachers who do not have any teaching experience. Therefore, it would be significant to examine the noticing skills of in-service teachers who have experience with students' invented strategies and in monitoring students' behaviors, learning, and understanding (Miller, 2011). Thus, we seek answers to the following research questions in the present study:

1. To what extent do primary school teachers attend to students' strategies in the context of whole number subtraction?
2. To what extent do primary school teachers interpret students' mathematical thinking in the context of whole number subtraction?
3. What is the nature of teachers' decisions to respond to students' mathematical thinking in the context of whole number subtraction?

Method

Design of the Study

Since this study aimed to investigate primary school teachers' professional noticing skills in the context of whole number subtraction, a qualitative case study method was used to reveal the findings and support the study's methodological perspective. The study focuses on one group of in-service, primary school teachers; thus, the study is a single case study. Moreover, since the aim is to investigate in-service primary school teachers' noticing skills in the context of only one mathematics subject, whole number subtraction, it includes only one unit of analysis. Thus, the study design is a single-case holistic design (Yin, 2009).

Context and Participants

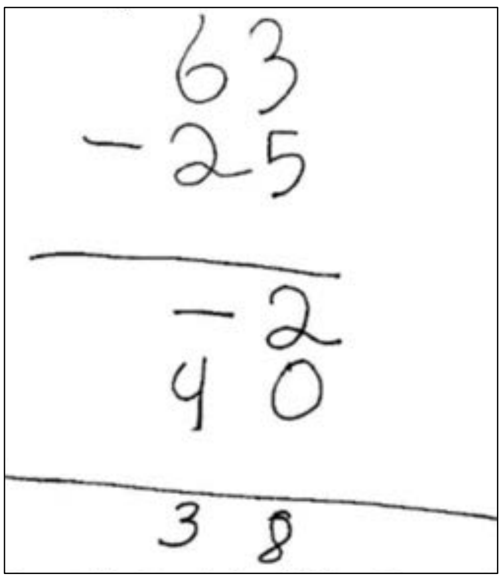
Turkey has a centralized national education system in which all public/private primary schools follow a primary school mathematics curriculum prepared by the MoNE (2018). In addition, primary school teachers graduated from four-year college programs from departments of primary education in Faculties of Education after the 1992-1993 academic year (Dursunoğlu, 2003). The participants in this study included 45 in-service primary school teachers. Three participants were male, and 42 were female, with teaching experiences ranging between five to 40 years. While most of them graduated from the faculty of education, participants became teachers by graduating from different sources, such as a bachelor's degree from any faculty, a two-year college, and a teacher high school. Of the 45 participants, ten worked in a public school, and the rest worked in private schools when the data was collected.

Student Invented Task

This study used a task involving a student-invented strategy adapted from the study of Philipp et al. (2008) to collect data. The scenario in this task also appeared in various sources (Campbell et al., 1998; Schifter et al., 1999; Son, 2016). The task involves a student's written response to a whole number subtraction problem and three further professional noticing questions in accordance with that response.

Figure 1

Mert's Solution to the Task



Note. In the scenario, a second-grade student, Mert, solves the problem of 63 – 25 = X correctly using an invented strategy, which is regarded as a decomposition strategy or partial differences strategy in whole number subtraction (Son, 2016).

For the study, three noticing questions connected to the scenario were developed from research on professional noticing of children's mathematical thinking (Jacobs et al., 2010). Each question corresponds to component skills of attending, interpreting, and deciding how to respond. The professional noticing questions are as follows:

1. Evaluate how Mert solved this problem and explain whether this method is appropriate for whole number subtraction in detail.
2. Explain what you learned from Mert's solution method about Mert's understanding on subtraction operation in detail.
3. Pretend that you are the teacher of Mert. What problem or problems would you pose next? Explain your rationale for posing that problem(s).

The first question was developed to identify the mathematically significant details of Mert's solution and determine whether it was correct. The aim of asking the second question was to assess teachers' professional skills in terms of interpreting children's understandings. Finally, in the third question, participants were asked to explain how they selected a further problem or problems to respond to the student.

Data Analysis

Data were analyzed qualitatively through thematic analysis regarding repeating coding and themes (Miles & Huberman, 1994). Additionally, the frequencies of categories were also described to detect patterns in themes, which are categories under the dimensions of the noticing. More specifically, to attain this study's goals, primary school teachers' written responses to the student's invented task questionnaire were analyzed according to the dimensions of the Professional Noticing of Children's Mathematical Thinking framework developed by Jacobs et al. (2010). While analyzing the first two dimensions, which are attending to children's strategies and interpreting children's understanding, teachers' explanations related to student approaches to the mathematical task and details of their strategy did not quite match the categories of Professional Noticing of Children's Mathematical Thinking framework. In other words, some responses were too detailed to be considered in the limited category, and too superficial to be considered in the robust category. Thus, more detailed categorization was needed to code the teachers' attending and interpreting skills. Accordingly, attending and interpreting skills were coded as Lack, Limited, Substantial, and Robust Evidence, which were presented by Tekin-Sitrava et al. (2021). However, besides these categories, it was suitable to code some teachers' interpretations as No Response, Wrong Interpretation, No Evidence of Interpretation, and Just Attention. About the deciding how to respond dimension, we did not categorize teachers' responses with respect to the level of evidence as indicated by Jacobs et al. (2010) and Tekin-Sitrava et al. (2021). Rather, we focused on the nature of the responses, so deciding how to respond skills of in-service teachers were investigated under the following categories: *Unrelated and General, Ignorance, Acknowledging,* and *Responding to child and incorporating***.** Details of each category are illustrated in the findings. After analyzing teachers' responses based on the Professional Noticing of Children's Mathematical Thinking framework dimensions, frequency analysis was performed to determine the number of teachers falling into categories in each dimension. Three mathematics educators (authors) analyzed data and discussed the inconsistencies until the coders reached 100% consensus.

Findings

Based on the research questions, the findings of the study are presented in three sections. The extent to which teachers attend and interpret is explained in the sections *Attending to Children's Strategies* and *Interpreting Children's Mathematical Understandings,* and what kind of decisions teachers make are explained in the section *Deciding How to Respond on the Basis of Children's Understandings*.

Attending to Children's Strategies

Attending to children's strategies is defined as teachers' explanations of students' approaches to the mathematical situation/activity/problem, students' usage of materials, students' strategies to solve the problem, and the details of these strategies (Jacobs et al., 2010). Based on the data analysis, the evidence of attention was categorized under four headings: *robust evidence of attention, substantial evidence of attention, limited evidence of attention,* and *lack of evidence of attention*. The details of each category and the frequency of responses for each category are given in Table 1. Then, evidence from teachers' responses is explained respectively.

Table 1

The Details of Attending to Children's Strategies Dimension and the Frequency of Each Category

Attending	Frequency
Lack of Evidence of Attention to Children's Strategies	
Identifying the solution/mathematical concepts correctly, but independent from tne students' answer Identifying the solution correctly, but the mathematical concepts are missing, or the explanation includes a general statement Identifying the solution as incorrect	15 (33.33 %)
Limited Evidence of Attention to Children's Strategies	
Correctly identifying the solution as true, but there is some naïve conceptions and misconceptions while describing the students' subtraction operation Correctly identifying the solution as true by using alternative numbers/alternative solution strategy without referring to the present situation	13 (28.89%)
Substantial Evidence of Attention to Children's Strategies	
Correctly identifying the solution as true and the subtraction between the numbers in the ones and tens place, but the connection between the numbers in the ones and tens place is missing Attention to solution is reasonable but not appropriate for that grade level (second grade)	16 (35.56%)
Robust Evidence of Attention to Children's Strategies	
Correct attention to solution through identifying the relationship between numbers in the ones and tens place.	1 (2.22%)

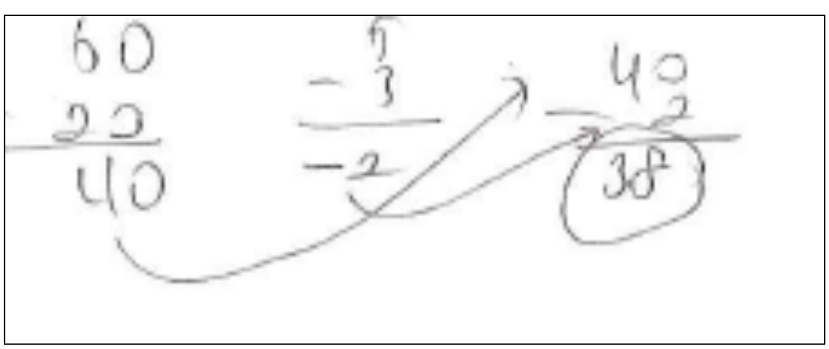
Robust Evidence of Attention to Children's Strategies

Data gathered from primary school teachers showed that only one of them could identify the solution as "true" by subtracting the ones and tens separately, and then establishing the relationship between results obtained from these subtractions. To accompany their work shown in Figure 2, Teacher 16 stated:

> The solution is correct. First, he found the difference between tens. Then, he calculated the difference between ones. Lastly, in order to find an answer, he subtracted this excessive amount (2) and found 38.

Figure 2

Teacher 16's Work



Teacher 16 identified the relationship between numbers in ones and tens place and explained the minus sign in front of 2 as an excessive amount which is subtracted from the difference between tens. Therefore, his attention to the children's strategy was regarded as *robust evidence of attention to children's strategies*.

Substantial Evidence of Attention to Children's Strategies

Most of the teachers (35.56%) correctly identified the subtraction operation by taking into account place value concepts and subtracted the ones from ones and tens from tens. However, the relationship between the numbers in the ones and tens place was missing. Also, the interpretation of '2' in the operation was neglected. Those responses were categorized as *substantial evidence of attention to children's strategies*. For instance, one of the teacher's expressions is as follows:

> He subtracted 3 (ones place of minuend) from the 5 (ones place of subtrahend) and wrote the result to the ones place. Then, he subtracted 2 (tens place of subtrahend) from the 6 (tens place of minuend) and wrote the result to the tens place. The answer is true. This is another way of thinking when you should subtract the small number from the larger number (Teacher 33).

As it could be realized from the above script, Teacher 33 did not relate the numbers obtained from the subtraction operation in the ones and tens place. In addition to this, some teachers tried to mention the role of the 2 in the subtraction operation; however, the connection between the numbers in the ones and tens place is still omitted. Why we subtract 2 from 40 or what -2 stands for is not obvious in the teachers' responses. For instance,

> He correctly solved the question, he subtracted ones from ones and found -2. Then, he subtracted tens from tens and found 40. Then, he calculated the difference of these two numbers (Teacher 19).

As could be deduced from the above response, the teacher correctly attended the subtraction operation by taking the difference between ones and tens separately. However, she did not interpret the (-2) and just explained the result by only taking the difference between ones and tens.

In this category, analysis of findings revealed another important issue regarding the student's solution. Some teachers' attention to the solution could be accepted as reasonable, but inappropriate for second grade. In other words, the concept of integer was not an appropriate explanation for the second-grade student. For instance,

> He subtracted 2 tens from 6 tens. Then, he thought about integers, and subtracted 5 from 3 and found -2. He subtracted the difference of ones from the difference of tens and found the answer (Teacher 1).

As can be seen from this example, Teacher 1 considered 2 as an integer. But, as a second grade student, Mert did not learn the subtraction operation of integers.

Limited Evidence of Attention to Children's Strategies

The analysis of in-service teachers' noticing skills revealed that 13 (28.89%) teachers' responses can be categorized as *limited evidence of attention to children's strategies*. Compared to the substantial evidence, the responses under this category consist of correctly identifying the student solution as true, but naïve conceptions regarding the subtraction operation. To exemplify,

> He conducted the operation mentally. He subtracted the tens and ones separately. Then, since he wrote the subtraction results in reverse order, he further subtracted two from 40 (Teacher 14).

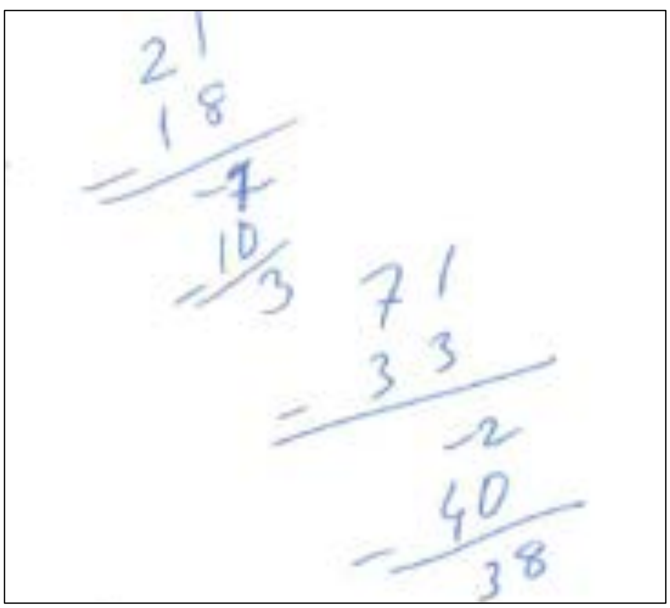
As could be deduced from the above response, the teacher could not interpret the difference between "-2" and "40". Furthermore, the teacher had a naïve conception that since the student wrote the operation in reverse order, he further conducted subtraction.

In some cases, the teachers used another number to check whether the solution was proper or not. In other words, the teachers correctly identified the solution as true by using alternative numbers without referring to the present situation. The teachers confirmed the operation by using alternative numbers. For instance, Teacher 12's explanation, and work shown in Figure 3, are as follows:

> He subtracted the ones and then tens from each other. Then he found the difference between these two results. The solution is correct when we tried the other number. Nice work! (Teacher 12).

Figure 3

Teacher 12's Work

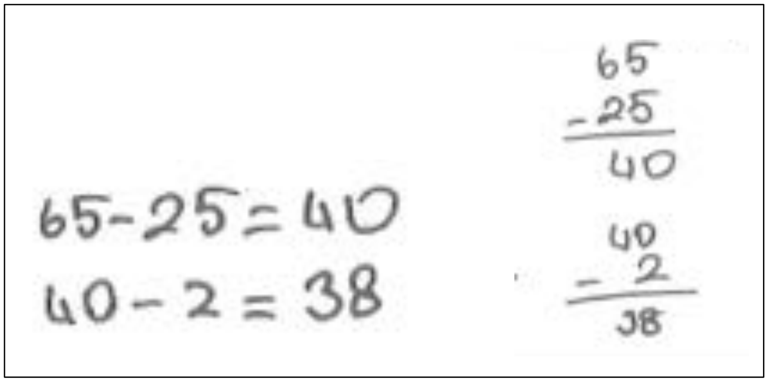


Identifying the solution without referring to the student's solution was popular among teachers' responses. In another example, another teacher identified the operation correctly, but he checked the correctness of the solution by using another solution strategy. Teacher 17's explanation and work in Figure 4 are as follows:

> I think the solution is true (Teacher 17).

Figure 4

Teacher 17's Work



Some of these teachers applied Mert's solution strategy using alternative numbers without referring to Mert's solution, and some presented an alternative solution. Thus, those responses are categorized as *limited evidence of attention to children's strategies.*

Lack of Evidence of Attention to Children's Strategies

Fifteen teachers' (33.33%) responses did not provide strong evidence of attention to the student's strategies. Indeed, in this category of response, teachers identified the solution/mathematical concepts correctly, but independently from the student's answer, or they used general strategies and ignored the details in the student's solution. Examples from teachers' answers are as follows:

> He conducted the operation in reverse order. The answer is correct. He just used the logic of regrouping that yields the correct solution (Teacher 31).
>
> He just subtracted the ones and tens from each other. Based on the subtraction operation rule (the way of writing) the operation is wrong, but it is true conceptually (Teacher 13).

As could be deduced from the responses, the teachers used general ideas and did not refer to the student's solution. In addition to the categorization of attention, one of the teachers directly rejected the student's solution and evaluated it as wrong. Teacher 37 stated that

> He solved the problem by using the subtraction operation. He most probably learned this methodology from his family. The technique is not correct. It is even an inhibitor for the following years (Teacher 37).

To summarize, more than one-third of the primary school teachers provided *lack of evidence of attention to children's strategy*. Except for one teacher, the other teachers could not identify the relationship between numbers in the ones and tens place, even though it is a vital issue to the subtraction operation.

Interpreting Children's Mathematical Understandings

Similar to the attention dimension, the evidence of interpretation of children's mathematical understanding was also categorized under five headings: *robust evidence of interpretation of children's understandings, substantial evidence of interpretation of children's understandings, limited evidence of interpretation of* *children's understandings, lack of evidence of interpretation of children's understanding,* and *no evidence of interpretation of children's understanding.* The details of each category and the frequencies of responses for each category are given in Table 2. Then, evidence from teachers' verbatim is given.

Table 2

The Details of Interpreting Children's Mathematical Understanding Dimension and the Frequency of Each Category

Interpretation	Frequency
No response	2 (4.44%)
Wrong Interpretation	3 (6.67%)
No evidence of Interpretation, Just Attention	8 (17.78%)
Lack of Evidence of Interpretation of Children's Understanding	
Interpreting the solution/usage of mathematical concepts correctly, but independently from student answer General statement	18 (40%)
Limited Evidence of Interpretation of Children's Understandings	
L1: Consisting of general statement on ones and tens interpretation	7 (15.56%)
L2: Correctly identifying the solution as true, but there is a misconception while interpreting the subtraction operation	5 (11.11%)
Substantial Evidence of Interpretation of Children's Understandings	
Correctly identify the solution as true, correctly identifying the subtraction between the numbers in the ones and tens place, but the interpretation of the connection between the numbers in the ones and tens place is missing Interpretation of solution is reasonable but not appropriate for that grade level (second grade)	2 (4.44%)
Robust Evidence of Interpretation of Children's Understandings	
A correct interpretation of the solution through identifying the relationship between numbers in the ones and tens place	0

Substantial Evidence of Interpretation of Children's Understandings

Findings revealed that there was no response regarding the robust evidence of interpretation of children's understandings. Only two (4.44%) teachers' responses were categorized under the heading of *substantial evidence of interpretation of children's understandings.* In this category, teachers correctly identified the solution as true and correctly interpreted the subtraction between the numbers in the ones and tens place. However, the interpretation of the connection between the numbers in the ones and tens place is missing. For instance,

> Mert conceptualized the subtraction operation. While finding the difference between ones and tens, he used natural numbers and instead of trading tens into ones, he treated each number as an integer. Indeed, this is the written way of the mental strategies that we use while calculating (Teacher 1).

As could be understood from the above script, the teacher interpreted the ones and tens correctly, but the interpretation of the relationship between the numbers in the ones and tens place is missing.

Limited Evidence of Interpretation of Children's Understandings

Seven teachers' (15.56%) responses consisted of general statements regarding ones and tens interpretation without referring to the student solution (L1). Teacher 12's response could be categorized under this heading.

> He definitely understood the subtraction operation. He separated the ones and tens and performed subtraction separately. Thus, I believe that he understands the operation (Teacher 12).

In some cases (11.11%), teachers correctly identified the solution as true, but had some naïve conceptions while interpreting the subtraction operation (L2). For instance,

> For each digit, he subtracted the smaller number from the larger number. Thus, for the ones digit, he found his own way to subtract the bigger number from the smaller number (Teacher 35).

As could be understood from the above script, Teacher 35's interpretations included some naïve conceptions like smaller numbers should be subtracted from the bigger number while subtracting.

Lack of Evidence of Interpretation of Children's Strategies

In this category of responses, 18 teachers (40%) interpreted the student solution in a general manner without referring to the ones and tens place. For instance,

> I think this is a really creative solution. This solution is invented by a student, and it is really reasonable (Teacher 20).

As could be understood from the above script, the teacher made general interpretations about the subtraction operation but did not discuss in detail the student's solution. For this reason, similar responses were categorized under the heading of *lack of evidence of interpretation of children's understandings*.

No Evidence of Interpretation, Just Attention

Data analysis revealed some teachers' (17.78 %) responses were directly related to the solution and involved no interpretation. In other words, these responses are similar to those in the attention dimension and directly focus on the students' solution. The following quotation illustrates this approach:

> He did it correctly, he subtracted ones and tens separately. After the subtraction operation, he combined the result (Teacher 3).

To sum up, more than half of the primary school teachers interpreted Mert's solution as true although most interpretations were not directly related to Mert's solution and consisted of general statements and misconceptions.

Deciding How to Respond on the Basis of Children's Understandings

Deciding how to respond on the basis of children's understanding dimension was categorized under four headings: *responding to child and incorporating, acknowledging, ignorance,* and *unrelated and general*. The details of each category and the frequencies of responses for each category are given in Table 3.

Table 3

The Details of Deciding How to Respond Dimension and the Frequency of Each Category

Deciding	Frequency
No response	5 (11.11%)
Unrelated and General	
L0: Misconception about knowledge of teaching subtraction, unrelated response	1 (2.22%)
L1: General pedagogy, including some mathematical concepts, models, representation	17(37.78%)
Ignorance	
L0: Ignorance of students thinking and presentation of unrelated evidence from curriculum	3 (6.67%)
L1: Ignorance of students thinking and reference to traditional algorithm	6 (13.33%)
Acknowledging	
L0: Asking the student to explain her/his strategy, trying to understand student strategy	5 (11.11%)
L1: Performing the same operation by using different numbers with/without giving any rationale.	7 (15.56%)
Responding to child and incorporating	
L0: Incorporating further understanding unrelated to the students' strategy/thinking	0
L1: Incorporating further understanding (e.g., to make some generalization) without rationale	1 (2.22%)
L2: Incorporating further understanding (e.g., to make some generalization) with rationale	0

Responding to Child and Incorporating

Only one teacher (2.22%), Teacher 36, incorporated further understanding regarding student's solution. This teacher's work in Figure 5 and explanation is as follows. She stated that

> I will ask him the following questions. Since I want him to generate a solution strategy for this kind of questions (Teacher 36).

Figure 5

Teacher 36's Work

As can be seen in the script, she tried to make some generalizations about the subtraction operation by asking questions where the minuend and subtrahend were not given.

Acknowledging

The acknowledging category of deciding how to respond on the basis of children's understanding is divided into two categories. In the first category (L0), five teachers (11.11%) stated that they asked the student to explain his/her strategy because they wanted to understand the student's solution. For instance

> I asked him why he performed the operation like this. By this way, I tried to learn the justification of his solution (Teacher 21).

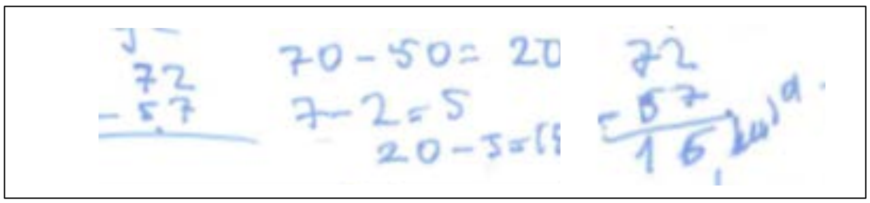
The teachers in this category did not aim to support or extend student's understanding. Instead, they wanted to understand the student's reasoning behind performing this kind of operation.

Teachers' responses under the second category (L1) were differentiated based on the provision of rationale. Five teachers (11.11%) stated that they performed the same operation using different numbers without giving any rationale. Teacher 10, who is in this group, stated the following and their work is shown in Figure 6:

> We have 72 cases of lemon, and we sold 57 of them. How many cases of lemon do we have at the end? (Teacher 10)

Figure 6

Teacher 10's Work

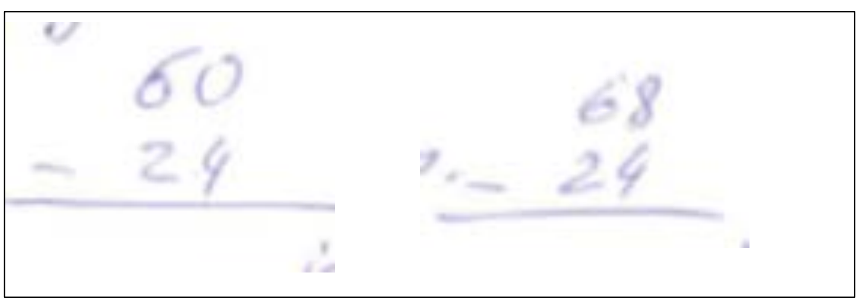


Apart from five teachers, two teachers (4.45%) stated that they performed the same operation using different numbers and giving further rationale. They reported as follows and their work shown in Figure 7:

> I asked him if we could operate more easily. Then, I asked the following questions because I tried to understand whether he had chosen the above method since he did not know the trading of tens into ones (Teacher 11).

Figure 7

Teacher 11's Work



Although Teacher 10 and four other teachers asked for the same operation with different numbers within word problems, they did not state any rationale related to changing the numbers and asking the operation in the form of a word problem. However, only two teachers explained the reasoning behind asking subtraction operations with different numbers.

Ignorance

Teachers' responses that are categorized under this heading are divided into two categories. More specifically, in the first category (L0), three teachers (6.67%) ignored student's thinking and presented evidence from the curriculum that was not directly related to the second-grade curriculum. For instance, one said:

> I asked him a similar question. Then, I asked if he can conduct the same operation by using three digit numbers(Teacher 30).

Six teachers (13.33%) ignored student's thinking and revisited the traditional algorithm (L1):

> Firstly, I asked Mert a subtraction operation involving one-digit numbers. Then, I asked a subtraction operation that did not call for trading tens into ones. Then, by using these operations to help them conceptualize the subtraction operation, I taught subtraction that requires trading of ten into ones (Teacher 19).

As it can be realized from these examples, Teacher 30 stated that he would ask a subtraction operation using three-digit numbers. However, second grade students perform subtraction operation with numbers up to 100 according to the mathematics curriculum (MoNE, 2015). Moreover, Teacher 19 did not attend to Mert's solution while deciding how to respond to him. Instead, she focused on teaching traditional algorithms. The teachers' explanations in this category were similar to those presented previously.

Unrelated and General

Data analysis revealed that 18 (40%) teachers responded to children without considering their solution strategy. Two categories emerged from these responses. Although the teachers did not consider the student's solution strategies in both categories, there is a discrepancy between them regarding the provision of mathematical concepts, terminology, and materials. In the first category (L0), one teacher (2.22 %) responded unrelatedly to Mert's solution. Her explanation is presented below.

> I asked the logic of the operation performed. I wanted him to share his solution with his friends. By this way, students can practically see that they can reach the solution through alternative methods (Teacher 3).

In the second category, 17 (37.78%) teachers expressed some mathematical ideas while responding to students, but they were too general and irrelevant to the student's solution. For instance,

> First of all, I asked Mert to explain his solution. I told him to write another problem. Then, I asked him to solve the written problem by using this method. As the teacher, I asked problems regarding the subtraction operation and asked him to solve the given problems by using different methods and compare the results (Teacher 8).

Of the 45 teachers, five (11.11%) could not provide any response to students regarding Mert's understandings.

To summarize, while responding to students, more than half of the primary school teachers disregarded Mert's solution. Instead, they came up with unrelated and general responses or revisited the traditional algorithm. On the other hand, a few teachers focused on Mert's solution strategy and aimed to get further understanding and make a generalization by performing the subtraction operation with different numbers.

Discussion and Conclusion

This study intended to investigate primary school teachers' noticing skills of students' mathematical thinking regarding whole number subtraction. The findings of the research questions will be discussed under three headings: attending to students' strategies, interpreting students' mathematical understanding, and deciding how to respond on the basis of children's understanding. Then, the descriptive findings will be discussed holistically, and implications will be made.

Attending to Children's Strategies

The findings of the study revealed that only one teacher provided robust remarkable evidence, while most of them showed a lack and limited evidence of attention. As teachers, they know how to solve the subtraction algorithm and can evaluate the correctness of the student's solution, but they do not understand how the students solve the problem. Thus, it can be concluded that solving the subtraction algorithm or evaluating whether the solution is correct was sufficient to identify relevant mathematical details in the student's solution. Consistent with these findings, previous studies resulted in teachers being able to solve the problems; however, they presented general statements about students' solutions rather than giving mathematical details in the solution (e.g., Sanchez-Matamoros et al., 2019). In a similar study conducted by Doğan-Coşkun et al. (2021) with pre-service teachers, more than half of the pre-service teachers demonstrated limited evidence of attention. Moreover, Fernandez et al. (2013) expressed that pre-service teachers had difficulty identifying the mathematically significant details involving proportional and non-proportional reasoning. However, Kılıç (2019) resulted that pre-service teachers could attend to students' thinking in the context of equations. Although the attending skill is regarded as the easiest skill (Jacobs et al., 2010), many studies concluded that in-service and pre-service teachers could not identify the details of students' solution strategies. Jacobs et al. stated that attending does not only require teachers' ability to determine noteworthy situations in a complex learning environment, but also requires having knowledge that enables teachers to determine mathematically significant details. Accordingly, LaRochelle et al. (2019) pointed out that to attend to students' strategies, teachers need to know different strategies that the students develop to solve the problems. From this point of view, the findings led us to conclude that the teachers providing lack of and partial evidence might have limited knowledge of students' strategies in the context of whole number subtraction.

Interpreting Children's Mathematical Understanding

Parallel to the attending expertise, approximately 65% of the teachers had interpreting skills under limited evidence, which included *lack of evidence*, *no evidence of interpretation, just attention*, and *wrong interpretation*. The teachers, who failed to identify the mathematical concepts, such as the ones and tens, did not interpret the relationship between these concepts. This important finding lets us conclude that attending to mathematical concepts in children's strategy plays a significant role in interpreting their mathematical understanding. This result is not surprising since it is necessary to give reasoning about students' strategies and understand how they perform the operation. However, even if a few teachers could explain the student's strategy, they could not interpret the student's understanding. This might be because teachers could focus on the procedural aspects of the operations rather than the conceptual aspects. Therefore, teachers might attach importance to explaining the steps of the strategy and disregard the students' understanding of their underlying reasoning. From this point of view, it can be concluded that although attending expertise is important for interpreting expertise, it does not guarantee to interpret students' understanding.

Deciding How to Respond on the Basis of Children's Understandings

The findings revealed that only one teacher (2.04%) incorporated further understanding to generalize the subtraction operation. Based on the suggested question, it can be realized that the teacher thought that if the unknowns are the minuend or subtrahend, then this question will make the students think deeply and generalize the subtraction operation. Although the teacher tried to enrich students' understanding by changing the unknown of the subtraction operation and to help the students to generalize, the question this teacher posed did not entice the students to think differently and invent new strategies. Because the question, which is generated by changing the unknown, does not necessitate conceptual knowledge of place value, knowledge of properties of operations, such as the associative, commutative, and distributive property, number relationships, connecting to subtraction operation, and other mathematical concepts. The students may only use the meaning and relationship between addition and subtraction to solve the subtraction operations whose minuend or subtrahend is unknown. In this case, the teacher had failed to suggest a question encouraging students to explore a new strategy.

On the other hand, the responses coded as acknowledging and ignorance did not include any questions provoking students to think about concepts deeply and extending their understanding. These teachers might be providing general responses because of difficulty attending to the students' strategies and interpreting students' understanding from their strategies. When a teacher does not understand students' strategies and interpret their understanding based on the important points of the strategies, they are likely to respond to the students in a general and superficial way (Amador et al., 2016). In order to support/ extend their current thinking, the teachers should notice how the students solve the problem and what knowledge/understanding allows them to solve the problem in this way. This result confirms the outcomes of the previous studies by concluding that responding expertise depends on both attending and interpreting expertise; thus, it can be regarded as the most complicated skill of teacher noticing (Crespo, 2002; Jacobs et al., 2010).

Another important reason for providing general responses might be their lack of content and pedagogical content knowledge. Indeed, Tyminski et al. (2014) emphasized that teachers should have coordinated and integrated knowledge to engage in deciding how to respond to students' thinking. As discussed earlier, to decide the best response to extend/support students' understanding, the teachers should attend to students' strategies, which require using their own knowledge related to the concepts. To this end, the teachers need to have rich specialized content knowledge (SCK) (Ball et al., 2008). Furthermore, to decide the best response, the teachers should interpret students' mathematical understanding, which necessitates knowledge of content and students (KCS). Therefore, providing no or limited evidence of attending and interpreting children's understanding might be attributed to teachers' deficiency in their mathematical content knowledge and knowledge of students (Son & Sinclair, 2010; Son, 2016). As attending, interpreting, and responding expertise are interrelated, SCK and KCS also play a foundational role in teachers' skills in responding expertise (Casey et al., 2018; Tyminski et al., 2014). Besides, knowledge of content and teaching (KCT) involves the knowledge needed to decide "which examples to use to take students deeper into the content" (Ball et al., 2008, p. 401), so responding expertise is closely related to teachers' KCT. As a result, the teachers' difficulty in attending, interpreting, and responding may arise from their lack of content and pedagogical content knowledge, and they necessitate having extensive teacher knowledge (Dreher & Kuntze, 2015; Jacobs et al., 2010).

Regarding the overall noticing skills, the descriptive findings suggest that the teaching experience may not positively impact teacher's noticing skills, which is different from what Jacobs et al. (2010) claim. The participants of the present study had teaching experience ranging from five to 40 years, but their noticing skills were not as robust as expected. Thus, it seems that teaching experience does not ensure having robust noticing skills. Instead of teaching experience, experience in students' invented strategies may have more effect on teacher noticing skills. The teachers, experienced in students' invented strategies, could analyze and interpret how the students make sense of the problem and solve it and what kind of knowledge they have. By doing so, they could respond to students using their invented strategies to support/extend their understanding (Crespo, 2002). From this point of view, it is significant to differentiate the terms of *teaching experience* and *experience in students' invented strategies*. Although teachers with at least five years of teaching experience are defined as experienced teachers (Berliner, 2001), they may not have any experience creating a classroom where students invent, share, and discuss their strategies. In such a case, the term experience has been misused, and it should be redefined since teacher noticing focuses more on children's understanding than the regular teaching that occurs during the mathematics lesson, which can be directly related to experience. Thus, in the literature of teacher noticing, the teaching experience might be regarded as being experienced in understanding/reasoning students' invented strategies rather than the number of years they have taught.

Last but not least is the study's contribution to the noticing literature in that it extended the categories of Professional Noticing of Children's Mathematical Thinking Framework developed by Jacobs and his colleagues (2010). Although the present study is grounded in Jacobs et al.'s study, our data, which was gathered by means of an invented strategy, has enabled us to add specific features to the framework. Since the invented strategies necessitate making sense of mathematics but do not require the usage of materials and are more flexible than the standard algorithm, we need to develop a more specific and analytic framework for teacher's noticing special to students' invented strategies

The main dimensions, *attending*, *interpreting,* and *responding*, were the same as those presented by Jacobs et al. (2010), and the names of the categories of attending and interpreting skills were the same as the categorization of Tekin-Sitrava et al.'s (2021). However, we have set the characteristics of each dimension, taking the domain specificity of whole number subtraction into consideration to ensure we contribute to the literature. Also, rather than presenting teachers' responding skills as robust, limited, and lack, we categorized them as *responding to child and incorporating*, *acknowledging*, *ignorance,* and *unrelated and general,* which gives greater insight into teachers' responses. With this categorization, we aimed to evaluate the teachers' responses in terms of whether the teachers consider the students' strategies while responding and whether they support/extend students' understanding. In conclusion, it could be emphasized that this framework enables one to analyze teacher noticing on the basis of students' invented strategies, which has a critical role in connecting students' strategies and the standard algorithm before introducing the standard algorithm.

Implications and Suggestions for Further Research

In light of the present study's findings, sharing some possible implications for teachers and teacher educators and recommendations for further research studies would be significant. Firstly, the findings revealed that most of the teachers had a lack of or limited evidence of noticing skills in the context of students' invented strategies. To enhance noticing skills, teachers might participate in professional development programs to deal with various students' invented strategies, interpret them, and decide how to respond to students based on their invented strategies. Those programs could be enriched by generating collaborative discussion environments among teachers on particular student solutions. This way, teachers find a chance to share their ideas with other teachers with various teaching experiences and could improve their noticing skills on student thinking.

This research was conducted with teachers; however, similar research can be conducted with prospective teachers, and valuable implications could be made for teacher education programs. Thus, further research is recommended to evaluate prospective teachers noticing skills through design courses where prospective teachers could discuss student-invented strategies through written scenarios or video clips. In this regard, teacher education programs might give theoretical and practical opportunities to pre-service teachers so that they will work through invented strategies and their noticing skills will be fostered. Studies with different content areas, including measurement, geometry, and statistics, could be conducted in order to depict teachers' noticing skills in alternative content areas in mathematics. In addition, from a research perspective, our framework is more specific than the current noticing frameworks since it focuses on students' invented strategies. Thus, it could be applied and tested in different contexts with participants from different contexts and backgrounds.

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