Mathematical Problem-Solving in two Teachers' Knowledge Models: A Critical Analysis

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Abstract: *Two of the teachers' knowledge models most widely used in the literature are the Mathematical Knowledge for Teaching (MKT) and the Knowledge Quartet (KQ). We develop an analysis of the limitations of the knowledge required for teaching problem-solving published during 1990-2018 which includes these models. This analysis revealed that MKT takes neither the nature of the process nor the knowledge accumulated by problem-solving research into consideration. While the KQ is subject to similar omissions, its major drawback is element overlap. We conclude that the knowledge required to teach problem-solving is not clearly envisaged in the theoretical teachers' knowledge models analysed.*

Introduction

Problem-solving (PS) is one of the fundamentals of classroom mathematics curricula (NCTM, 2000). As the parties responsible for delivering that curriculum, teachers must be more than mere competent solvers of the problems used in the lessons taught. Teachers' ability to solve complex, cognitively demanding problems does not suffice to guarantee appropriate PS instruction (Lester, 2013). They must also have specific PS knowledge (Chapman, 2015; Piñeiro, 2019), a conceit that has arisen from reflection and previous research (e.g., Weber & Leikin, 2016). For instance, the problems selected to teach mathematics and how they are posed in the classroom are influenced by teachers' own understanding of the mathematical content involved, the educational aims pursued and their beliefs around mathematics, its instruction and their students' capacities (Weber & Leikin, 2016). That state of affairs determines the need to elucidate the factors of PS not associated with teachers' PS skills but that should form part of mathematics teachers' acquis (Lester, 2013).

In light of the slow progress made in the field, research linking mathematics teachers' knowledge to PS has been identified as an area in need of attention (Weber & Leikin, 2016). The studies conducted to date focus primarily on teachers as problem solvers, with a paucity of papers addressing PS from the perspective of their knowledge (Lester, 2013). Earlier research along these lines shows that pre-service trainees' and in-service primary education teachers' limited knowledge of PS impacts their students' PS proficiency (e.g., Depaepe et al., 2010).

Shulman (1986), commonly regarded to have laid the grounds for this area of research, theorised that teachers' knowledge is characterised by seven dimensions. The element of his theory with the greatest impact on the subsequent research is the characterisation of a special type of knowledge specific to teachers that enables them to teach: the pedagogical knowledge of the content. The need for more specific dimensions have prompted researchers to re-interpret his model, however. Rowland, Huckstep and Thwaites's (2005) Knowledge Quartet; Davis and Simmt's (2006) Teachers' Mathematics-for-Teaching; the Michigan Group's (Ball et al., 2008) MKT; among others, are models that build on Shulman's theories. All have focused primarily on the two domains of teachers' knowledge highlighted in his articles: knowledge of content and pedagogical knowledge of content.

Teachers' knowledge models are "framed predominantly around mathematical concepts" (Foster, et al., 2014, p. 98) which, as some researchers contend, prompts significant omissions in the role of processes such as PS. Lin & Rowland (2016) note:

Papers presented at PME include a number of proposals for the elaboration, or modification, of extant theories of mathematics teacher knowledge... While such studies usually add to acronym-overload in the field, some draw attention to gaps or conflicts in the mainstream teacher knowledge discourse. Both Chapman (2012) and Foster, Wake and Swan (2014) take up a critique that Shulman's framework and its derivatives focus on knowledge of mathematical concepts at the expense of PS proficiency (p. 489).

Chapman's and Foster et al.'s alerts constitute the basis of the work presented here, in the sense of shedding light on the differences between knowledge about mathematical' concepts and processes.

We differentiate processes and concepts in the sense of NCTM (2000). From that perspective the process aims to find solutions for "something or some situation [which] is a problem only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity" (Mason, 2016, p. 263). The inference is that knowledge of mathematical problems cannot be positioned in any dimension unrelated to the solver, whereas it is normally positioned in the knowledge of content dimension, from which students' role is absent. The previous example shows that, unlike concepts, which are normally associated with mathematical structure, representation, and contexts or modes of use (e.g., Castro-Rodríguez et al., 2016), processes are entities in themselves. Therefore, one of the perspectives that can be adopted is that there is a difference between concepts (knowing) and processes (doing) (NCTM, 2000). Therefore, the notions related to the processes are not necessarily mathematical knowledge about some specific concept. Thus, given that the knowledge models most widely used in mathematics education may not capture elements inherent in the nature of processes, we posed the following research question: How do the most widely used teachers' knowledge models address PS-related knowledge? We approach this issue from a theoretical perspective. First we identified knowledge required for teaching PS. We then reviewed the literature to select the teachers' knowledge models cited most extensively by the community of researchers. The third step consisted in identifying the limitations related to PS in the to the two most often cited theoretical models. The present description of those steps is followed by a discussion of the implications of our analysis for the research and teacher training.

Knowledge for teaching PS: Theoretical perspective

The specific knowledge required for teaching PS can only be accurately identified if broaching not only from the solver's perspective but in terms of the theoretical particulars of the PS process, which in turn calls for a clear understanding of what PS involves. From that perspective PS instruction calls for different types of knowledge. Chapman (2015) proposed a specific framework or theoretical model on the grounds of a review of the literature from 1922 to 2013. She noted that PS is not organised around the same categories as proposed in other teachers' knowledge models. In her model teachers' PS skill is deemed a primary asset on which knowledge for teaching builds, described as a complex network of interdependent types of knowledge. The model components are summarised in Table1.

Table 1. Components of MPSKT (Chapman, 2016, p. 141)

Knowledge of:	Description
Mathematical PS proficiency	Understanding what is needed for successful mathematical PS
Mathematical problems	Understanding of the nature of meaningful problems; structure and purpose of different types of problems; impact of problem characteristics on learners
Mathematical PS	Being proficient in PS Understanding of mathematical PS as a thought process; PS models and the meaning and use of heuristics; how to interpret students' unusual solutions; and implications of students' different approaches
Problem posing	Understanding of problem posing before, during and after PS
Students as mathematical problem solvers	Understanding what a student knows, can do, and is willing to do (e.g. students' difficulties with PS; characteristics of good problem solvers; students' PS thinking)

Table 1. Components of MPSKT (Chapman, 2016, p. 141)

Knowledge of:	Description
Instructional practices for PS	Understanding how and what it means to help students to become better problem solvers (e.g. instructional techniques for heuristics/strategies, metacognition, use of technology, and assessment of students' PS progress; when and how to intervene during students' PS).
Affective factors and beliefs	Understanding nature and impact of productive and unproductive affective factors and beliefs on learning and teaching PS and teaching

Further to Chapman's (2015) theoretical model, from the perspective of PS as professional knowledge, the issues posed around problems include understanding what a problem is, what PS is and what learning and teaching PS involves.

Teaching PS: What knowledge does it require from teachers?

In order to frame the authors' answer, we discussed in the following three sub-sections, will subsequently serve to analyse the teachers' knowledge models addressed.

Mathematical problems, their solution and teachers' knowledge

Professional PS knowledge entails a command of problems per se and problem types. The classroom use of different problems calls for specific knowledge of the possible types that can be defined (routine or non-routine, for instance). Capitalising on the potential of a problem necessitates a knowledge of the mathematical complexities involved in the problem and its solution. Consequently, teachers must understand problem types and their properties. While none of the several classifications of problems in place has merited full consensus, researchers concur on the acceptability of certain dichotomies, such as applied/non-applied, routine/non-routine or open/closed.

Like all other problems, routine non-applied problems, also called exercises, require teachers to have mathematical knowledge and of their students. In our definition of problem, the learner is unaware of the pathway to solve it. Therefore, if an exercise of the type (27x5)-18 = □ is to be deemed a problem, the student must be unfamiliar with some step in the algorithms involved or with the use of parentheses. Otherwise, the exercise would not constitute a problem. In other words, in this type of tasks problem conceptualisation necessitates teachers' knowledge of content and of their students.

An applied routine problem of the type “María has 12 apples that she wants to set out on plates. Each plate must have 3 apples. How many plates does she need?” requires teachers to know various multiplicative arithmetic structures and the strategies students might use to solve the problem. At the same time, they must know how the problem variables may interfere with one another, creating difficulties that would translate into student error. Errors might arise around the complexity of the multiplication, for instance, and the roles of dividend and divisor in asymmetrical problems; around language, relative to the position of the unknown in the wording or data sequencing; or inversion.

Non-routine non-applied problems require teachers to have profound mathematical knowledge. Although their use has been confined to entertainment or brain-teasing, they can be vehicles for developing independent mathematical thinking. They therefore call for problem-solving knowledge that enables teachers to prompt discussion of and verify predictions or conjectures.

Applied non-routine problems, whether closed as in “how many squares are there on a chessboard?” or open as in “how much paper is used in your school in a week?” are normally deemed to have greatest potential for developing mathematical skill (Lester & Cai, 2016). Such problems require a much more complex spectrum of knowledge, focused less on mathematical concepts and more on the solving process. Teachers must provide opportunities to generate and discuss strategies, help learners apply mathematical knowledge and induce discussion of both mathematical knowledge and the assessment of the strategies used. Such problems also require teachers to express beliefs and conceptions that favour the use of the problem to its full potential: they must believe that the problem may be solved via different valid pathways, have several acceptable answers or no answer at all. In problems of this nature teachers must likewise know how to deliver a lesson in which strategies can be discussed, their greater suitability than others defended or their relationships to previous problems drawn. The ultimate aim would be to generalise properties that may be of aid to learners when faced with similar situations in future.

Learning PS and teachers' knowledge

Teaching PS necessarily defines students as solvers. Using a problem in a teaching situation involves two types of teacher knowledge: an understanding of the mathematical content required to solve the problem and the knowledge inherent in the notion of problem and its solution. When choosing a problem for classroom use teachers are influenced by their knowledge of their students and the extent to which it will be a problem for each. In particular, theoretical knowledge of PS is described either by the stages students follow in that process or the various pathways they chart to reach a solution. Such understanding enables teachers to mediate in PS by posing focused questions that provide a sound scaffolding to help students build their mathematical thinking.

By way of example, let us take a teacher who aims to teach a two-step arithmetic word problem. The problem might be as follows: "Rosa bought some sweets. She ate half and then gave five to her best friend. After that she had seven left. How many sweets did Rosa eat?" (possible calculation- or mathematical concept-related difficulties are not addressed here). In such contexts, teachers must be aware of the strategies their students may deploy to reach a solution (diagrams, dividing the problem into separate steps or other). They must also bear the difficulties in mind, which in this case may be due to the change in structure or inversion of the operation, i.e., the structural variables. Such knowledge prepares teachers to suggest alternative strategies or representations to help their students overcome such difficulties.

Teachers need to understand students' behaviour when faced with certain problems and their possible difficulties, for such knowledge defines the limits of what can be demanded of their students. The characteristics of successful problem solvers revealed by PS research help teachers establish learning expectations (Chapman, 2015).

Teaching PS and teachers' knowledge

In addition to the knowledge of mathematical tasks as problems and learning to solve them described above, teaching PS entails an understanding of how to plan and orchestrate a lesson. One element related to such understanding consists in the access pathways in PS teaching (Schroeder & Lester, 1989), commonly known as teaching about, for and via PS. Castro and Ruiz-Hidalgo (2015) contend that "the first two [for and about] deem problem solving as a learning objective and the third [via] a vehicle to teach or develop other content" (p. 95).

For instance, the aim of the mathematics teaching for PS approach is for students to acquire the ability to apply mathematical knowledge when solving problems. Teachers must therefore know how to sequence a series of tasks, initially around concepts and subsequently around transferring knowledge between contexts (Castro & Ruíz-Hidalgo, 2015). This approach requires teachers to be conversant with a type of problems that offer students different contexts in which to apply their mathematical knowledge.

In teaching about PS the goal is problem-solving process instruction, characterised by two elements. The first is related to solving models as described by Pólya (1945), which translates into a knowledge of the stages involved and their implications for the actions to be performed by students in each. Other processes such as communicating solving strategies or representing mathematical ideas acquire relevance in this context. Teachers must also be able ensure that the transition from one stage to the next takes place naturally. They must also understand PS as a dynamic, non-linear process in which solvers may turn back to an earlier stage where necessary. A second element has to do with specific strategies (such as look for a pattern or make a table). Teaching about them calls for teachers' knowledge of the types of problems that foster the strategy they want their students to learn but do not mandate or constrain the freedom to choose or invent another. Teachers must also understand how affective factors may impact the use of the strategy they aim to teach. The effect of the mandatory use of a certain strategy under teacher instructions differs from that of encouragement and discussion of its use and assessment of its relevance or efficacy.

The via PS approach, in turn, is used as a teaching method and a vehicle for learning classroom mathematics (Castro & Ruíz-Hidalgo, 2015). The specific aim is for students to build classroom mathematics via problematisation. Teachers must be acquainted with the for and about approaches, but primarily be good problem selectors or designer, for the mathematics their students will be able to build will depend largely on those skills. They must envisage the representations resorted to by students to solve problems and how to guide them if they run into difficulties. This is a complex approach, for a number of factors intervene in its application, some relating to mathematics and others to processes but primarily to teachers' beliefs about how mathematics should be learned. Non-cognitive factors are essential for structuring this approach and allowing students to express the knowledge acquired by exploring, discussing and defending their work.

Each approach requires a specific type of teachers' knowledge, as clearly shown by the foregoing analysis of problems, their solution and learning, as well as of the approaches themselves.

Choice of the knowledge models analysed

To choose the models addressed in the present analysis we reviewed the literature, "it makes clear where new ground has to be broken in the field and indicates where, how and why the proposed research will break that new ground" (Cohen et al., 2018, p. 162). The review identified the teachers' knowledge models most widely used in mathematics education research. The characteristics of the review and the extent to which they afforded a response to our research question are described below in terms of the taxonomy proposed by Randolph (2009).

The focus is a critical discussion of teachers' knowledge for teaching PS. We position this study in the framework of traditional reviews—which usually adopts a critical approach (Jesson et al., 2011). It differs, however, in that it furnishes information on how the sources were identified, what was included, what excluded and why. That modus operandi helps identify weaknesses in or reveal the insufficiency of today's theories or document the absence of theory, which would justify putting forward a new theory (Randolph, 2009).

As Randolph (2009) suggests with regard to qualitative reviews and following our research question, we have adopted a perspective in which we conjecture that the mathematics teachers' knowledge models used do not address the fundamental factors for PS. That premise led us to draw a sampling of typical cases (Hernández et al., 2014), because we called for "an abundance of in-depth, high quality information rather than quantity or standardisation" (Hernández et al., 2014,

p. 387). The outcome was the identification of the theoretical models for teachers' knowledge described in mathematics education handbooks published in 1990-2018. The chapters explicitly referring to teachers' knowledge in the title or abstract are listed in the column headed "Chapter/s" in Table 2. The 14 models resulting from the review are shown under the "Knowledge model" column in the same table.

Table 2. Knowledge models present in handbooks

Handbook	Chapter/s	Knowledge model
Handbook of Research on Mathematics Teaching and Learning (Grouws, 1992)	Chapter 8	Teachers' knowledge: Developing in context (Fennema & Franke, 1992)
International Handbook of Mathematics Education (Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996)	Chapter 29	None described
Handbook of International Research in Mathematics Education – 1st ed. (English, 2002)	Chapter 10	None described
Second International Handbook of Mathematics Education (Bishop et al., 2003)	Chapter 22 Chapter 23	Topology of professional knowledge (Bromme, 1994)
Handbook of Research on the Psychology of Mathematics Education - Past, Present and Future (Gutiérrez & Boero, 2006)	Chapter 14 Chapter 15	KQ (Rowland et al., 2005) Mathematics teachers' professional knowledge (Ponte, 1994)
Second Handbook of Research on Mathematics Teaching and Learning (Lester, 2007)	Chapter 4	MKT (Ball et al., 2008)
Handbook of International Research in Mathematics Education – 2nd ed. (English, 2008)	Chapter 10 Chapter 11	MKT (Ball et al., 2008)
The International Handbook of mathematics teacher education.	All chapters	MKT (Ball et al., 2008)

Table 2. Knowledge models present in handbooks

Handbook	Chapter/s	Knowledge model
Volume 1: Knowledge and beliefs in mathematics teaching and teaching development (Sullivan & Wood, 2008)		Teachers' mathematics-for-teaching (Davis & Simmt, 2006) Teacher knowledge and mathematics teaching (Chinnappan & Lawson, 2005) Teachers' knowledge: developing in context (Fennema & Franke, 1992) Mathematics teachers' pedagogical content knowledge (An et al., 2004) Pedagogical content knowledge in mathematics (Marks, 1990) The COACTIV Project (Kunter et al., 2013) KQ (Rowland et al., 2005)
Third International Handbook of Mathematics Education (Clements et al., 2013)	Chapter 12 Chapter 13	MKT (Ball et al., 2008) KQ (Rowland et al., 2005)
The Second Handbook of Research on the Psychology of Mathematics Education. The Journey Continues (Gutiérrez et al., 2016)	Chapter 14	MKT (Ball et al., 2008) KQ (Rowland et al., 2005) Mathematics-for-teaching (Davis & Simmt, 2006) The COACTIV Project (Kunter et al., 2013) TEDS-M framework (Tatto et al., 2008) Mathematical Discourse for Teaching (Cooper, 2014)

Table 2. Knowledge models present in handbooks

Handbook	Chapter/s	Knowledge model
		Mathematical PS knowledge for teaching (Chapman, 2012)
		MKT concepts and processes rewrite (Foster et al., 2014)
		Specialised technological and mathematics pedagogical knowledge (Getenet et al., 2015)
		Framework for analysing pedagogical content knowledge (Chick, et al., 2006)
Handbook of International Research in Mathematics Education – 3rd ed. (English & Kirshner, 2016)	Chapter 2	MKT (Ball et al., 2008)
		KQ (Rowland et al., 2005)
		Mathematics-for-teaching (Davis & Simmt, 2006)
	Chapter 10	Mathematics for teaching (Adler & Davis, 2006)
		TEDS-M framework (Tatto et al., 2008)
		The COACTIV Project (Kunter et al., 2013)

The criterion for determining the most widely used models was their presence in at least two of three databases / search engine: Google Scholar (GS), Web of Science (WoS), and Scopus. We have chosen these three tools because, as Williams and Leatham (2017) point out, the indexes contained in it are often used to measure impact from a citation-based perspective. On the other hand, by drawing from more than one database, we were able to minimise the effects of issues such as: (a) citations not necessarily related to the use of models; (b) the absence in WoS and Scopus of some papers included in GS; and (c) the less demanding requirements for a GS than a WoS or Scopus listing. After identifying the key paper in which each knowledge model was proposed, we determined the number of citations received by database (see Table 3).

With the highest number of citations in all three sources, MKT was deemed the model most widely used. The next highest scores in GS were found for Teachers' Knowledge in Context, Pedagogical Content Knowledge in Mathematics and the KQ. As the second mentioned, Pedagogical Content Knowledge in Mathematics, does not envisage all the dimensions of teachers' knowledge, it was excluded. The first, Teachers' Knowledge in Context, appeared in only one database, whilst KQ appeared in two of the three consulted (in second place in Scopus). The KQ was consequently determined the second most widely used teachers' knowledge model.

In addition to drawing from the same source (Shulman, 1986), one of the two models chosen, MKT, adopts the perspective of in-service teachers, whilst the KQ analyses teachers' pre-service training. That distinction ensured that they did not distort but rather complemented the aim of revealing the limitations of these theoretical frameworks in connection with mathematical processes such as PS.

Table 3. *Teacher knowledge model citations*

Knowledge model	Citations in GS	Citations in WoS	Citations in Scopus
Pedagogical Content Knowledge in Mathematics (Marks, 1990)	825	164	189
Teachers' Knowledge: Developing in context (Fennema & Franke, 1992)	1567		
Topology of teachers' professional knowledge (Bromme, 1994)	253		
Mathematics teachers' professional knowledge (Ponte, 1994)	235		
Mathematics Teachers' Pedagogical Content Knowledge (An et al., 2004)	490		16
KQ (Rowland et al., 2005)	536		155
Teacher Knowledge and Mathematics Teaching (Chinnappan & Lawson, 2005)	90		26
Mathematics-for-Teaching (Davis & Simmt, 2006)	448		109
Mathematics for Teaching (Adler & Davis, 2006)	287	89	85
Framework for analysing Pedagogical Content Knowledge (Chick et al., 2006)	127	21	

Table 3. *Teacher knowledge model citations*

Knowledge model	Citations in GS	Citations in WoS	Citations in Scopus
MKT (Ball et al., 2008)	5291	1397	1275
TEDS-M Framework (Tatto et al., 2008)	228		
The COACTIV Project (Baumert & Kunter, 2013)	149		32
Mathematical Discourse for Teaching (Cooper, 2014)	14		
Specialised Technological and Mathematics Pedagogical Knowledge (Getenet et al., 2015)	5		

Note: blank cells mean the publication was not included in the database.

After identifying the models to be reviewed, MKT and the KQ, we analysed the knowledge components or dimensions explicitly defined and where in each dimension PS knowledge was positioned. More specifically, we conducted a detailed analysis of the two knowledge models, testing the capacity of each to identify PS knowledge based on knowledge about problems, PS and their instruction.

Results

In the following sections, we discuss how PS knowledge is addressed in the two models and their respective dimensions and categories in the subsections below.

MKT viewed from the PS perspective

MKT (Ball & Bass, 2009; Ball et al., 2008; Hill & Ball, 2009; Thames & Ball, 2010), comprise two dimensions or domains: content knowledge and pedagogical content knowledge.

Content knowledge: Those authors defined content knowledge as the classroom mathematics needed to solve the problems posed to students (Ball et al., 2008). A first subdomain within this domain is common content knowledge, defined as "the mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p. 399), recognising that "some of the mathematical resources that teaching requires are similar to the mathematical knowledge used in settings other than classrooms" (Thames & Ball, 2010, p. 223), that enable teachers to know "whether a student's answer is correct" (Hill & Ball, 2009, p. 70).

In addition to common content knowledge, Ball et al. (2008) identified teacher-specific or specialised content knowledge. That subdomain, acknowledged to be one of this model's major contributions, can be equated to a way of understanding mathematics from a classroom perspective. It differs from the command of other more scientific or technical knowledge and is independent of students, instruction and curriculum (Thames & Ball, 2010). It is knowledge that views the concept from a different perspective that is useful and necessary for understanding it.

The third subdomain is knowledge of the mathematical horizon, described as "peripheral vision" (Thames & Ball, 2010, p. 224) that affords a broad view of the implications of and interconnections among the concepts taught (Hill & Ball, 2009).

In the PS context, common content knowledge as construed by the authors would consist in teachers' own PS skills. The question that might be posed here is: which particulars of PS are common content and which specialised content knowledge? Whilst the argument is clearly logical, authors such as Carrillo et al. (2018) note the difficulties involved in differentiating the two types of knowledge. For instance, which types of PS constitute common and which specialised knowledge? Plausibly, a command of one type might be thought to be common, and awareness of the existence of all types to be specialised, knowledge. The determination of which types are deemed common and which specialised knowledge is unclear, however.

The question that might be posed around specialised knowledge is: which aspects of PS constitute teachers' knowledge unrelated to the ability to solve a problem or to teach PS? One of the fundamental characteristics of problems, that the task to be performed must be challenging for the solver (Chapman, 2015; Lester, 2013), necessarily involves students, which clashes with the description given by the model's authors for this subcategory.

Lastly this process evolves over time and constitutes a personal construct involving both cognitive and non-cognitive elements that cannot be unequivocally determined for certain ages. Those two factors render its analysis in terms of the mathematical horizon category particularly complex.

Pedagogical content knowledge: A second domain is pedagogical knowledge, described as a combination of content knowledge and general pedagogical knowledge (Thames & Ball, 2010). The authors note that certain "subdomains that combine knowledge of content with knowledge of students, teaching and curriculum" (p. 223) can be identified in this domain. The first subdomain, knowledge of content and students, focuses on students' most common conceptions, errors and difficulties around given types of mathematical content. That entails not only identifying the error and its nature, but recognising it as a common difficulty and planning tasks accordingly (Ball et al., 2008). A second subdomain is knowledge of content and teaching, defined as a combination of knowing how to teach and knowing mathematics (Ball et al., 2008). The authors describe it as knowledge underlying suitable decision making in terms of examples, tasks or assessment with which a concept can be learned. The third subdomain, knowledge of content and curriculum, entails familiarity with official curricular proposals geared to student learning. Ball & Bass (2009) explain that it is a detailed view of the school curriculum, narrower than the peripheral vision that describes knowledge of the mathematical horizon, for it focuses on standards, their interconnections and supplementary materials.

In PS, the description of content and students naturally leads to the various characterisations of solvers. Research in this area not only premises erroneous ideas, however, but focuses on good solvers. Chapman (2015) stresses the need to differentiate between knowledge of the characteristics of good solvers and knowledge of possible difficulties, bearing in mind solvers' thought processes. The author emphasises that these elements are useful when understood from the students' perspective and highlights the importance of making what they know meaningful as they solve problems.

Knowledge of content and teaching would include a knowledge associated with problems, PS, teaching approaches, and assessment. Such breadth renders it difficult to broach, for all these particulars would arise in classroom situations, reducing the feasibility of detailed analysis useful for understanding both the nature of such knowledge and its implementation by teachers.

A knowledge of PS in the curriculum would entail a command of the levels or grades associated with certain skills involved in PS, how they connect with other areas of the curriculum or how they are addressed in textbooks. PS, however, is a personal, non-standardisable process that develops slowly.

Although as Foster et al. (2014) contend, the MKT model can identify elements of knowledge, inasmuch as its logic stems from a mathematical concept certain elements are omitted from the perspective of PS. Given the way they are structured, these knowledge domains fail to envisage essential elements of the process, such as the solver perspective, generating difficulties in categorisation. Moreover, certain pivotal elements in PS teaching are absent when PS is discussed on the grounds of this model. Class orchestration, an imperative for authors such as Lester and Cai (2016) for instance, is excluded from this classification.

The KQ viewed from the PS perspective

Like MKT, the KQ builds on Shulman's (1986) ideas. Having been spawned in a context of classroom practice under a pre-service training programme, it is evaluative in nature and the dimensions used differ from the traditional distinction between content knowledge and content teaching. As it envisages teachers' knowledge as knowledge-in-action, its dimensions are apt for identifying knowledge-in-use (Rowland et al., 2005; Turner & Rowland, 2011).

The first dimension under this model, foundation, is related to teachers' previous knowledge and beliefs. Turner & Rowland (2011) note that it refers to knowledge, understanding and the resources learnt in different stages of training. It differs from the other three dimensions because it is knowledge (intentionally or unintentionally) held. The authors explain that in this dimension the three key subdivisions of knowledge and understanding are mathematics per se; significant factors resulting from research; and beliefs about mathematics, including how and why it is learned (Rowland et al., 2005).

The other three dimensions relate to knowledge-in-use. The second, transformation, describes action geared to students pursuant to judgement based on the first dimension, foundation (Rowland et al., 2005). That would translate into choosing examples, representations and proofs (Turner & Rowland, 2011). Connection, the third dimension in this model, defines the relationships between mathematical elements that are coherent with their internal logic (Turner & Rowland, 2011). It covers anticipation of complexity, decisions around sequencing, possible connections and recognition of conceptual suitability. The fourth dimension, contingency "concerns the teacher's response to classroom events that were not anticipated in the planning" (Turner & Rowland, 2011, p. 202), in other words deviations from planning, possible replies to students' ideas and their use as learning opportunities.

From the PS perspective, foundation-related knowledge would comprise teachers' own PS skills, theoretical factors stemming from research on PS and their beliefs about the process. It would also include elements associated with approaches or access pathways. This single dimension consequently mixes elements relating to problems and to teaching and beliefs about both. While we feel that addressing beliefs about PS is important, this approach seems too broad for any useful analysis of the knowledge included in this dimension.

Transformation, in turn, is related to problem selection, possible ways to solve them and so on, factors that overlap with the elements included under foundation. In keeping with the authors' description of transformation, elements associated with solver characteristics should appear, but that does not occur naturally. The third dimension, connection, refers to the relations between process and concepts. That description also prompts overlap, however, in terms for instance of the teaching approaches that implicitly favour certain sequences or types of lessons and appear in other model dimensions. Lastly, whilst contingency would identify knowledge related to orchestrating a PS lesson, elements such as discourse (Lester & Cai, 2016) are omitted.

One positive factor is that KQ identifies elements associated with PS lesson orchestrating and teachers' beliefs, both of which are relevant to PS teaching (Lester & Cai, 2016). Nonetheless, this framework may omit some elements identified earlier, such as the characteristics of successful problem solvers.

Discussion

PS research, a fruitful field of study, has established useful knowledge on the role of PS in teaching and learning in mathematics classrooms. Its contributions are introduced into classrooms very slowly, however, and at times only partially or incompletely. The results of this study reveal a steady proliferation of new frameworks, as set out in the handbooks published over the years. However, this area of research has led to the relegation of mathematical processes. Theoretical models of teachers' knowledge have not focused on this concept. Specifically, recent and non-well known proposals have been forthcoming for explicit reflection on PS as a component of teachers' knowledge. The proliferation of frameworks such as MKT and the KQ constitute progress that has not yet been extrapolated to PS.

This present fine-grain focus identifies the shortcomings in teachers' knowledge of PS in the two most commonly used and studied models. As both are guided by mathematical concepts, they omit pivotal factors that underlie processes as PS. They therefore fail to envision how processes, which differ in nature from concepts, align with the components of teachers' knowledge, and to establish proposals in which PS is explicitly analysed as part of teachers' knowledge. Against that backdrop, the present findings support the following premises.

- Positioning content knowledge in a dimension outside teaching itself proves useful for analysing the knowledge of concepts, but not of PS or how it is taught. For instance, a number of studies, conducted from different perspectives, have analysed what teachers know about the concept of fraction (Castro-Rodríguez et al., 2016). That cannot be extrapolated to problems, however, for a problem acquires meaning through the relationships established with the student solving it. The present analysis reveals a limitation of the most commonly used models to focus on the student-content relationship and therefore on the imperative relationship between student and problem. Therefore, failure to consider the student can lead to an authentic teaching of problem solving not being done, and to reduce the teaching to solving exercises.
- In the MKT model a task is not identified as a problem for teachers based on the student-problem relationship. That may have contributed to the lack of research on teachers' knowledge of the matter, even where a problem is deemed a task with no direct pathway for the solver to follow. As contended by Carrillo et al. (2018), conceptualising teachers' knowledge globally as specialised is believed by the present authors to be more suitable than separating knowledge of content from knowledge of teaching content. With such a perspective teachers' knowledge of problems could be broached in terms of their students rather than only of themselves.
- The primary shortcoming to the KQ model is element overlap. Some aspects of the approaches to PS teaching arise in all its dimensions and in connection with sequencing classroom tasks, for instance. Nonetheless, research has shown that teachers primarily deploy the teaching *for* PS approach (Pansell & Andrews, 2017). Pansell and Andrews (2017) in fact contend that teaching *through* PS emerges spontaneously. In that same study they point out that when the aim is for students to explicitly learn a concept through PS,

the characteristics of that approach are not maintained throughout the lesson. This is not to defend one approach as more suitable than the other, but only to argue that teachers need knowledge enabling them to prioritises one over the other depending on the objectives sought in a given lesson. Element overlap in such a context obstructs a specific, clear view of how a lesson is organised depending on the approach, each of which is governed by a conceptualisation of mathematics, a consideration omitted in the KQ model. Hence the need for in-depth review of the types of professional knowledge brought to bear and their coordination in classroom scenarios.

To date part of the literature notes that teachers' expert knowledge is generally determined by the student-problem relationship and assumes the existence of a core or anchor category of knowledge deemed critical to PS teaching that imbues the other categories of knowledge with meaning (Chapman, 2016). As the present findings show, however, the theoretical frameworks in place omit the importance of teachers' knowledge in that relationship. That would explain why research on the subject has paid so little attention to teachers' knowledge of the relationship between students and problems. Further study is required to help understand how teachers envisage that relationship, their degree of awareness of its existence and how they use it in the classroom practice.

Processes and concepts are conceptualised differently (NCTM, 2000) and the knowledge models reviewed here fail to capture the elements of the former, for their logic focuses on the components of concept knowledge. As an example, we have analysed the teacher knowledge involved in teaching the different types of problems, the PS process and learning and teaching PS. One of the implications of that failure is that it limits the reach of research findings. One way to overcome that shortcoming is for the research perspective to move from teachers' thinking to a collective view of teachers and students, focalising on mathematical processes such as PS. More specifically, that analysis should be conducted at two levels, i.e., macro (classroom events) and micro, with sights trained on teacher and student. Adopting that approach would help identify when a teacher realises that a student is struggling with a problem and when with an exercise while at the same time overseeing the other students participating in a PS lesson. Such research calls for two inputs: a theoretical perspective from which to focus on teachers' knowledge of PS; and active participation by teachers themselves. That would yield a fuller understanding of what the classroom use of knowledge means from a teacher's perspective.

Conclusions

Despite the present meaningful, exhaustive analysis-supported results for research on teachers' knowledge of PS, the present study is subject to some limitations. The primary drawback has to do with general applicability, given the number of models analysed. Whilst the analysis omits other prominent frameworks of knowledge, the selection criteria deployed suffice to glean essential information, although extending the analysis to other models would enhance applicability of the approach adopted. However, we are aware that this choice is influenced by the assessment criteria of the journals. A second limitation is related to the theoretical perspective that this work takes. Although we have described the PS knowledge that would be necessary for practice, more research is still needed to help us understand how knowledge about PS is actually used in classroom practice.

Teachers' knowledge in mathematics education has unquestionably acquired prominence in the pursuit of a general understanding of teachers' knowledge. The present findings show, for instance, that since publication of handbooks on mathematics education in the nineteen nineties, a number of models has been put forward to ascertain what sort of knowledge is needed to teach mathematics. The present study takes a magnifying glass to such models, identifying their weaknesses in connection with the ability to detect elements characteristic of a process such as PS (rather than mathematical concepts) with a view to encouraging research on PS-related issues as an object of teaching. Thus, one of the primary conclusions of this study is the identification of a need to explicitly include such issues in the dimensions or domains of teachers' knowledge. Such envisagement must likewise address the meaning and nature of PS as a mathematical skill demanded in today's society.

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