Mathematical Flexibility of Degree of Primary Education students in solving an area problem: Pick's Theorem

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Abstract: *The development in mathematical flexibility should be included in the mathematics teaching training of students in Primary Education Degree. Teachers in training have to acquire the skill to modify the problem resolution and be able to break with stereotyped methods.*

This document presents an analysis of spontaneous mathematical flexibility developed by teachers in training against problems in which the calculation of the area is requested. At the same time what type of statement can promote is analysed, in a more effective way, a flexible thought and the comparison of the possible mathematical flexibility between the different problems is established.

Keywords: Mathematical flexibility; area; pre-service teaching learning; Primary Education; Pick's Theorem

INTRODUCTION

Many students propose a single way of solving, replicating the same procedure over and over again, in the different mathematical problems posed. Possibly this is due to the absence of flexible mathematical reasoning during their school years (Joglar-Prieto, Abánades & Star, 2018). This happens at different educational levels and in different branches of knowledge of mathematics. On the other hand, the use of manipulative materials can encourage the consideration of works or strategies that had not been thought of before, awakening curiosity and reflection in teachers in training about different mathematical concepts that they will teach in the future (García-Lázaro, Garrido-Abia & Marcos-Calvo, 2020).

In particular, this study analyses the answers of 63 students of a Mathematics subject and its Didactics of the Primary Education Degree of a Spanish public university to a problem of calculation of the area of a simple polygon on an orthometric geoplane or grid, in both cases made in paper and pencil.

To do this, the different types of geoplane had to be introduced beforehand: isometric, ortometric and circular grid. In view of the impossibility of having enough geoplanes, and due to its dynamism, it was decided to use the free digital orthometric geoplane from Math Learning Center.

Later, the students were proposed to calculate the areas of different concave polygons, always on square frames. In the classroom, from the different activities of area calculation on a square grid, as already collected by Jiménez-Gestal and Blanco (2017), procedures of decomposition, complementation and the use of the formula of the area of the triangle emerged. Immediately afterwards, the Pick's Theorem was introduced as an additional strategy, in which the application of a formula is sufficient.

This is intended to achieve the following objectives:

- To analyze the spontaneous mathematical flexibility of the teaching staff in training.
- To analyze what kind of statements promote flexible thinking.
- To compare mathematical flexibility in the face of different area calculation problems.

In order to achieve them, a bibliographic review of research on learning of the area magnitude and the theoretical framework that supports mathematical flexibility is carried out. Through a written and individual evaluation, two problems of calculation of areas of a simple polygon are proposed, to later carry out the analysis of the results.

THEORETICAL FRAMEWORK

This section is composed by two subsections: the first one is dedicated to the bibliographical review of the researches that analyze the area magnitude; and the second one to the mathematical flexibility, the fundamental pillar of the present research.

Learning the area magnitude

The magnitudes have a practical application in the resolution of problems, daily life situations, commercial exchanges, needs of the technical trades, that means, needs of quantification present throughout history in the different civilizations and programs of obligatory education (Chamorro, 2001). Although there is no unanimity in the different curricula, the following have always been included: usual and legal units of measurement, and knowledge and handling of measuring instruments.

Determining the area of a given figure is interesting because it involves the coordination of two dimensions. This coordination makes it possible to provide various examples: relationship with the concept of a unit and its iteration, the number of units and calculation with formulas (Outhred & Mitchelmore, 2000). In the teaching-learning processes, a qualitative and quantitative treatment of the area must be carried out. These processes are sometimes reduced to poor instruction, reducing their determination to the use of formulas (Freudenthal, 1983).

Caviedes-Barrera, Gamboa-Rojas and Badillo-Rojas (2019), when analysing the procedures or justification, carried out by teachers in training, showed a tendency to associate the area with the use of routine calculations and formulas, even if they had to spend more time on solving the task. For this reason, these authors state that teachers in training in Primary Education have a little knowledge of some mathematical elements: unit of measurement, conservation of area, additive and multiplicative relationships. Moreover, the numerical context was easier for them than the intuitive geometric context.

This same trend is followed by Codina, Romero and Abellán (2017), who state that the importance given to the use of formulas, even at very early levels, to the detriment of understanding, hinders the development of measurement in students. This statement is supported by previous studies, such as that of Segovia, Castro and Flores (1996), who pointed out that reducing the calculation of areas in teaching to the use of formulas may justify the lack of significance of students in surface units.

The fact is that learning the area magnitude is not only memorizing formulas, but it is a complex process that requires a series of concepts, processes and skills, such as: perception, comparison, measurement and estimation (Zapata & Cano, 2008).

Mathematical flexibility

Flexibility has been studied from both psychology and mathematical education (Callejo & Zapatera, 2014). These authors, compile the different meanings of the term: as the amount of variations that a person can introduce in the notions and mental operations (Demetriou, 2004); as the ability of a person to modify the resolution of a problem by modifying the task (Krems, 1995); and as the ability to solve problems by breaking with stereotypical methods of resolution (Krutetskii, 1976). In our study we will address the second of these meanings.

Going a step further, Joglar, Abanades and Star (2017) opt for the meaning of mathematical flexibility as the capacity to produce different strategies to solve a problem and to distinguish between them the most effective for each case. However, Star and Rittle-Johnson (2008) determined that knowledge of different strategies, including the most effective one, does not imply that students are able to choose the most appropriate one for each problem posed. Therefore, a distinction is made between competence -knowledge of different strategies- and performance -choosing the most appropriate strategy for a specific circumstance.

In addition, it is important to note that flexibility can occur spontaneously or induced. The first of these refers to performance: the individual provides an innovative strategy in the first of his responses; while the second refers to competence: the person offers an innovative solution when asked to provide other solutions. Non-flexible students are those who do not provide any innovative response (Xu et al., 2017).

Among the researches that analyze mathematical flexibility, we highlight those that have teachers in training, such as the one carried out by Lee (2017) for fraction division problems. In addition, Aguilar and Telese (2018) highlight that trainee teachers who solve a problem in different ways can better help their students. The mathematical flexibility must be made latent in the teaching staff as well. This occurs when in certain situations teachers change their plan according to the unexpected responses of their students (Leikin & Dinur, 2007).

METHODOLOGY

In this study we have collected, in a written form, the performance of two mathematical activities in which the area of a quadrilateral is involved.

During this section, the sample of participants, the context in which the students were, and the protocol of action followed in the theoretical classes and in the evaluation session are collected.

Participants

The sample has been elaborated with 63 students of a subject Mathematics and its Didactics of the Degree of Primary Education of a Spanish public university.

Design

In this subject, students are expected to acquire the following knowledge: measurement of a magnitude and reality; origin and historical evolution of measurement; concepts and procedures related to magnitudes and their measurement; stages in the measurement process; study of some magnitudes; and basic considerations in teaching and learning about measurement. In addition, these students enroll in the subject after passing the previous two courses in this educational branch associated with arithmetic and geometry. In particular, they must acquire the mathematical and didactic knowledge associated with the area magnitude. Among this knowledge is the decomposition into simpler geometric forms and the Pick's Theorem.

As a reminder, Pick's Theorem determines the area of a simple polygon, that is, its sides do not cut into each other, and its vertices are nodes in a square-weave geoplane. The area is determined by the formula[1] $i + \frac{b}{2} - 1$ where i is the number of interior points and b is the number of points on one side of the polygon.

For the latter, and in view of the impossibility of having enough geo-planes in the classroom, it was decided to use the free digital application of Math Learning Center, giving it great dynamism. It allows working with orthometric geo-planes, that is, with a square grid.

[1] Students use $d + \frac{f}{2} - 1$ express themselves in Spanish language.

Instrument

The way to collect the development given by the students to the proposed mathematical activities was through a written test. There were two quadrilaterals from which their area had to be determined. At no time was it requested which procedure should be used to resolve it.

In the first of the exercises, a quadrilateral located on an orthometric geoplane was contemplated, and the area inside was requested to that geometric form. See Figure 1.

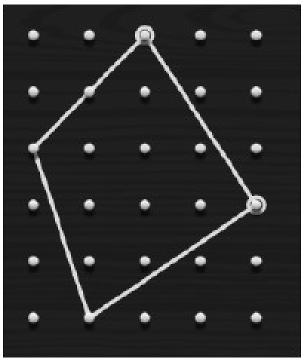

Figure 1. Area activity of a quadrilateral on an orthometric geoplane.

In the second one, the calculation of the inner area of a rectangle on a grid was proposed, where the latter was not visible inside the geometrical shape (see Figure 2). This task had already been proposed during the Spring Contest for 5th-6th grade Primary Education students before.

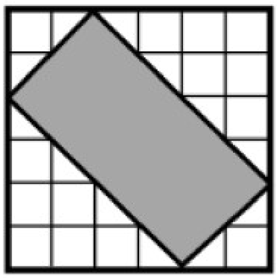

Figure 2. Area activity of a rectangle on a grid.

In addition, it should be noted that no calculation mistakes are contemplated in the resolution of the different exercises by the teachers in training, since they had a calculator in that test.

Procedure

Qualitative techniques were used in order to establish the categories in the resolution of the two tasks described, from the procedure used by the student, and thus be able to determine the spontaneous flexibility with which the teachers-in-training act. Subsequently, quantitative techniques were used to be able to analyze the frequency of each of these categories, with percentages.

RESULTS

In the presentation of the results obtained, it has been considered necessary to establish the presentation of each of the tasks separately in order to subsequently address their similarities and differences.

Results of the geoplanning activity

In the task in which it is asked to calculate the area of a quadrilateral on a square-frame geoplane, a total of two different resolutions are observed: application of the Pick's Theorem and decomposition into right triangles of the complementary figure. In addition to these two resolution categories, it is necessary to point out the existence of teachers in training who are not capable of solving this exercise. One of the possible resolutions contemplated, because it was developed assiduously in the classroom (decomposition into simpler geometric shapes, such as right triangles and squares), was not used by any of the students.

For the first case, in which the exercise was solved using the Pick's Theorem (see Figure 3), the students count the number of interior points, denoted as i (inside) or b (border); and the number of border points, those points through which the segments pass. They then apply the formula $i + \frac{b}{2} - 1$, giving rise to the inner area of the geometric form.

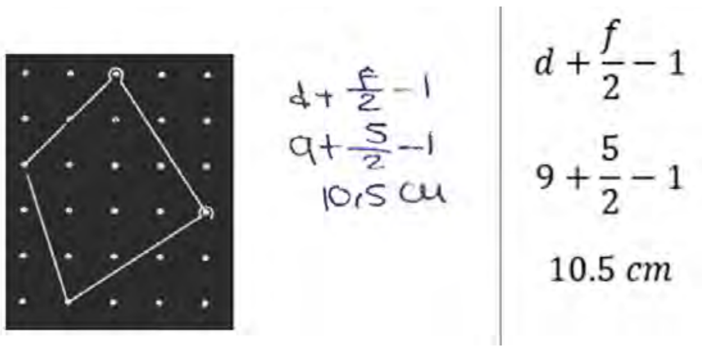

Figure 3. Student Resolution. Use of Pick's Theorem.

In the second case, the teachers in training calculated the area of the entire geoplane and then determined the area of the four triangles outside the quadrilateral (see Figure 4). Since the latter are complementary, it was sufficient to subtract both amounts.

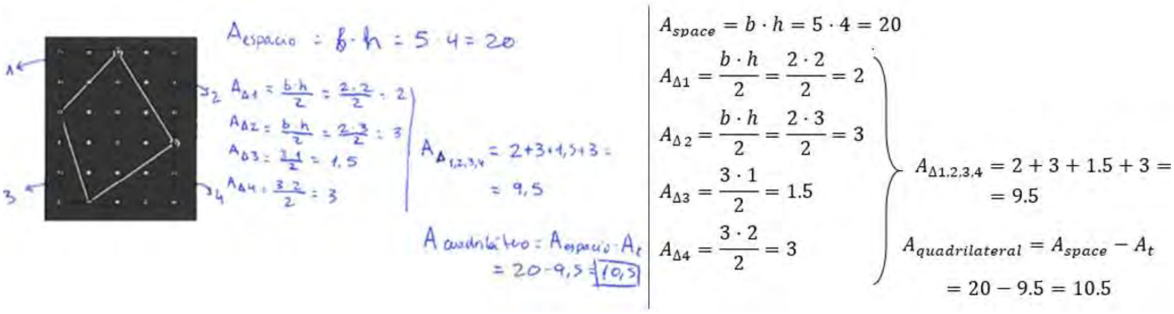

Figure 4. Student Resolution. Calculation of the area with complementary triangles.

Among the students who failed to solve the problem we find two situations: those who did not find did not even try to find any solution to the problem and those who made a mistake. Among the mistakes we find the confusion between magnitudes, Figure 5, where the hypotenuses of triangles are calculated from the Pythagoras' Theorem to determine the area.

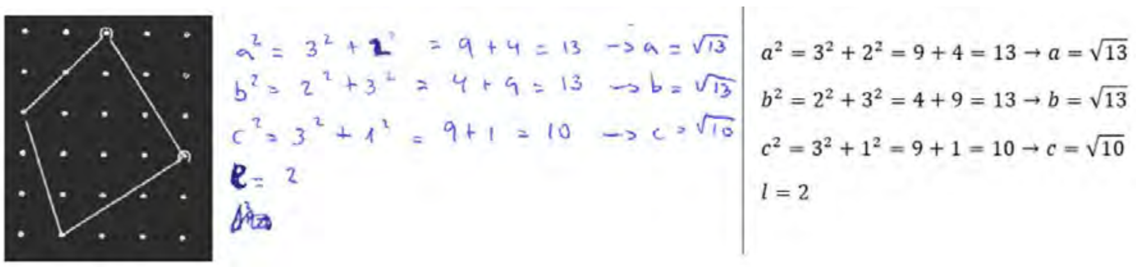

Figure 5. Wrong student resolution.

The frequency of these resolutions is not uniform. 41.27% of students solve the exercise using Pick's Theorem, 7.94% with the area of complementary triangles, and 50.79% of teachers in training do not successfully solve this task.

Figure 6. Frequency of each resolution.

Among all the answers, this exercise considers spontaneous mathematical flexibility to the calculation of the area through complementary triangles, because the student must perform a reflection task. However, the application of Pick's Theorem is not considered mathematical flexibility since it is the direct use of a mathematical formula.

Results of the grid activity

In the second task, in which it is asked to calculate the area of a rectangle on a grid and its hidden interior, a total of six different resolutions are contemplated: application of Pick's Theorem, decomposition of the complementary figure into right triangles, use of Pythagoras' Theorem, sum of the squares and triangles, sum of the squares and complementary triangles, and from the diagonal. In addition, as in the exercise described above, there are teachers in training who were not able to solve this exercise. The first two resolutions already appeared for the first exercise, while the other four came up in response to this task.

In the first of the resolutions addressed, students used Pick's Theorem (see Figure 7). To do this, they had to draw the grid inside the rectangle, as it was hidden, and then consider this grid as an orthometric geoplane. The following steps are identical to those already explained for the first task: application of the formula $d + \frac{f}{2} - 1$.

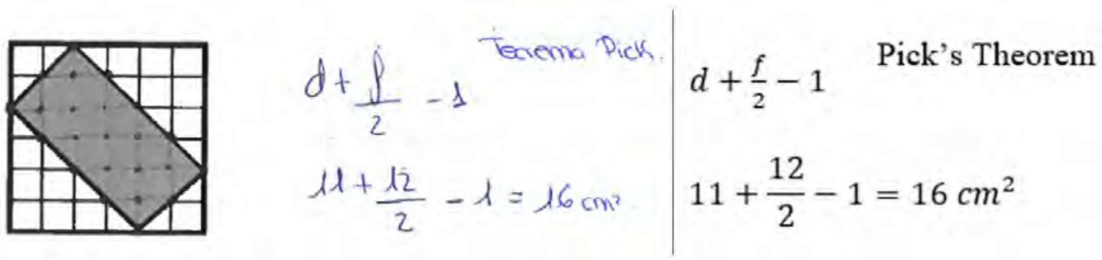

Figure 7. Student Resolution. Use of Pick's Theorem.

The second of the resolutions analyzed had also been found in the first exercise. The students calculated the triangles outside the rectangle and subtracted the sum of all of them from the area of the outer square (see Figure 8). This strategy has been presented in students who did not solve the first task, and who could have replicated it in an almost analogous manner.

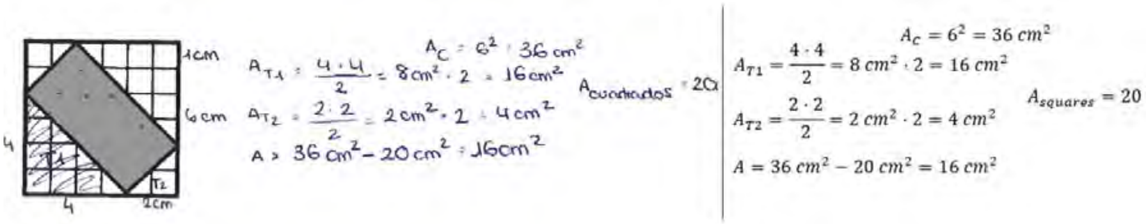

Figure 8. Student Resolution. Calculating the area with complementary triangles.

In the third resolution we find the use of the Pythagoras' Theorem (see Figure 9). Although none of the students proved that the geometric shape was a rectangle, some considered it as such directly because of its visual appearance. They applied this theorem to determine the base and height of the rectangle in order to apply the formula they are most used to from the figure: base per height, $b \cdot h$.

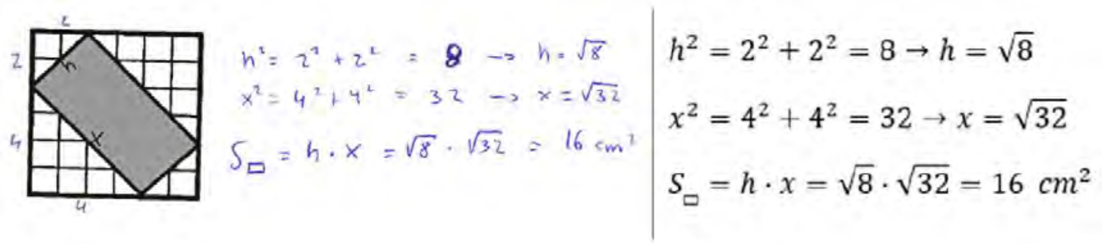

Figure 9. Student Resolution. Use Pythagoras' Theorem.

The fourth of the resolutions could have been made without any mathematical knowledge, because you need to know about the symmetry of the square. Since the colored rectangle is made up of full and half squares, it would be sufficient to count them (see Figure 10).

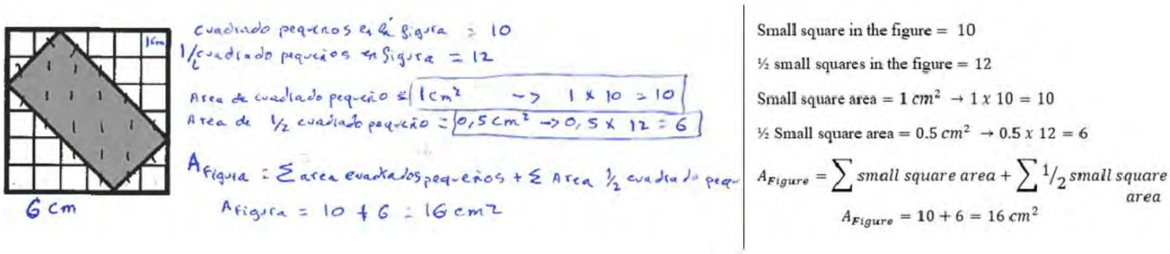

Figure 10. Student Resolution. Calculating the area as the sum of squares and triangles.

The same applies to the fifth of the resolutions. The students who chose this form of calculation counted all the squares on the grid and subtracted the exterior squares that were visible, without having to reconstruct the grid on the inside (see Figure 11).

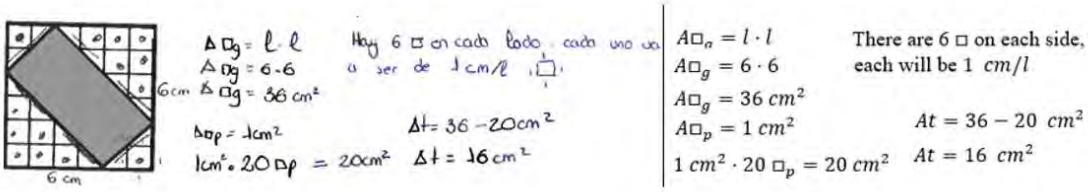

Figure 11. Student Resolution. Calculation of the area with sum of squares and complementary triangles.

The sixth of the resolutions offered by the teachers in training was not foreseen by the teachers of the subject, having to develop also the mathematical flexibility for its correction. In it, the rectangle is divided into two equal squares and the diagonal of the outer square is calculated. Later, the side of the square is determined, being this the third part of the diagonal. Finally, the area of one of the squares into which the rectangle has been divided is calculated and multiplied by two. See Figure 12.

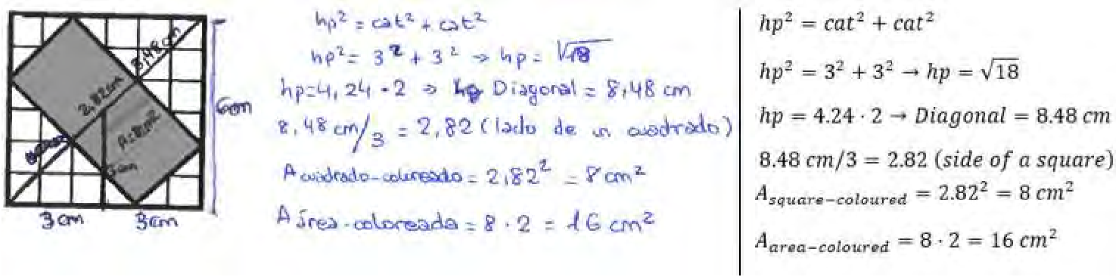

Figure 12. Student Resolution. Calculation of the area from the diagonal.

As was the case in the first exercise, with the square-frame geo-plan, the frequency of these resolutions is not uniformly presented. In this case, the highest percentage of students is found in those using the Pythagoras' Theorem, 26.98%, followed by those using the Pick's Theorem, 17.46%. The rest of the resolutions are given with a low incidence: 6.35% from complementary triangles, 4.76%, 3 people, with the sum of squares and triangles, 4.76% with the sum of squares and complementary triangles, and 1.59%, 1 person, with the calculation of the diagonal. Once again, a high percentage of students do not successfully complete the exercise, 38.1% despite the fact that it was designed for 5th and 6th grade Primary School students who participated in the Spring Contest[2]. See Figure 13).

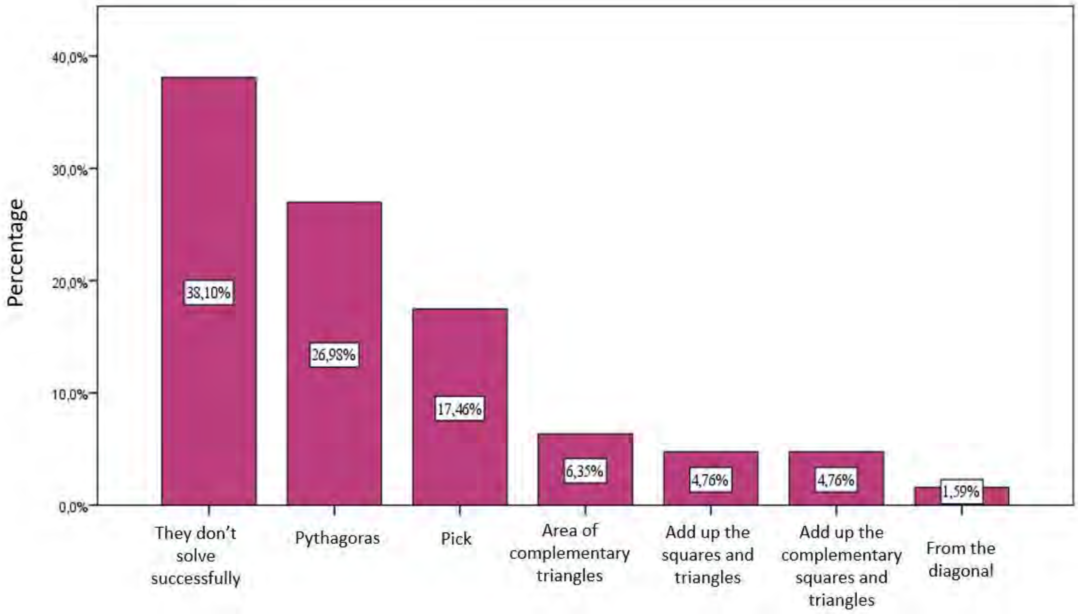

Figure 13. Percentages of each resolution.

In this exercise, spontaneous mathematical flexibility is considered in the calculation of area through the complementary triangles, as it already happened in the first exercise; and also that which involves the diagonal of the outer square in its development. In addition, with the latter, the subject's teachers also had to develop a mathematical flexibility, as mentioned above. The counting of squares and triangles (half squares) and the counting of complementary squares and triangles are not considered spontaneous mathematical flexibility, since they could be done without any mathematical knowledge. Neither is the resolution using Pick's Theorem or Pythagoras' Theorem

[2] Mathematical Competition for 10-18 years old students

since the aim is to use a mathematical formula and not a reflection on its resolution, despite having to perform a greater number of calculations than for other options.

Comparison between the results of the two tasks

Despite the fact that we found two exercises of similar characteristics and difficulty -even the first exercise is a little easier- the results in both have been very different, as far as mathematical flexibility is concerned.

While in the first exercise two different resolutions have been observed, in the second one up to six different ways of reaching a correct and reasoned solution to the proposed problem have been counted. Furthermore, the two resolutions of the first exercise have also been perceived in the second one.

Not all students solved both exercises in the same way; in fact, while Pick's Theorem was used in the first exercise by 41.27% of the students, this percentage was reduced to 17.46% in the second. This percentage reduction is justified because during the explanation of Pick's Theorem in the classroom it was used with a problem similar to the first case, with the use of the orthometric geoplane and the interior of the visible polygon. Its use in both cases, despite the percentage difference, was a non-flexible resolution, even more so for the first case because it is an exact replica of an exercise already carried out.

Similar percentage is presented for the resolution from the complementary triangles: 7.94% in the first exercise and 6.35% in the second one. The steps to be carried out in both cases were similar, and with them the mathematical flexibility was developed.

In the case of the second task, students did perform the decomposition into simpler geometric shapes: 4.76%, 3 of them each, directly and 4.76% with the complementary ones. This strategy, moreover, has been presented in students who did not solve the first task, and who could have replicated it in an almost analogous manner.

With the characteristics of the first problem, for the calculation of the area of a quadrilateral it was not possible to apply the Pythagoras' Theorem or the calculation of the diagonal, although there could be other strategies not contemplated by the teacher and the research team.

The high percentages of unsuccessful resolution of both problems are striking. It should be remembered that the level for which they are intended is 5th-6th grade Primary Education. Specifically, the first of the problems was not carried out correctly or was left blank for 50.79% of the students. Although the percentage in the second case is lower, it also reaches a high value, 38.1%. In both cases, these percentages are higher than those of any resolution strategy.

If we analyze the percentages together and contemplating the resolution strategies (see Figure 14), flexible or not flexible, we obtain that, while in the first exercise the students using the Pick's Theorem and calculating the area of the complementary triangles represent 49.21% of the total, for the second, these same two cases represent 23.81%. From this we can deduce that at least 25.4% of students in the Primary Education grade are able to modify their resolution strategy, adapting it to the data and geometric forms provided. In addition, 12.69% are able to carry out a strategy if any of the data or geometric forms vary, despite not being able to solve other similar problems.

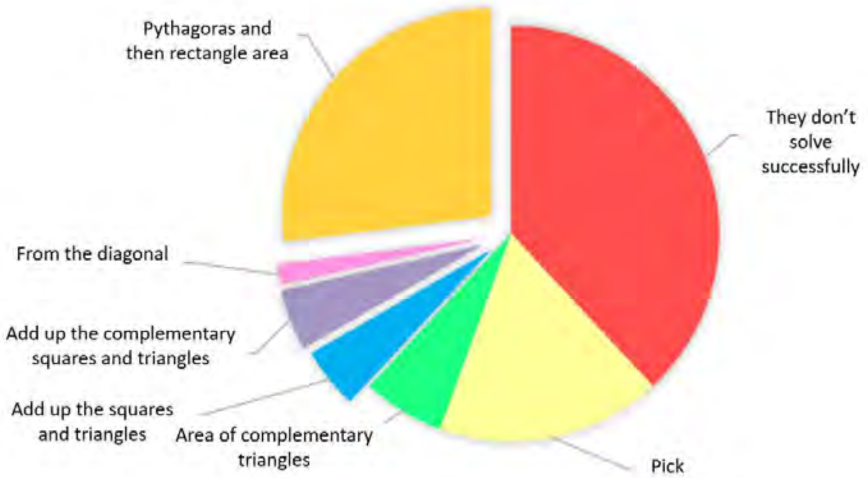

Figure 14. Sectorial chart second problem.

Conclusions

In this research, we have been able to deepen our understanding of the spontaneous mathematical flexibility that students in the Primary Education grade have when they face an area calculation problem. The purpose of this theoretical framework is not only to address the fact that students correctly and justifiably perform a certain problem, but also that they are capable of adapting this resolution to each problem and innovate with the development of their answer, even going so far as to perform procedures not expected by their teachers (Xu et al., 2017).

Analysis of the responses of teachers in training shows that only in 7.94% (area by complementary triangles) of cases is there mathematical flexibility for the calculation of the area requested in the first problem, and 25.4% (Pick, complementary and diagonal triangles) of students are flexible for the second problem.

In addition, among all the individuals that make up the sample, we highlight the 12.69% that are capable of modifying the resolution strategy for the data and geometric shapes provided.

With the difference between the results of both exercises we can affirm that mathematical flexibility is encouraged depending on the type of problem requested. In the first case, it was an exercise similar to the explanation offered for Pick's Theorem in the classroom; consequently, a good number of students, 41.27%, opted for this strategy, without having to make any additional reflection to that of the application of a previously memorized formula. In the second case, a reflection by the university student is further enhanced by not exactly replicating a problem in the classroom, even if only a geoplane is modified by a grid and the visibility of the interior of the geometric form.

This fact should promote a reflection by active teachers on how statements condition resolution strategies and can inhibit flexible reasoning.

This study has some limitations, as already occurred in the analysis of mathematical flexibility for fractions in Lee's research (Lee, 2017). This happens because the problems posed have some particularities that can influence the strategies used by students, having to adapt, in a cyclical manner, the tests carried out on the students.

The research foresees as a future perspective to deal with the evaluation of the analyzed resolutions by other teachers in training, being able to analyze the mathematical flexibility that they develop as teachers and not only as students.

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