Why should we speak about a complementarity of sense and reference?

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Abstract: *Reflection on the theme of the paper makes one to remember Sir Snow's 1959 lecture on The Two Cultures in which Snow had advanced the thesis that the sciences and the humanities had become split into two cultures and had argued that this division had become a major handicap to solving the world's problems. Mathematics education itself has always been negatively be influenced by this rift. For example, in educational context it is frequently claimed that "mathematics is a language since it provides both a conveyance for and a substantiation of our thoughts. It is that aspect of mathematics that explains the key role it plays in modern science" (EFFROS 1998, p.132). But mathematics is no mere language, as everyone will find out if he tries to order a new door to his apartment with words alone and without measurements. The paper will argue historically and in semiotic terms proposing a methodology based on the conception of a complementarity of meaning and reference of knowledge representations.*

Keywords: Semiotics, Complementarity of Sense and Meaning, Peirce.

INTRODUCTION

Reuben Hersh, distinguished mathematical author asked *What is Mathematics really?* (1997). Hersh proposes to consider mathematical objects as *social entities* and to acknowledge that mathematics is essential a social reality. He wants to avoid the alternative of mentalism vs. empiricism. Social entities, he says, "are neither mental nor physical" (Hersh, R., 1997, p.14), but they have "[…] mental and physical aspects […]" (Ibid.). Hersh claims, in fact, that questions about the nature of mathematical objects can be answered from a social perspective only. By the way this belief brings the digital computer as soon as mass societies have replaced local communities and traditions.

Hence comes the next question: *What is human society really?* The answers to this latter question are commonly framed in terms of two alternative schemas of comprehension: the paradigm of communication and the paradigm of production (Markus, 1986).

A reference to the influential 1959 lecture by the Sir C.P. Snow, entitled: *The Two Cultures and the Scientific Revolution* comes to mind. Snow's lecture sparked a tremendous discussion spanning almost the whole world, in which ultimately literary and technical intelligence appeared as irreconcilably opposed. Language refers to language and technology refers to technology. The logic of sense or meaning dominates the life of language and therefore metaphor and logical contradictions abound. The eminent linguist Karl Vossler once said in a speech to students in Munich:

"The true artists of language remain aware of the metaphorical nature of all of their words. They always correct and supplement one metaphor with the other, they let the words contradict each other and only pay attention to the unity and certainty of the thought."

Mathematics and the natural sciences, on the other hand, start with objects, rather than with concepts. Every schoolboy knows how to make the yet unknown, the famous *X* of algebra, an object of mathematical reasoning. These indexical signs represent kind of existence claims and existent objects must require logical consistency, while saying that *yellow* unicorns are actually *green,* does not cause any harm as long as unicorns do not exist. Logical consistency is essential to the sciences and to mathematics. And "the hallmark of consistency is redundancy. If I say *p is true than p is true!* … What is wrong with this? It merely says that a proposition reflects its own truth. What can disturb the equanimity of logical perfection, Why a Cretan can" (Churchman 1968, p.108). His name was Epimenides. This Cretan said that all Cretans are liars. More specifically he says, "I am now lying". We have to forbid *Epimenides* to speak about himself and in particular about his own lying.

No kind of language can be universal or unlimited in expressive power and yet be consistent. Logicism and in particular Frege's conception of logic as a universal language and of mathematics as a part of logic became unfeasible as soon as Russell had informed Frege about the semantic paradoxes in his system.

Mathematics as a science started in Antiquity already with Euclid's *Elements.* These are nowadays considered as a first example of a theory and a science. However, Euclid systemized the solutions of geometric construction problems, rather than the relations between mathematical propositions. A proof in Euclid's *Elements* is nothing but the demonstration that if certain operations or constructions are licensed, something can be constructed. One might think of the strange first theorem: "On a given straight line to construct an equilateral triangle". It is always a matter of solving certain construction problems, which in turn require the solutions of other problems. Euclid's *Elements* was created as a kind of multifunctional problem-solving machine or technology, rather than as a theory (for more details: Otte 2018, p.73).

Structure and Context

There is a branch of mathematical philosophy – fictionalism (L. Tharp) which shows such a reaction, and which compares mathematical discourse to poetry or story-telling. As an illustration of the role of fiction in his argument Tharp presents the following very short story: "The only people in our story are Gertrude and Hamlet. Gertrude is a queen. Hamlet is a prince, and Gertrude is Hamlets mother". Tharp continues:

"Given these two stipulations which constitute our story, various consequences follow from the meanings of the concepts 'prince', 'queen' and 'mother', and are evidently true-in-the-story: for example no princes are queens; Gertrude and Hamlet are distinct; Hamlet is not Gertrude's mother. None of these conclusions follow logically from the given story, however" (Tharp 1989, 168/169).

Fictionalism believes that mathematical statements such as "8 +5 = 13" and "π is irrational", literally interpreted, are wrong because these statements imply the existence of mathematical objects, and according to fictionalism, there are no such objects. The difference between "8 +5 = 13" and "8 +5 = 15" is now that the first equation taken as part of the "great narrative" of ordinary arithmetic is true and the second is false, just as in Leslie Tharp's history about Hamlet and his mother Gertrude the sentence "*Gertrude is Hamlet's son*" would be wrong.

What is the difference between the two cases? In the story the truth or falsity follows from the meaning of the concepts involved. In mathematics or physics, we do not know, as a rule, the meaning of the concepts involved, but do know whether $A = B$ for two numbers or other entities. For example, one need not know, what the *mass* of a body really is. The balance will tell us immediately whether two masses A and B are equal or not (although the equality of heavy and inert mass remained a mystery until Einstein's theory of relativity). What we do in studying mathematics or science is learn about structures, be those syntactical or otherwise.

In Tharp's example we argue from the meaning of the concepts involved. But again, here too we need not know what meanings or intensions are but know whether two given concepts mean the same or something different. We must now the code. In Tharp's story we know code and context, but in the arithmetical example we know the structure, which is defined in axiomatic terms, but not the context. We assume, - perhaps falsely - that Gertrude and Hamlet are existent people, a woman and her son, while we do not have objects called 8 or 5. We do not really know beforehand whether Peano's axioms are applicable in the context of a given objective universe of discourse. In language we know intuitively the meaning but have to learn the correct syntax.

If we have to interpret some encrypted message the situation could vary. Either we know the code, that is, we understand the language in terms of which the message had been constructed, or we do not know the language and have to rely on some structural analysis and other types of methods, like pattern recognition.. Jacobson describes the different situations as follows:

"The addressee of a coded message is assumed to be in possession of the code and through it he interprets the massage. Unlike this decoder the cryptanalysts come into possession of a message with no prior knowledge of the code and must break this code through dexterous manipulations of the message" (Jakobson/Halle, 1956, p.28)

The complementarity of structure or rather syntax and metaphor appears also in Jacobson's classification of *speech aphasia*. Roman Jakobson has classified all linguistic behavior as referring to either code or context. One could call one type of aphasia a loss of predication, or, using semiotic terminology, a lack of metaphor, or loss of meta-language and the other, a loss of structure. The syntactical rules of organizing words into higher units are lost (Jakobson/Halle, 1956, p.79-86).

For aphasics of the *first type* context or the purpose of an activity is the indispensable and decisive factor. Such a person may never utter the word *knife* alone, but, according to its use and surroundings, alternately call the knife *pencil-sharpener, apple-parer, bread-knife.* In the second type of aphasia, the ability of organizing words into higher units are lost. Patients with this type of aphasia tend to give rise to infantile one-sentence utterances and one-word sentences.

The fact that syntax is context free favors automated procedure of classification and even translation. Even with the very limited computer power of 40-50 years ago, astonishing things may appear (Bennett, 1977). In some experiments the frequencies of letter sequences in the work of, say Shakespeare or Goethe were read into the computer, to cause the computer to create "works" in the style of Shakespeare or Goethe. The results were absolutely convincing. The reader perceived the differences and similarities of style of texts and could identify stylistic similarities and differences of various, even though the texts produced by the computer were of course semantically meaningless. And Structural analogy plays some role and has sometimes a greater share in many mathematical or scientific discoveries (Polya, 1973).

Language and Geometry

At the 1972 congress in Exeter, *Rene Thom* made a suggestion on how language and algebra (school mathematics is essentially algebra) could be related and developed together. There have in fact frequently been attempts to do this in school practice. In ordinary language the meaning as a rule is intuitively clear while the syntax is poor. With algebra the opposite is true. The meaning of an algebraic equation is due to the fact that we should be able to solve it, that is, it is algorithmic and instrumental. If an equation is not solvable it becomes meaningless. But the syntax of algebra is rich and free and in principle even limitless. Thom suggests that spatial intuition and geometry might be useful when trying to mediate between common language and algebra, that is between meaning and structure. He writes:

"This problem recurs in a form 'everywhere dense' in mathematics, where the mathematician has to communicate his intuitions to others. In this sense, the spirit of geometry circulates almost everywhere in the immense body of mathematics, and it is a major pedagogical error to seek to eliminate it" (Thom,1972, pp.206f.)

Geometry has always been the epitome of the objective in mathematics, even after it has been displaced from pure mathematics by arithmetization and algebraization. The mathematician speaks of all possible types of spaces (functional spaces, Banach spaces, etc.). And some linguist have claimed that "most of our fundamental concepts are organized in terms of spatialized metaphors (Lakoff/Johnson, 1980, p.17). This is certainly an exaggeration due to a kind of conceptual reductionism. However, in the end or rather in the beginning all recommendations like those of Thom or Lakoff should make us aware of the fact that the human subject itself is just a sign among signs. All our cognitive processes and every contact with the surrounding environment be its social or material is mediated by signs. Why was mathematics education so hesitant in accepting semiotics and the insight that *man is a semiotic animal* (John Deeley).

In 1868 Charles Peirce (1839-1914), still a young man, published a paper under the title *Some Consequences of Four Incapacities* (Peirce, W2, pp. 211-241, CP 264ff) where he said the following:

"Whenever we think, we have present to the consciousness some feeling, image, conception, or other representation, which serves as a sign. But it follows from our own existence that everything which is present to us is a phenomenal manifestation of ourselves. This does not prevent its being a phenomenon of something without us, just as a rainbow is at once a manifestation both of the sun and of the rain. When we think, then, we ourselves, as we are at that moment, appear as a sign. Now a sign has, as such, three references: first, it is a sign ***to*** some thought which interprets it; second, it is a sign ***for*** some object to which in that thought it is equivalent; third, it is a sign, ***in*** some respect or quality, which brings it into connection with its object" (Peirce W2, p.223; CP, 5.283).

Mathematics is, as Peirce defines it, "the science which draws necessary conclusions" and this implies that it occupies itself with mere hypothetical statements and their logical consequences. In 1870 Charles Peirce had published *Description of a Notation for the Logic of Relatives, resulting from an Amplification of the Conceptions of Boole's Calculus of Logic* (Works, W2, p.359). Joseph Brent tells us in his biography of Charles Peirce:

"The paper may well have arisen out of an argument between Charles and his father Benjamin Peirce (1809-1880), a distinguished mathematician, over which discipline held primacy – logic as Charles held, or geometry as Benjamin held" (Brent,1993, p.132)

Already an old man Charles Peirce himself commented on the difference in 1902:

"The philosophical mathematician, Dr. Richard Dedekind, holds mathematics to be a branch of logic. This would not result from my father's definition, which runs, not that mathematics is the science of ***drawing*** necessary conclusions -- which would be deductive logic -- but that it is the science which ***draws*** necessary conclusions. It is evident, and I know as a fact, that my father had this distinction in view. At the time when he thought out this definition, he, a mathematician, and I, a logician, held daily discussions about a large subject which interested us both; and he was struck, as I was, with the contrary nature of his interest and mine in the same propositions. The logician does not care particularly about this or that hypothesis or its consequences, except so far as these things may throw a light upon the nature of reasoning. The mathematician is intensely interested in efficient methods of reasoning, with a view to their possible extension to new problems" (Peirce, CP 4.239).

So, logic is meta-mathematics and as such a kind of language, while mathematics is a science, at least as far as it is applied. On many occasions it is not sufficient to have an idea. One has to act and to apply it. An algebraic equation should be solvable, a theory should be applicable, a machine should work, and a mathematical or scientific concept is essentially a function. Mathematical or scientific representations gain their significance only through their applications.

Structure and Interpretation

If we were still inclined to compare mathematics with language, we might interpret pure mathematics as a kind of grammar for dealing with the world and the intended applications as depending on the choice of productive metaphors. Syntax and semantics are complementary to each other. ″Colorless green ideas sleep furiously″, is a sentence composed by Noam Chomsky in 1957, as an example of a sentence whose grammar is correct, but whose meaning is nonsensical (Chomsky 1957, 15). Chomsky's sentence may, however, appear as a completely acceptable phrase within a piece of poetry. In 1985, a competition was held at the University of Stanford, in fact, the purpose of which was to present Chomsky's sentence as part of a poem, giving it a meaning as a metaphor. And one of the easiest ways of making a joke is to take a word and use grammar to play with its meaning.

We are suggesting in fact that all knowledge is constructed by choosing a syntax, a kind of operative structure and the appropriate metaphors to apply the formal structure on some kind of reality. One may start to make modifications at both ends, either preserving the structure or the concepts. For instance, there are commutative and non-commutative group-structures and both types may be interpreted or applied variously and frequently.

In 1845 Grassmann criticized Ampere's *Law of Electrodynamical Force,* because Ampere had transferred assumptions valid in the area of gravitational force to a very different area where the elements are vectors rather than points. Grassmann accepts Ampere's view that the form of the law, which describes the mutual influence of two electrical currents, is to be the same as the form of Newton's law of gravitational interaction between material points. But Grassmann believed that this analogy between gravitation and electrodynamics had to be based on a new interpretation of the mathematical objects and operations in question, taking into account that the entities between which electromagnetic forces act are directed quantities, vectors, not scalar quantities. Grassmann had accordingly to generalize the idea of a product from numerical quantities to directed ones, to vectors, giving up commutativity. He defines the idea of a product in a completely new and structural way and by this is able to preserve analogy between gravitation and electro-magnetism (Otte, 2003, pp.189-190).

Formal axiomatic structures or theories become instruments in the same way as maps or diagrams or symbols, etc. But differently form maps they are the result of a combination of empirical and reflective abstraction from mathematical activity itself. When algebra was transformed from a 'language' into a science of structures the operative conceptual schemata themselves, had to become an object of thinking. An important case is given by the interpretation of the imaginary numbers in terms of formal vector space presentation.

Algebra became meta-algebra, that is, 'algebra on algebra', as Sylvester (1814-1874) once remarked with particular reference to the algebra of determinants. Analytical geometry works with arithmetic coordinate systems. And the coordinate transformations themselves are given in terms of coordinates (matrices). As Salomon Bochner observes:

"In Greek mathematics, whatever its originality and reputation, symbolization ... did not advance beyond a first stage, namely, beyond the process of idealization, which is a process of abstraction from direct actuality, ... However, ... full-scale symbolization is much more than mere idealization. It involves, in particular, untrammeled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth" (Bochner,1966, pp.18, 51, 57).

And the fact that axioms in the modern sense in this way become mere hypotheses, implies that the truth and the foundations of an axiomatic theory lie *in the future,* in the intended applications. Any formal axiomatic theory has quite a number of different intended applications. What the axioms describe are concepts or classes of objects, rather than particular objects themselves. Peano's axioms do not answer the question "What are numbers, what is the number 1 or 2?" Numbers could be anything, even games (Conway-Numbers, Hackenbusch-Games, Chessboard-Computer, etc.). Let us simplify things a little. Let our theory of arithmetic consist of only **one** single axiom:

If a and b are numbers, then $(a+b) = (b+a)$

Suppose two countries. One endorses a discrete world-view and considers numbers to be equivalence classes of sets.

So 3 is the class of sets with just three elements $$$! Can numbers be such sets?

Test: ($$) +($$$) = ($$$$$) = ($$$) + ($$)

So sets are numbers!

The people of the other country say "No, numbers are translations (classes of vectors). The reader may verify with ease using pencil and paper that vector-addition obeys our axiom such that in this country numbers are vectors! Theories are just kind of structures with an indefinite and sometimes nearly universal range of possible applications or interpretations.

Language, Space and Indexicality

Mathematics is no language, for the simple reason that language does not contain indexical signs. All linguistic relations to reality are descriptive and arbitrary. The difference between language, philosophy and the humanities, on the one hand, and mathematics and natural sciences, on the other hand, is in spatial intuition and in the indexical sign. In his famous *Tractatus* Wittgenstein maintained that,

"*The world is the totality of facts, not of things. The world is determined by the facts. The facts in logical space are the world*" (*Tractatus*, 1.1).

And he concluded that if there were no language there would be no logic. And this implied that there were no necessity, since all necessity is linguistic necessity. His friend Frank Ramsey pointed out to him, however, that the impossibility of a particle being in two places at the same time expresses a feature of the world, rather than of language. Ramsey`s argument works only if we assume space in the objective sense as a set of locations (points), rather than as something ideal, like a set of signs or perspectives. So, $A = B$ could mean a relation of things or a relation of signs. Descartes as well as Newton or Kant held the first view, Leibniz the second.

Peirce found that there are three kinds of signs, that are all indispensable in reasoning:

"The first is the diagrammatic sign or icon, which exhibits a similarity or analogy to the subject of discourse; the second is the index, which like a pronoun, forces the attention to the particular object intended without describing it; the third or symbol ... signifies its object by means of an association of ideas or habitual connection between the name and the character signified" (Peirce, CP 1.370).

The icon provides the qualities of its object, but does not contain any existence claim with respect to the latter. The index, in contrast, is just an existence claim without providing a description; it is in general physically connected with its object. The symbol is connected with its object by virtue of a convention like using the words **cadeira** or **chair** depending to the linguistic community. This classification of signs takes the sign-object relation as its starting point.

Catherine Elgin objects to Peirce's classification. She does not believe that there are any signs, "that are simpler and more easily grasped than conventional symbols" (Elgin 1997, 146).

Differently from symbols the other signs, she says, seem to bear the same type of relationship to their objects "whether they were interpreted as doing so or not. …. The difficulty is that resemblances and natural correlations are ubiquitous. Every two entities bear some likeness to each other, and some correspondence in fact. Yet we do not consider every object a sign, much less an icon or index of every other … . Something is an icon or index only if it functions as such. … But being taken to signify requires an interpretant. So, icons and indices, like conventional signs, are symbols. … Icon, index and symbol threaten to collapse into an undifferentiated heap" (Elgin, 1997, p.143).

But this is a consequence of philosophical nominalism, that is, of the view that the subject determines the nature of a sign and thus becomes the source of signification, rather than the object. Obviously, this implies that questions of truth are ignored. In this manner, language and communication are separated from objective knowledge processes. Semiotically speaking: existence pure and simple is ruled out. If you bump with your head against something hard in the dark, however, you become convinced that there is something resisting your head, even though you might not know what it is, a bar or a warning from God and destiny. Fever is an index of illness, usually interpreted as a symptom of inflammation, even though one might not be able to really diagnose the disease. The fever is, semiotically speaking, an *indexical sign* of the disease. Fever indicates the fact of sickness but does not describe its character. The footprint on the beach, informed Robinson Crusoe of the arrival of another human being on his lonely island, rather than of the presence of a lion or goat. But it did not provide him with sufficient information to know how to react.

And one might certainly use these symptoms, *heat, redness, trembling, hectic activity* etc. and refer to them in terms of metaphorical speech as “fever” of some kind and as a consequence the word “fever” is applied in quite a number of different contexts. But this does not justify to call fever an icon or a symbol and to conclude that “icons and indices, like conventional symbols, are symbols” (Elgin,1997, p.139).

Charles Peirce had also pointed out that it is exactly analytical reasoning which depends upon "relations of reason", or, as one might say, on associations of *similarity*, that is analogy or metaphor, while “synthetical reasoning depends upon associations of *contiguity*" (Peirce, CP. 6.595). Contiguity relations are objective, real, - like inflammation and fever, or a tree and a bump on the head - while metaphorical relations or analogies are due to the representation, rather than in the things represented. Metaphors serve to make one see something in a certain light or from a certain perspective, that is, to see an *A* as a *B*. The philosopher of art A. Danto illustrates this point as follows:

“When Napoleon is represented as a Roman emperor, the sculptor is not just representing Napoleon in an antiquated get-up, the costumes believed to have been worn by the Roman emperors. Rather the sculptor is anxious to get the viewer to take towards the subject – Napoleon – the attitudes appropriate to the more exalted Roman emperors.... That figure so garbed is a metaphor of dignity, authority, grandeur, power and political utterness. Indeed, the description or depiction of *A* as *B* has always this metaphoric structure” (Danto 1981, 167).

When the artist presents Napoleon as Roman emperor the viewer must “perceive the metaphor as an answer to the question why that man has been put by the artist in those clothes – a different question entirely from that which asks why Napoleon is dressed that way, the answer to which might not be metaphorical at all … -the locus of the metaphorical expression is in the representation – in Napoleon –as Roman-emperor – rather than in the reality represented, namely Napoleon wearing those clothes " (Danto 1981, p.171).

This becomes even more clearly visible in some examples of modern Art like Andy Warhol's *Campbell Soup* cans or his *Brillo Boxes* as works of art. The fascinating thing about Andy Warhol's art is just the transportation of an ordinary article of daily use into a new context. Metaphor relies neither on the similarity of objects nor on proximity of concepts but is a result of the decisions use some words in a certain way. Donald Davidson thinks that linguistic metaphor "is something brought off by the imaginative employment of words and sentences and depends entirely on the ordinary meanings of those words and hence on the ordinary meanings of the sentences they comprise" (Davidson, 1984, p.29-46).

Different theories of reference

A companion to the belief that logic is a universal language and that mathematics is based on logic consists in what is called a descriptive theory of reference. Logicians, like Russell or Frege or philosophers, like Wittgenstein define *exists* as a second order predicate, that is as a predicate applied to concepts, rather than objects. McGinn illustrates this view of existence as follows:

"When you think that tigers exist you do not think of certain feline objects that each has the property of existence; rather, you think, of the property of tigerhood, that it has instances. ... The concept of an object existing simply is the concept of a property having instances" (McGinn, 2000, p.18).

If in this manner, existence pure and simple is ruled out and cognition as an explorative activity that confronts the yet unknown and uncategorized becomes ignored or excluded. Let us ask with John Searle, "Is there a linguistic distinction between referential and attributive uses of labels?" (Searle, 1979, p.137ff). Searle denies this. He believes that all the reality about which we are supposed to speak has already been substituted by a representation of it. Searle makes an even stronger claim believing that the distinction of attributive and referential descriptions is made on account of the claim that definite descriptions have an ambiguity, in so far as a they are used in a twofold sense, to refer as well as to describe. On might say: "That man over there with the champagne in his glass is happy".

Searles explains: "Suppose the man over there only had water in his glass; still what I said might be true of that *man over there*, even though the definite description I used to identify him is not true of him" (Searle, 1979, p.146).

In this situation the difference of referential and attributive use consists only in the circumstance that "in the cases of the so-called referential use the reference is made under a secondary aspect and in the so-called attributive cases it is made under a primary aspect" (Searle,1979, p.150).

The suggestion, “Go and congratulate him”, would apply to the man I am seeing over there, even though the description I have been given of him, is incorrect. But the situation is different and the function of the phrase *the man with the champagne in his glass* changes as soon as I am told: “Go inside the house and look for the man with a glass of champagne in his hands and do this or that…”

In this case again there might not be any man with a glass of champagne inside the house. And as the description is the only means to identify the man, I might not be able to do what I have been told. In the first case the phrase “with the glass of champagne” is secondary and I would be able to meet the man because of the direct indication, while in the second case it is of primary importance that my description is correct and is part of a referring description.

What Searle wants to say is, that the words “with the glass of champagne” are used descriptively in both cases independently of the success. But their function depends on whether they represent some truth or not. In addition, the actual reference is established even in Searle’s example by pointing at that man, rather than identifying him just descriptively.

Frege too beliefs that all references are established by means of descriptions. As a consequence, Frege interpreted an equation $A = B$ exclusively as a relation between signs, not between objects. The meaning of a sign or representation is considered as a perspective (among others) on some object, or as, as Frege puts it, as "mode of presentation of an object". In Frege’s famous essay on *Sinn und Bedeutung*, the author quotes some examples from elementary geometry. Frege writes:

“Let *a, b, c* be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of *a* and *b* is then the same as the point of intersection of *b* and *c*. So, we have different designations for the same point, and these names (‘point of intersection of a and b’; ‘point of intersection of b and c’) likewise indicate the mode of presentation, and hence the statement contains actual knowledge”. (Frege,1969, p.40; our translation).

However, a description like *‘point of intersection of b and c’* does only work because indices like *b* or *c* are employed. And in order to introduce a coordinate system one has to directly select three points in a suitable position in the plane. Everything else can then indirectly be described in terms of arithmetic coordinates. And an essential feature of measurement is the difference between the ‘determination’ of an object by individual specification and the determination of the same object by some conceptual means. The latter is only possible relative to objects which must be defined directly” (Weyl, 1921).

Mere conceptual descriptions cannot capture all there is to spatial relations. Already Kant’s example of the two gloves demonstrates this. If you take a right-handed glove and a left-handed glove, all the relations between the various parts of the two gloves are exactly the same, but they represent different orientations in space (Kant, Immanuel (1768): *Von dem ersten Grunde des Unterschiedes der Gegenden im Raum*, Werke vol. I., Frankfurt Suhrkamp Verlag, p.993-1002).

Formalism and Individualism

During the French Revolution, *Louis XVI* (1754 – 1793), the last King of France before the fall of the monarchy, could not understand how Robespierre and the Jacobins might put him on trial, because traditionally the king himself was the state and the law. How can one bring the king's case to court, if he himself embodies the law? The modern state emerged in the aftermath of the French Revolution and were characterized by individualistic ideals (freedom, equality), on the one hand, and a formal structure of laws and of social relations and the legal system, on the other hand. The *Code Civil des Français* as well as the *Code penal* transformed traditional society which until then was organized around formal social stratification such as caste or class into modern society. With its publication on March 23, 1804, the *Civil Code* came into force. In 1807 it was renamed *Code Napoleon*. It served on the entire European continent as a basis and paradigm of law up to the 20th century. In Germany the *Code Civil* became replaced by the *Bürgerliche Gesetzbuch* not until January 1900.

Formal law was a product of views that the Romantics in France and Germany came to adopt and that transformed knowledge and especially mathematics as much as views about our place in society and in the universe. For lack of space we quote two witnesses, Novalis (1772-1801) and Georg Hamman (1730-1788), respectively.

Novalis:

> The designation by tones and strokes is an admirable abstraction. Three letters signify *God* to me; a few strokes a million things. How easy is the handling of the Universe, how vividly the concentricity of the spiritual world! Language theory is the dynamics of the spiritual kingdom. A command moves armies; the word *freedom* nations (1960, p.412, our translation).

And Hamann wrote:

> A law is never as disturbing and insulting as a verdict based on convenient approval (Billigkeit). The first does not touch my self-esteem at all, and extends to my action alone, therefore equates all those who are in the same situation. An arbitrary decision without a law is always a bondage for us (in: MAJETSCHAK, S. (ed.), 1988, p.59).

Language for Hamann is the exact opposite to what was claimed by the linguistic theories of the *Enlightenment*. Both Novalis as well as Hamann emphasized the creativity of language and symbolism. Norbert Wiener (1894 – 1964) elucidates these transformations from a functional view to theoretical structuralism, when characterizing the new intellectual individualism saying that,

> […] he who concentrates on his own mental states will concentrate, when he becomes a mathematician, on the proof of mathematical theorems, rather than on the theorems themselves, and will be compelled to object to inadequate proofs of adequate theorems. […] To us, nowadays, the chief theme of the mathematicians of the Romantic period may sound most unromantic and repelling. The new mathematics devoted itself to rigor, […] What the new generation in mathematics had discovered was the *mathematician*; just as what the Romantics had discovered in poetry was the poet and what they discovered in music was the musician". (WIENER, 1951, p.92).

Wieners description is somewhat biased, because he fails to mention the symmetry in the relations between man and symbol. The subject is a symbol and the symbol becomes the world. For example, John Keats (1795 – 1821), who is considered the most important English romantic poet, wrote a letter to Robert Woodhouse on 27 October 1818 saying:

"The poetic mind has no self – it is everything and nothing – It has no character – it enjoys light and shade; […] What shocks the virtuous philosopher, delights the chameleon Poet. It does no harm from its relish of the dark side of things any more than from its taste for the bright one; because they both end in speculation. A Poet is the most unpoetical of anything in existence; because he has no Identity – he is continually in for and filling some other Body. The Sun, the Moon, the Sea and Men and Women who are creatures of impulse are poetical and have about them an unchangeable attribute. The poet has none; no identity. He is certainly the most unpoetical of all God's Creatures" (HIRSCH, 2001, p.489)

Edward Hirsch in his Introduction to Keats poems comments on this letter, observing that: […] the displacement of the poet's protean self into another existence was for Keats a key feature of the highest poetic imagination. […] He took to heart the ideal of disinterestedness of the wholly adaptable poet and he went on to contrast Wordsworth's capacity for self-projection, which he names the *egotistical sublime* with Shakespeare's essential selflessness. (Ibid., p.xxv).

Romanticism and the new self-understanding of the individual subject seems to be directly related to Protestantism and in particular to Luther's *sola scriptura* principle. But for the Romantics Luther manifests antithetical aspects of the isolated force of free reason. In embodying the principle of rationalism, he is the reinstator of absolutism, the reactionary. In incarnating the principle of progressive reason, he is the herald of individual freedom, the revolutionary.

Novalis accuses Luther of having founded a religion without a "sense for the invisible." Protestantism overvalued the "letter," without understanding its "sense" or "spirit." In fact, Luther's *sola scriptura* principle and the associated reorganization of (Protestant) Christians into written typographic communication devalued the inner world of the medieval man. Inner convictions were less true to the extent to which the textualization distanced consciousness and communication farther from each other.

For example, Alexander Osiander (1498–1552) a very prominent Lutheran theologian and preacher at the *Lorenz-Church* in Nuremberg, the biggest Lutheran town in Germany, in 1525 led an inquisition against the schoolmaster at the St. Sebald Church in Nuremberg, who had gathered around himself a mystical community of followers. This process did not move forward because the defendant was talking unintelligibly, just telling that he had the truth in his heart and that he wanted to listen to God, who speaks to him. But he himself could not comment or say anything about it (Giesecke,1992, pp227-239).

What we see here is a development of social relations, which only becomes fully established after the *French Revolution*. To the extent that the official church built its structures around the printed word, man's interiority became overpowered by standardized texts and formal laws. Intuitive and implicit knowledge and the explicit and logical presentation separated to the same degree as the social relations became formalized.

It is said that the intellectual as well as the social and the industrial revolutions of modern Europe have their origin "in the religious Reformation of the 16th century and that the Protestant reformers, either directly by their theology or indirectly by the new social forms they created, …. prepared the way for the transformation of the world " (Trevor-Roper, 1967, p. 179).

Words and Things

During the Middle Ages the whole world was meaningful *Non solum voces, sed et res significativae sunt!* Such was the belief: not only the words but the things themselves had meaning (Ohly, 1977, p.5).

Then at a certain period in history it happened that words and things parted ways and the common interpretation of our sense impressions seemed to become utterly unreliable. It is a merit of Michel Foucault to have brought this turn to the center of our attention in his "*The Order of Things*".

"At the beginning of the 17th century, during the period that has been termed … the Baroque, thought ceases to move in the element of resemblance. Similitude is no longer the form of knowledge, but rather the occasion of error. …. 'It is a frequent habit', says Descartes, in the first lines of his *Regulae*, 'when we discover several resemblances between things, to attribute to both equally, even on points in which they are really different, that which we have recognized to be true of only one of them'. The age of resemblance is drawing to a close. …. And just as interpretation in the sixteenth century … was essentially a knowledge based upon similitude, so the ordering of things by means of signs constitutes all empirical forms of knowledge as knowledge based upon identity and difference" (Foucault 1973, p. 47 -51 and pp. 56-57).

And "the sign ceases to be a form of the world; and it ceases to be bound to what it marks by the solid bounds of resemblance or affinity" (Foucault 1973, p. 58). Sense and reference of human discourse became independent from each other, meaning and truth separated. Stated in the common language of the philosophy of science we could indicate the changing relationship between concepts and objects and between philosophy and science as being expressions of what we call the *Scientific Revolution* of modern times.

According to the traditional view that reigned from Aristotle till Leibniz all, true propositions were analytically true, because a concept, being abstracted from its object, consisted in no more than a set of properties that really did belong to the object in question.

All criticism of formal logic since the 18th century is therefore comprised either in the criticism of the doctrine of the formation of concepts or of judgements and propositions. Concepts became functions of the mind, rather than being abstractions from objectively given things. And the decisive insight changing the Leibnizian view of propositions as mere aggregates of terms that represent concepts or classes, came when people like Peirce or Frege for example, realized that the subject and predicate of a sentence play different roles in the contexts of the proposition. Two of those basic roles were that of representing relations or concepts and that of denoting individuals. Neither of these nor their representations are interchangeable. The subject of a sentence is therefore represented – using the terminology of Peirce, - by an *index* and the predicate by an *icon*. Kant pursued the first option and Bolzano the second. Gottfried Martin concludes:

"One can characterize the difference between Kant and Bolzano meaning that for Kant axiomatization, and that for the Bolzano arithmetization has been the ultimate goal….. By the keywords arithmetization and axiomatization the viewpoints are given for a specific assessment of the researchers involved in these investigations. " (Martin, 1956, p. 103).

These "ultimate goals" did appear, however, during the 19th century only

Implicit and Explicit Knowledge

Theories and works of art are forms, that is, they are realities in their own right. A work of art is just a work of art; a theory is just a theory. It must be grasped as a form *sui generis*, rather than as a pale and passive reflection of the empirically given world. Before we can inquire into its possible meanings or applications. In the artistic drawing what we achieve is a line, and the line does all the work, and if it fails to do so, no philosophical commentary will rescue or repair a bad work of art. In literature or philosophy, it is the word or the sentence, in mathematics the formula or the diagram, which carry the entire weight, etc. etc.

When the Europeans came to China first, they were astonished by the wealth of China's culture. And when they collected Chinese art – the painting of a sea piece for example - they were not interested in the construction or appearance of Chinese ships but wanted to see how the Chinese artists saw and interpreted these realities. However, to convey their technical knowledge of Chinese ships back to their people in England was not meant to describing how the Chinese people and artists saw or felt about ships. Describing one's knowledge of something is not to describe mental processes be they one's own or those of others.

Now people who deny the existence of something like the intentionality of the mind at all, affirm that all our conscious thinking occurs in terms of signs and symbolic representations. There exists *a unity of sign and thought* (Vygotsky), such that the sign gives the meaning, rather than the other way around. Others, being afraid that this leads to the view of human knowledge as mere activity or even of the mind as a data processing machine, oppose this belief. These people are as a rule nominalist, rather than realists, believing that the human mind is the source of all representations, rather than conceiving the sign as being connected with some object.

The controversy becomes somewhat fuzzy, as soon as we have to deal with implicit or tacit knowledge. Michael Polanyi (1891–1976), a very well-respected physical chemist came to create a theory of the dynamics of cognitive processes, developing the concept of *implicit* or *tacit knowledge*. Polanyi argued that much of the scientist's success depends upon knowledge that is acquired through practice and that cannot be articulated explicitly. One of the most convincing examples from everyday experience is facial recognition (Polanyi, 1958).

We know a person's face and can recognize it among a thousand. Yet we usually cannot tell how we recognize a face we know. Neither could we tell why we believe that sea piece has been painted by say, a Chinese artist, rather than by a British painter. And Caravaggio (1571-1610) painted two versions of the subject *Judith beheading Holofernes*, one before 1600 and the other about 10 years later in Naples. His friend Louis Finson (1580 - 1617) copied the second version and this copy remained – despite all efforts of experts - considered the original until the latter was discovered in 2014.

The *Nobel Prize* laureate Herbert Simon too considers intuition as recollection or recognition. He writes:

"The situation has provided a cue; this cue has given the expert access to information stored in memory, and the information provides the answer. Intuition is nothing more and nothing less than recognition. We are not surprised when a two-year-old looks at a dog and says *doggie*! because we are used to the miracle of children learning to recognize and name things".

A very famous example is represented by the so-called *Madeleine* episode from Marcel Proust's novel *In Search of Lost Time* (*À la recherche du temps perdu*). Proust's narrator involuntarily recalls his childhood with in Combray after tasting a *Madeleine* dipped in tea, because this had been a regular habit of his aunt. Simon's point is that the miracles of expert intuition have the same character.

"Valid intuitions develop when experts have learned to recognize familiar elements in a new situation and to act in a manner that is appropriate to it. Good intuitive judgments come to mind with the same immediacy as *doggie*!" (Simon, 1992, p.150-161).

Historically, this debate around the relationship between idea and sign, or of thought and sign, has intensified during the *Enlightenment* and at the beginning of the *Romantic* period. One can name Goethe (1749-1832) and Schiller (1759-1805) as protagonists of the discussion. Interestingly, Schiller and Goethe, our Germany`s poetic heroes, seem to realize both different types of poetic talent and style. Several letters from Schiller to Goethe in August 1794 indicate this. For example, on August 31, 1794, Schiller wrote: "Your mind is extremely intuitive and all your thinking powers seem to be imaginative .... Basically, this is the highest thing that man can make of himself. ...You strive for it. … My mind is actually more symbolic."

A week earlier, Schiller had generally commented on the difference between the difference between the intuitive and the analytical mind:

"At first sight, there seems to be no greater opposition than the speculative mind that starts with unity and the intuitive one that comes from diversity. But if the first seeks experience with faith, and the last seeks the law with free thinking, it cannot be missing that both will meet each other halfway" (Goethe / Schiller, 1966, p. 43 + 35).

The contrast between Goethe and Schiller is also expressed in the following passage from Egon Friedell's monumental *Cultural History of the Modern Era*:

"On June 5, 1825, when he spoke of the definitions of poetry, Goethe said: *What is there to define? Vivid feeling of the states and ability to express it makes the poet.* In contrast, Schiller wrote: *What has happened never and nowhere that alone is poetry*" (Friedell, 2012, p. 889). Incidentally, both Schiller and Goethe sincerely hated each other (Friedell, 2012, p.886).

A different example of this contrast is the following:

In the film *Forrester Found* (2000, USA), the elderly lonely author who had only written one novel in his life, a book that had made him famous teaches a 16-year-old boy from the Bronx how to become a writer. And that only works, he tells the young boy through an intimate acquaintance with language and the act of writing. The old man advises the boy and shows it by example him that he should just punch the typewriter and write on it. Reading and critical reflection come in second place.

Ironically, the film critics interpreted the “teacher” as a figure modelled after *Jerome Salinger* (1919 - 2010). Salinger has published only a single novel - *The Catcher in the Rye* - and a series of short stories. *The Catcher in the Rye* has sold more than 25 million copies worldwide. From 1953 onwards, just like Forrester in the film, Salinger lived completely withdrawn: he became famous for not wanting to be famous. Nevertheless, Salinger is considered one of the most important writers of the 20th century. Some critics went so far as to call the years 1948 to 1959 in the United States the "Salinger Era", because his novel reproduces the spirit of the post-war period so intimately and intensely.

The profile of Jerome Salinger - the poet of one book - actually appears more clearly in another different film: *The Croupier* (Great Britain, 1998). The story begins when the protagonist of the film, Jack Manfred (*Clive Owen*) - an aspiring writer who is going nowhere - is being offered the job as croupier at a London Casino and consequently finds himself completely drawn into the casino world. The job is gradually filling his head, in fact, taking over his life. So much that one day he decides to write a novel exactly about the life he knows, the life of a croupier at a casino.

The two films draw complementary images of creative writing.

Incidentally, Goethe's writing and poetry stamped literary culture between 1770 and 1800 even more than Salinger's *The Catcher in the Rye* his lifetime. Especially in Goethe's novel *The Sorrows of Young Werther* published in 1774, became a huge success through all of Europe. Napoleon seems to have read the book seven times (Friedell, 2012, p.752).

Relations and Functions

Turning now to the history of algebra, one may notice something similar. Euler's *Complete Guide to Algebra*, (1770) begins by explaining the notion of *quantity* and then Euler goes on to explain: “Arithmetic or the art of calculation deals with the numbers. It only affects activities in ordinary life. In contrast, algebra or analytics generally includes everything that can occur with numbers and their determination” (EULER, chapter 1, §7, our translation).

All the difficulties the Greeks have had with the fact that numbers are actually not quantities, are circumvented by identifying magnitudes with their measures, instead of seeing numbers as **relations** between measured magnitude and unit of measurement. Like in geometry this philosophical retrogression enabled some progress on the active side of mathematics. It was more or less a result of Descartes’s conception of analytical geometry. In this way one always operates on the symbolic level and all progress was attributed to the invention of new methods and new symbols.

But algebra and mathematics in general seems to have a double face. The terms axiomatic theory and algorithmic language indicate the two sides.

Poncelet identified the secret of algebraic generality and the principle of continuity, that is, in the possibility of operating with variables. Frege expressed a similar idea believing that the ascent from arithmetic to algebra is due to functional thinking and on the introduction of the function concept (FREGE, 1969, p.21).

If we write down the following sequence of arithmetic expressions:

$2.1^3 + 1$

$2.2^3 + 2$

$2.3^3 + 3$

We could come up – concentrating on the common form – with the expression:

$2.x^3 + x$

That is, we might encounter the idea of a *function*. Frege thereby re-introduced relational thinking into arithmetic. This appears as somewhat strange, because the function came from technology and physics. The concept of a mathematical function, on which the notion of natural law is based, "applied to physical phenomena, appeared for the first time in the literature of mankind in a prescription for gunners in 1546" (Zilsel, 2003, p. 110), eighteen years before the birth of Galileo and exactly half a century before the birth of Descartes. Peirce summarizes the history of the function concept rather succinctly:

"The word *function* (a sort of semi-synonym of "operation") was first used in something like its present mathematical sense in 1692, by a writer who was doubtless Leibniz. It soon came into use with the circle of analysts of whom Leibniz was the center. But the first attempt at a definition of it was by John Bernoulli, in 1718. There has since been much discussion as to what precise meaning can most advantageously be applied to it; but the most general definition, that of Dirichlet, is confined to a system of numerical values" (Peirce, CP 4.253).

The peculiarity comes from the fact that the function concept has double root. Either the function is identified with an algorithm or with some kind of free variable as part of a law of nature or like in the "famous" general triangle or like in expressions like "An apple is a fruit". In a proposition like "an apple is a fruit" it would be unnatural to interpret "an apple" as a placeholder, like Frege, because this presupposes that we have identified and given individual names to all the apples in this world. There are ideas of an apple or a triangle in general, or of a function but they turn out to be representations of particular ideas, put to a certain use.

Pierre Boutroux, Henri Poincare or Charles Peirce have adopted the intensional perspective on functions, taking a function as a concept in its own right and emphasizing the principle of continuity thereby expressed, while Cauchy, Dirichlet, Cantor, Russell or Frege reduced functions to sets. Boutroux, for example, believed that the relative independence of meaning or sense with respect to reference, should be pursued. He writes:

"La notion de fonction est avant tout, pour le mathématicien, un indéfini, un indéterminé. L'idée que nous en avons est plus riche et plus pleine que toutes les définitions ou expressions que nous pouvons donner ou construire » (Boutroux, 1920, p. 167).

The classical example that clearly demonstrates the importance of the function concept conceived of in terms of a complementarity of arithmetic and geometry or of the discrete and the continuous comes from the tentative to cope with Zeno's paradox of Achilles and the tortoise. Let us see: Achilles runs ten times as fast as the tortoise, though the tortoise has a one stadium start. For *each* of the stages, $x (x > 0)$, covered by Achilles, the tortoise has crawled the distance

$f(x) = 1/10x + 1$ *stadium.*

This model of the relative movement of the tortoise in relation to the position of Achilles enables us to reproduce the paradox on a new level, because of the function's double character: The continuous aspect of the movement does not contradict the discrete perspective. It remains correct that the tortoise is at $x_{(n+1)}$ as soon as Achilles has reached x_n, but the representation using the function concept enables us to liberate Achilles' movement from the one-sided fixation on the discrete set x_i $(i = 0, 1, ...)$, conceiving of the movement as a whole. The question "At what point does Achilles really catch up with the tortoise?" becomes now: "What is the fix point of $f(x)$?"

The fix point can be calculated simply as a function of the constants a and b:

$x = f(x) = ax + b.$

We seemingly have solved the problem by taking a relational point of view, that means by adopting a "world view" which provides objects and relations between objects with an equal ontological status. This transition took place at the end of the 18th century only. The paradox of the movement leads to a complementarity in the function concept. It shows the necessity of having the concept of the functional relation as a model or as a single mathematical object. And secondly, to have available the effectiveness of symbolic calculations, that allow us to write down the meeting point (Otte, M., 1990, Arithmetic and Geometry: Some Remarks on the Concept of Complementarity, *Studies in Philosophy and Education*, 10; 37-62, p.55f).

Functions are a distinguishing attribute of modern mathematics, and are considered as the most profoundly distinguishing of all. Bochner says:

"In its innermost structure Greek mathematics was a mathematics entirely without functions and without any orientation towards functions. […] By outward appearance Greek mathematics was geometrical rather than analytical and by inward structure it was representational rather than operational" (Bochner,1966, p.217).

Conclusion

Andre Leroi-Gourhan sketched a picture of the development to today's human being in which the coordination of hand and eye plays an essential role:

"Erect posture ... free hand during locomotion and possession of movable implements - those are truly criteria of humanity. Ultimately, the human capacity for abstract knowledge is attributed to the combination of locomotion and the associated mobility of hand and eye. Although it came more spontaneously to early researchers to characterize humans by the size of their brain by their intelligence than their mobility… this cerebral view of evolution now appears mistaken" (Leroi-Gourhan, 1993, pp19-26).

The complementary of meaning and reference reflects and even tries to instrumentalize the distinction, between pointing at something and describing it. Descartes rediscovered this when he recognized the meaning of the analogy of space and number. It is based on the concept of the coordinate system and thus on a combination of pointing and describing. in order to introduce a coordinate system, one has to directly select three points in a suitable position in the plane. Everything else can then indirectly be described in terms of arithmetic coordinates. And an essential feature of measurement is the difference between the 'determination' of an object by individual specification and the determination of the same object by some conceptual means. The latter is only possible relative to objects which must be defined directly" (Weyl, 1921).

Logic did not recognize the importance of the distinction between context-sensitive symbol systems and context free languages until the 20th century when Russell drew Frege's attention to the antinomies that nullified the Fregian idea of logic as a universal language. The origin lies in the x of the algebra that entered our thinking with the invention of printing. It represented the ability of algebra to go beyond mere descriptions of what was given by making the unknown itself x the subject of mathematical investigation. We owe this context-sensitive mathematical language to mathematics the freedom of the individual as it has become known to everyone since the French Revolution through the slogan *liberté, égalité, fraternité*..

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