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Towards a critical mathematics

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Abstract: *The goal of this paper is to express necessary conditions for arithmetic in ways that are compatible with the unity of being and knowing understood within first-person experience. In psychological literature, this experience of unity is discussed as flow, but the epistemological and ontological unity is prior to the observer's position from which psychology unfolds, making this project essentially non-psychological. Instead of an externalized psychological conceptualization of mind, the idea explored is internal to first-person experience that experiences consciousness as movement, with key elements of that movement explicable as inference and algorithm. The notion of first-person experience is often taken to be transparent and irrelevant, so an exercise for orienting readers to the I-feeling and its connection to movement is elaborated. A mathematical notion of history as algorithmic elaboration is introduced in order to explicate how to move from talk about judgments in general to doing arithmetic. In this preliminary and exploratory paper, I forgo many possible linkages with modern mathematics education research literature to focus on articulating what I think are some of the foundational elements of a critical mathematics. The purpose of this preliminary work is to orient readers to an ongoing project in critical mathematics, not to provide instrumental notions of how one might improve teaching or research, though brief discussions of how this mathematical theory could be applied in mathematics education research and practice are included.*

Introduction

Critical mathematics has included projects that center the socio-political effects of mathematics (i.e. Powell & Brantlinger, 2008), are explicitly ethnomathematical in nature (i.e. Gutstein, Lipman, Hernandez, & De los Reyes, 1997), or focus on critical mathematics education (i.e. Skovsmose, 2013). My goal is not to upend these applications of critical philosophy to mathematics education, but rather to contribute to their foundations by conducting an exploration of the mathematics side of critical mathematics education within a philosophy of language that cuts across these broad interests.

At its essence, I take critical philosophy to inform a philosophy of mathematics by pointing towards the unity of being and knowing that is prior to mathematical thought. This unity can be

felt from within first-person action and understood when first-person action is foregrounded. Many of us are conditioned to think of knowing as necessarily, strictly, and intrinsically separated from being, in part because of the fundamental role their separation plays in our normal ways of thinking about and using those terms. Both their unity and their difference will be illustrated below. The assumption that they are strictly different is explainable in part because we inherit the notion that valid knowledge comes from within a third-person observer position, typified by empiricism. In the empiricist tradition, what counts as valid knowing is disinterested knowing (often called 'objective'), but there is no way to become disinterested without an interest in disinterest. The implication of this interest in disinterest is that other knowledge types can be suppressed. In particular, the historical/hermeneutic knowledge type, defined by an interest in reaching understanding, and the critical/emancipatory knowledge type, defined by an interest in freedom and self-actualization (Carspecken, 2009), can be completely erased as valid knowledge types within a specific domain. The interest structure is how early Habermas joins subject and object (1971 p. 314) so the interest in disinterest that typifies empirical/analytic knowledge is secondary to their primary unity. In domains like mathematics, the interest in disinterest has led to the nearly complete suppression of these other knowledge types to the point where we have almost completely lost the grammar through which we might express those interests. Such interests are simply 'not math.' In mathematics education communities, conversations about mathematical communication predominate, and with the advent of the socio-political turn (i.e. Gutiérrez, 2013), conversations about emancipatory mathematics education are quite common but tend to leave the mathematics alone, or gesture towards a mathematics of others where it can be challenging to locate the emancipation of individuals.

In Western academic communities, we might conceptually access part of the unity of being and knowing from within the concept of flow (i.e. Csikszentmihalyi, 1990), but the unity of being and knowing has deeper roots in German idealism than is often recognized. Flow is a state typified by an undifferentiated feeling of being/knowing that is ruptured when it is recognized. The purpose of this paper is not to dwell on the psychological concept of a flow state that is often explored from within the empiricist tradition, so I do not wish to be tied to that literature too tightly; I just mention it as an access point for readers who might be more familiar with that concept than they are with critical philosophy, where expressions like the unity of epistemology and ontology are taken up with more clarity than the psychological tradition really can provide.

In the rest of the introduction, I will first describe the experience of error. I will then describe the synthetic and transcendental unities of apperception that map onto the existential needs to be recognized as a normative subject (the need to be recognized as a good person) and the need to be recognized as someone whose identity exceeds the finite identity claims that constitute the self as

a normative subject (the need to be recognized as infinite). The next section deals primarily with language. The goal here is to articulate the language structures that are necessary to produce a mathematics within. The next section is primarily mathematical, where I first demonstrate how to make numerical structures that are implicit in judgments of distinctness explicit. I then present an interlude on embodiment that I am calling The Exercise. From this I turn to arithmetic. The last section is an explicit discussion of the theory of number that is less directly addressed but in use throughout this paper.

The experience of error

The other main reason besides our conditioning to think that knowing is separate from being is the simple fact that we make and experience errors frequently. Error involves the difference between appearance and reality. Appearances are what reality is for consciousness, as the kind of perceptual raw materials of awareness produced by the relevant biological machinery, but they are not consciousness itself. Instead, consciousness is the difference between appearance and reality. Hegel expresses this as what things are for-consciousness what they are in-themselves, and the difference between those as what they are to-consciousness (Brandom, 2019).

For example, I recently moved and decided to store a heavy cast-iron pan in the drawer under my oven. The pan is dark and matte, and so unless one is paying close attention it appears the same whether hot or cold. In my previous home, the oven drawer was for storage. It turns out that my new oven drawer is actually the broiler and so gets exceptionally hot. When I invited a friend over to cook dinner with me, I asked them to please get the pan out from under the oven, which was on. We both saw it sitting in the oven drawer and for both of us the pan just appeared to be a pan. Moreover, what the pan was to both of our consciousnesses was not meaningfully distinct from what it was for our consciousnesses because neither of us were considering the difference between appearance and reality. My friend picked up the pan and dropped it, exclaiming, “That pan is hot!” What the pan was for my friend’s consciousness was now a painfully hot pan. Its appearance for my friend changed, but the underlying reality of the pan in-itself had not changed meaningfully. This asymmetry leads to the feeling of separation of mind from world, because the world, unequivocally, is not what it appears to be. This is a separating moment when appearances are considered as a separate kind of stuff from what they are appearances of, and so these kinds of experiences fund the relatively common understanding of knowing as separate from being. But this overlooks a key feature whereby we might explain how the judgments this feeling of separation produces are actually in error and so overcome the problem of separate substances. The principle question to ask is whether the resulting separation of mind from world is also subject to the experience of error, because we do not stay in a state of separation permanently. Rather, we

re-enter flow-states easily once the error has been rectified. Any claim to the unity of being and knowing must contend with the importance of the experience of error.

The way that Brandom resolves this is to remark on what something is to-consciousness. What the pan is to my friend's consciousness is now the-pan-that-looks-safe-but-is-dangerous. There is a meaningful distinction between appearance and reality, and that difference between the two is how we can begin to understand the {I} as negation. The {I} is not a separate kind of stuff; it is, for the moment, just this difference between appearance and reality. The {I} shall take on a bit more significance later in the section called the Exercise.

As essentially social creatures, we do not just experience errors based on perception. My experience of error in the same episode was not purely perceptual. I, like most people, have a general commitment to not cause physical suffering in my friends. Consequently, I experienced dissonance as the result of a local incompatibility between this general commitment and the implicit commitment I held that the pan was safe to pick up. In fact, I felt terrible about it, the way one feels when one might lose a friend. This feeling of shame was the normative correlate to perceptual pain—the pain of contradictions that rupture recognitive communities. For my friend, the error was with respect to the pan in-itself. The error for me was an ethical error that threatened my sense of being a valid person, a good person, someone whose actions accord with their commitments. We have a deep existential need to be valid, to live up to the assertions we have internalized as commitments. When our actions introduce local incompatibilities between these commitments, we have a rational obligation to repair those commitments in order to be taken as valid by other people, but we feel these local incompatibilities in our bodies and feel the contradictions even in the absence of other people. The local incoherence in this case was between what I did when I asked my friend to pick up the pan and my general commitment not to cause suffering in my friends. Incoherencies in constellations of commitments threaten our status as normative subjects, which is the outward facing component of communicative rationality. The inner facing component of communicative rationality is necessity at the level of feeling.

Hegel sees a single episode of experiencing error as beginning with the registration of an anomaly: the acknowledgment that one finds oneself with commitments that are incompatible, in the sense that one cannot become entitled to them both (or to all of them). They preclude jointly fulfilling one's justificatory responsibility. Practically acknowledging that incompatibility is taking oneself to be obliged to do something, change something. This is the obligation to engage in a process of repair of the anomaly, to replace rational discord with rational harmony, by altering or giving up some of the offending commitments. (Brandom, 2019, p. 679)

I will discuss the justificatory responsibility later in this paper, but in this instance, it was easy to repair the local incompatibility by acknowledging the commitment to the pan-as-safe was what needed to be modified, rather than the general commitment to not induce suffering in my friends. The feeling of shame was fleeting, in part because there was no suppression of the grammar by which I could acknowledge the incompatibility in my commitments and repair those commitments in a way that included justification. In terms of justification, I attempted to figure out why the pan was hot at all and discovered that I had a broiler drawer, rather than a storage drawer. It was also easy to repair the relationship through forgiveness. The burn was minor, my friend forgave me when I asked them to, and I took on the new commitment to the pan being the-pan-that-looks-safe-but-is-dangerous.

Apperception

The story that unfolds here is a story about the normative more than a story about perception. A way to integrate sensory judgment with judgments based on commitments and entitlements is better told by Brandom (2019) or Hegel (1977) than I can offer here. I take it that there are two fundamental existential needs related to identity formation that are met in different ways depending on where an individual is on a developmental trajectory. These needs are contradictory to each other until that contradiction is resolved dialectically. One of those needs is the need to be recognized as normative subjects. That is, we need to be recognized as people who honor our commitments and repair our system of commitments when incompatibilities are made explicit. This is essentially the need to be recognized as good people, where goodness takes on different qualities depending on one's normative horizons, including where one is on some developmental trajectory, but trends towards universality as those horizons are fused by internalizing the attitudes of others. This is the finite side of human identity where the self is understood as a collection of commitments that are often incompatible with one another, but its self-consciousness is its incompleteness in actual universality – the infinite. That is, there is some 'thing' called the good that in all our striving to understand one another we approach but never quite attain because we, who are it, are necessarily incomplete. The first existential need is to be recognized as good, though these partial sums are finite. I shall call the first the synthetic unity of apperception or the normative self.

The other existential need is the self-consciousness of the first and is the need to be recognized as infinite. Our normative selves, understood as a finite collection of identity claims, is never adequate when considered within the present moment in which all is interconnected. The self-consciousness of this side of self, which I will call the transcendental unity of apperception or the infinite self, is its temporal incompleteness. The idea is that the transcendental unity is infinite

within the now but incomplete with respect to some future. New commitments shall come along that, in turn, are inadequate.

These two moments of self are reciprocally self-conscious of each other, depending on one another for their individual meanings, and so form an identity – a unity of unities. However, this identity is not one that is really representable as the sum of its parts, since each part truly depends on the other. They contradict one another until their contradiction is dialectically resolved. Consequently, it is an identity drawn out over long pages that cannot be summarized or pictured. I shall recollect this identity as the {I} or as representational nullity (∅). The representational nullity (∅) of the {I} is how I introduce number later in this paper. When I use the first-personal pronoun "I" without braces, I am referring to the empirical self, which in this case is just the author.

The Synthetic Unity of the Normative Self

We have an existential need to be taken as good people and we feel it when we are unable to take ourselves as such due to incompatible commitments. In the following, I discuss the goodness upon which we stake our social selves as the normative synthetic unity of apperception. Much of the content for the synthetic unity that I describe actually shows up in the section on judgment that follows. Specifically, the integrative task responsibilities that Brandom (2019) describes give crucial details to the concept. In this section I am more focused on the trend towards universality.

The moment of the {I} that this normative unity of apperception is identical with is a claimed universal subject position. It is a kind of claimed "we," though it does not necessarily show up in empirical speech with this pronoun attached. For example, "stealing is wrong" has no particular pronoun attached but is issued from a claimed universal subject position. Similarly, the assertion that "the pan is hot" emanates from a claimed universal subject position. In that case, the universality is explicit in the sense that both my friend and I actually agreed that the pan was hot and so could have asserted, "We agree the pan is hot." This is getting closer to the claimed universal third-person experience that can be understood as the observer position, but it is important to understand that the {I} is within this "we," not standing outside and observing. There is nothing disinterested or indifferent in the assertion "stealing is wrong" when it is understood as a personal commitment.

Relating a point Sellars makes about talk about appearances and talk about reality, Brandom writes, "because he thinks part of what one is doing in saying how things merely appear is withholding a commitment to their actually being that way, and because one cannot be understood as withholding a commitment that one cannot undertake, Sellars concludes that one cannot have the ability to say or think how things seem or appear unless one also has the ability to make claims about how things actually are" (Brandom, 2008, p. 12). Claimed universality is unavoidable. The

claimed universal subject position is nominally evacuated by adding "in my opinion, stealing is wrong" or "in the culture I was raised in stealing is considered wrong" or "the pan appears to be hot," but these evacuations are affixed in anticipation of possible challenges from other interlocutors rather than truly evacuating the claimed universal position. That is, they are indications of withheld commitments, rather than alterations of the universality claimed within the assertions. To truly evacuate the universal position that "stealing is wrong" requires more complex work like what one finds in Les Misérables, as an appeal to empathy as a virtue, or in Marxist arguments for the dissolution of private property. That is, undermining the universality of one claim is only accomplished through justification that appeals to other universals.

The Transcendental Unity

The other existential need is discussed more in the section below called the Exercise. It is essentially the need to be recognized as individuals along the axis of self called the transcendental unity of apperception, given to us by Kant as the "I think." The notion is that we who use first-personal pronouns in a meaningful way are always beyond any finite identity claims that are explicated as commitments articulated in the normative unity of apperception. For example, we might say things like "I am a musician, a teacher, a cisgender white male…but I am so much more than just these claims." We have an existential need to not be totalized by our normative identity, the need to be recognized as infinite. The reciprocity of these two axes of identity, the finite and the infinite, even the self-consciousness that each moment of apperception is to the other, impel the masterpiece in motion that is the self. In communicative action, rather than instrumental action, both axes of identity are recognized. Habermas writes:

The specific characteristic of this linguistically structured community is that individuated persons communicate in it. On the foundation of intersubjectivity they accord in something general in such a way that they identify with one another and reciprocally know as well as acknowledge one another as homogeneous subjects. At the same time, however, in communication individuals can also keep a distance from one another and assert against each other the inalienable identity of their egos. The community that is based on the intersubjective validity of linguistic symbols makes both possible: reciprocal identification and preservation of the non-identity of one another. (Habermas, 1971 p. 157)

What Habermas is expressing here is that when we treat each other as subjects, we enter into "we" formulations. Communicative competency requires the assumption that people use universals, "linguistic symbols," and so in order to be recognized as normative subjects, individuals have to use those universals correctly. In first-person plural claims, the risk is that the {I} becomes completely dissolved in the "we." When we talk about establishing communities that actually

support communicative actions, as classrooms ought to do, both axes of identity, the "we" and the {I}, need to be mutually asserted. In mathematics classrooms, given the almost pure rulishness of mathematics, "the preservation of the non-identity of one another" is precarious. One of the purposes of this paper is to explicate how the {I} is present in mathematical judgments in order to counteract the deformation that is the result of the suppression of the {I}.

The {I}

To do this necessitates the re-articulation of mathematical knowledge from the first-person actor's perspective (Carspecken, 1999), with the understanding that the third-person universal subject position that typifies most articulations of mathematical knowledge is epistemologically and developmentally secondary to this first-person subject position. Precisely how this affords a unified epistemology and ontology commensurate with a non-arbitrary mathematics shall require multiple trips around the hermeneutic circle, and this paper is just one of those tellings. The most important understanding for this paper is that the unity of being and knowing can be understood in terms of movement. When movement is explicated, it can sometimes be explicated as inference (a so b) or algorithm (a so b so c). For example, one might have a first-person experience of consciousness that involves a perceptual judgment like "I am hungry" which is followed by a different thought, "I should get something to eat." This is expressible as an inference: "I am hungry so I should get something to eat" (a so b). Perhaps this is followed by the thought "I should go to the kitchen" (a so b so c). Insofar as the latter is repeatable, it is an algorithm.

At its heart, what I am critiquing is one of the main assumptions of people who think about mathematical knowledge for educational or empirical purposes: that what counts as mathematical knowledge should be accessible from within a universalized third-person detached observer position. This is the basic subject position claimed through empiricism, but it is also reproduced in many anti-empiricist frameworks (Carspecken, 2015). Consider the cartesian coordinate system. In order to understand how the cartesian coordinate system works, one must imagine an anonymous subject that is completely outside of the representation, looking down on the graph. When this observer position could be anyone, it is universalized. When we think about shared experiences of such a graph, perhaps while looking across the table at a student's work so that the graph is upside-down, we have to imagine our way to such a position, mentally rotating the graph in order to draw correct inferences. When we think of science in general, we often think about the 'disinterested' observer who sets up an experiment and then merely collects the data. But this disinterested position has an interest in disinterest, it is not an original subject position in development. Rather, we develop into third-person thinking from first- and second-person thinking. When we think about mathematics, we often think about a universal language that any sapient could learn and use to make correct inferences. So there are differences between the

physical sciences, where an 'empty' subject passively records data, and the mathematical sciences, where the content of the universalized subject position are axioms, but both point towards a universalized position that anyone could step into.

To say this another way, mathematical knowing and knowing about mathematical knowing are almost exclusively understood within a kind of picture-thinking, where the subject, the I, is left implicit. In picture-thinking "knowledge is like a picture of what is, taken from a position any possible sapient being can in principle occupy" (Carspecken, 2015). That is, the kind of knowledge that 'counts' involves a generalized third-person "god's eye" view that anyone could occupy and judge, assert, or think within. This is understandable because the subject "is easy to hide since it is fundamentally not an object, not objective, not picturable, and impossible to model" (Carspecken, 2015). The {I} is necessary for understanding what it means to observe something in the physical sciences, or to understand a graph, or to understand what it means to use axioms to do mathematics, but it is not and can never be adequately included within the representation because it is not a thing. Rather it "enters into representational frameworks as the negative" (Carspecken, 2009). The negative is not a new kind of thing, but is more like a process, or movement. Within experience, consciousness is movement within the now. When we exit experience, we flatten that movement when we think/judge/assert. These are the representations that punctuate the movement of experience and are necessary for understanding that movement.

Cartesian dualism enforces a strict separation between subject and object that results in two ontological categories. On the side of the subject we have representations which are known directly as the stuff of consciousness, and on the side of representeds we have some kind of non-consciousness-stuff. That is, there is a "gulf of intelligibility" (Brandom, 2019) between representations, which can be known directly, and representeds, which are known only through representations. Kant can be understood as troubling the ontological dualism between the two different kinds of stuff (mental and physical) but reproduces the gulf of intelligibility through the assertion of a fundamentally unknowable thing-in-itself alongside phenomena, which are knowable though not a separate class of stuff. "Descartes understood the distinction between minded creatures and everything else in terms of a distinction between two kinds of stuff: mental and physical. Kant's normative reconceiving of sapience replaces Descartes's ontological distinction with a deontological one. Discursive creatures are distinguished by having rational obligations" (Brandom, 2019). This understanding of our responsibility to concepts is inclusive of the responsibility that representations have to what they purport to represent. Understanding the deontological nature of conceptual representations in consciousness means that Kant has introduced the social or normative element of conceptual use, rather than understanding those sorts of things as a second type of substance. The synthetic unity of apperception is a normative unity,

what Brandom calls a set of task-responsibilities to integrate a new commitment into a normative system of commitments and entitlements.

Hegel continued Kant's ontological critique by essentially arguing that whatever Kant meant by the thing-in-itself must be conceptually articulated. That is, we can know reality, not just our representations of reality, because that reality is conceptually articulated. This claim can easily be misunderstood as presupposing a kind of supernatural, metaphysical, animistic entity. This misunderstanding is propagated by interpreting "conceptual" as essentially "psychological." But Brandom offers an alternative conceptualization of the conceptual, understanding "Hegel's nonpsychological conception of the conceptual as what is articulated by relations of material incompatibility and consequence" (Brandom, 2019). I discuss the terms material incompatibility and consequence below, but essentially we can understand the natural world as conceptual in the sense that we can say things like "that is a dog so it is not a cat" and "that is a dog so it is a mammal" and mean precisely what we say. These words are not inscribed on being, so it is impossible to explicate these relationships of material incompatibility and consequence without notions like vocabulary.

In order to explicate these relationships, Brandom introduces the idea of bi-modal hylomorphic conceptual realism. This is essentially the claim that Kant's deontological conceptualization of the conceptual can describe what it means to articulate objective modal relations. That is, the objective world can be described through the normative world. More precisely, the deontic normative vocabulary of {commitments, entitlements, authority, responsibility}, the essentially social vocabulary, can describe what one must do in order to deploy the alethic modal vocabulary of {possibility, impossibility, necessity} which is the language of objectivity. Material inferences, which I discuss more thoroughly below, have the quality of being explicable in both the deontic normative modality and the alethic modality.

We are accustomed to thinking about mathematics as purely articulated in the alethic modality, cutting the deontic normative modality out completely. In the alethic modality, knowledge is essentially externalized in the sense that when we say things like "it is impossible for 5 to be an even number," the identity of the person who utters such a statement is not caught up in that statement. The claim in the alethic modality is essentially that any subject would agree that this is the case, insofar as they understand what the terms involved refer to. This is what we mean by "objective," and it is related to picture thinking. The actual process that goes into thinking our way to committing to the assertion that "5 is an even number" has a much finer texture that we often ignore that depends on identity formation.

For instance, we must determine whether or not such a sentence is even a thought. Is it so much as thinkable, or is it merely wind whistling through our lips? That is, is it well-formed? Moreover, we often consider whether or not stating such a thing would be a shameful speech action. Would uttering this threaten my status as a normative subject? Would my identity as a good person be threatened by such an utterance? This feels silly when considering a 'true' statement, but if I were to say in all sincerity that "2 + 2 =3," people could very well take me as an inferior sort of subject. The cliché "2 + 2 = 4" is as much a statement that someone is playing by the rules, is a normative subject, or is not a jerk driven by their own naked solipsism – someone to be trusted – as it is a statement of alethic modal fact. In uttering "2 + 2 = 3," the risk is that others might take the person speaking as someone who does not play by the rules, someone who is not responsible to the norms at play, and so not a normative subject. How we treat people who are not taken as normative subjects is often abhorrent (i.e. the solitary confinement of people who are imprisoned, racism, ableism, cisgenderism, sexism, etc.), and so there is a kind of fear of shame at play when judgment moves towards commitment. In acting and judging, our status as normative subjects is always at risk, but we also have an existential need to be recognized as normative subjects. Fulfilling the need to be recognized as a normative subject feels good, so this should not be interpreted as strictly saying that we are motivated solely by fear of shame.

The other axis on which identity turns in a general sense, besides the existential need to be taken as normative subjects as noted earlier in this essay, is the need to be recognized as infinite. We are all totally singular within the {I, here, this, now}, and yet united in that commonality. This is a tricky notion, because the assumption is often that when we appeal to the infinite nature of people there is some concomitant commitment to a metaphysical substance, a kind of 'soul stuff.' This is not at all what I am trying to get at. In order to help make the distinction, I turn back towards a more familiar way to understand the unity of being and knowing which is human identity.

In the context of learning how to formulate a good "elevator speech" when going on the academic job market, I was once asked to pick five universals that would intersect to determine who I am as a unique individual. Readers might take a moment to attempt this. It is an impossible task not simply because five universals are insufficient to disambiguate one subject from all others but also because it in no way exhausts the identity claims one might make. I could have said "I am a musician, I am interested in rehabilitating students' relationship with mathematics, I am a native to Indiana, I am churlish when hungry, and I am a dog-lover." Equally, I might have said "I am not from England, I am not a weaver, etc." There is nothing exhaustible about what I am, as evidenced by the plain infinity of all that I am not.

I am deliberately choosing facile identity claims here but the more interesting claims relate to our gender, sexual, racial, academic, and political identities because it generally feels much worse

when one's self-validity is threatened by local incompatibilities produced along those axes of identity. The reason I am choosing facile identity claims is because the interesting ones are complicated by notions of whether or not one can take responsibility for one's identity along those axes. For example, being white is an important part of how others perceive me and so an important part of my being, but I do not really have a choice in being white, only the choice in making that identity explicit. That being said, it feels bad to have incoherencies in facile identity claims as well. The task of picking out just five identity claims entails the suppression of the countless others, something that I hope readers who attempted this exercise also felt.

These are identity claims that are how other people perceive me. We are essentially social creatures, so how we identify is part of our being. "In Hegel's terms, what a self-consciousness is in itself…depends on both what it is for itself and what it is for others" (Brandom, 2019, p. 23). That is, when I am recognized by another as a dog-lover or whatever, the way in which that other describes me is an element of how they are perceiving being. We recognize each other through identity claims like how we recognize a rose by its thorns. Moreover, what feels like discovery when another says something that is an identity claim that had only been implicit for oneself is really a moment of self-actualization, where one becomes more through the explication of implicit identity claims. For example, imagine that my identity claims were suppressed to the extent that I only acknowledged the original five claims. If someone were to point out some of the things I was not, the self that is the finite normative identity claims could expand infinitely. Generally, while one is implicitly certain of one's infinite nature, the existential need is to be recognized as such can go unfulfilled for long periods of time. Consequently, it can feel very satisfying when another explicitly recognizes oneself as infinite. I take it that most educators are familiar with this kind of recognition, but it is often distorted by being understood as the potential for a student to become more than they currently are, rather than the recognition of their current self as infinite. This is distorting because it produces the feeling of inadequacy as it relates to potential, a kind of withholding of recognition until such time as one is worthy of it, instead of referring to the extant status of the individual as infinite. I argue for this in the section on the second moment of the judgment.

To return to the "2 + 2 = 3" moment, students might ask, "will this make me seem like a know-it-all?" Or "do my peers really not understand that 2 +2 = 4, or are they withholding their judgments because it is uncool to be an active participant in a math class or because trolling a teacher by insisting that 2 +2 = 3 is the height of good fun?" Or "are my peers withholding the fact that 2 + 2 = 4 because they are protesting how the teacher/school/state instrumentalizes everything about them and so are refusing to participate?" When we flatten all of this reasoning into the alethic modality, the notion that mathematics content itself is wrapped up in self-formative processes and

identity development is almost absurd. We boil it down to something like "well, everyone should agree that 'it is impossible that 5 is an even number.'" The point of the present paper is not necessarily to get into the possible causes and remediations of shame in mathematics classrooms. Instead, the purpose is to articulate a mathematics that is close to first-person experiences of mathematical judgment to articulate the full inferential movement of mathematical judgment, not just the flattened result which is the mathematical utterance.

This all presumably feels a little loose for most mathematical audiences, who probably tune into the idea of contradiction and assume it is of the explosive variety that allows anything to follow from anything else, but as Priest (2014) argues, not all contradictions are explosive. This is just the familiar one, discovered through reflection, that a "me" exists as a social being who makes identity claims and is recognized accordingly but the {I} also exists. This I does not exist as an object, is not bounded and instrumentalized, but moves beyond any boundaries that are put up to attempt to objectify it.

To orient readers to the structure of this paper, I have just established the sense in which I am using the term "critical," which is the assertion that being is knowing and knowing is being. Next, I shall describe judgment. Judgment is essentially the smallest unit of thought or language that we can take responsibility for (Brandom, 2000a), and so is the smallest unit of thought that we can understand ourselves as liberated within. One of the chambers of the beating heart of German idealism is that freedom is responsibility, not expressions of independence from authority. The various kinds of mental phenomena that are part of the background noise of experience are not at issue in the present understanding of mathematics I am trying to explicate. A threefold structure for judgment is presented that allows for judgment to stand apart from those fleeting phenomena that can arise that we cannot take responsibility for. This interpretation of judgment allows for a rational recollection of the number 2 in the vocabulary of object collections.

I follow this impulse from object collection to operation. In order to accomplish this, I use the framework of analytic pragmatism (Brandom, 2008). Analytic pragmatism allows for rational recollections to precisely understand how basic mathematical vocabularies, like talk about object collections, can be elaborated into more sophisticated vocabularies like four function arithmetic. I do not perform this elaboration to its conclusion but just begin the process by retelling a section of the story that Lakoff and Núñez (2000) tell in pragmatist rather than cognitivist terms. The last piece of theory that I produce is a story of how certain notions like linearity that are often considered "laws" or "properties" can instead be understood as algorithms. The purpose in all of this is to internalize what is often considered external. We might follow laws, but we are not "law." To say as much would be to deny our existential need to be taken as infinite and reduce action to behavior and preclude the possibility of emancipation. Instead, we are being/knowing, and so must

find ways to express what are commonly related as mathematical laws as internal to the first-person actor's experience.

Language

Because one of the primary goals of this project is to explain how mathematics can arise out of language use, rather than necessarily existing in a formal ontological realm, a fair amount of detail is required to tell a convincing story. Below I discuss different types of inference, material incompatibility, material consequence, substitution, and anaphora briefly. I also describe Brandom's (2008) analytic pragmatism.

Material Inference, Incompatibility, and Consequence

Before we can discern arithmetic structures, we must discuss the role of material inferences in Brandom's inferentialism. Material inferences are normatively regulated pre-logical or pre-formal inferences. Rather than relying on either form of inference, like modus ponens, or definitions of the terms involved, the idea is that material inferences are those whose "correctnesses determine the conceptual contents of (their) premises and conclusions" (Brandom, 2000, p. 52). For example, in the word "Bright," a possible material inference is "B is to the left of T, so T is to the right of B." The correctness of this inference, in the sense of its status as a shared commitment by a community of interlocutors (its status as a norm), lends content to the terms "left" and "right." Defining "left" without invoking its material relationships with "right" is certainly possible, as in "designating that side of the human body which is to the west when a person is facing north" (Dictionary, 2019), but such definitions expand conceptual horizons rather than grounding them (Carspecken, 2015).

Material inferences can be expressed in either the deontic normative or the alethic modality. For example, the above inference could be inscribed in the deontic normative vocabulary as "Commitment to 'B is to the left of T', entitles 'T is to the right of B'." It could also be inscribed in the alethic modality as "B is to the left of T so T is necessarily to the right of B." In this paper, I am interested in just the deontic normative modality because the alethic mode is well-trodden by traditional mathematics, as it is essentially the modality of proof. That is, if I wanted to prove that 6 was an even number, the modality I would be invoking is alethic. It is necessarily the case that 6 is an even number because it is divisible by 2, or it is impossible that 6 is odd. The deontic-normative expression would be more along the lines of it is proper to infer from the divisibility of 6 by 2 that 6 is even.

Another key term to explicate before counting can occur is material incompatibility. With the language of commitments and entitlements, Brandom introduces incompatibility, stating, "to treat claims as materially incompatible is to take commitment to one to preclude entitlement to the

other" (2008, XV). Material incompatibility does not function like formal negation but is instead along the lines of Aristotelian contrariety: "**Square** and **circular** are exclusively different properties, because possession by a plane figure of the one excludes, rules out, or is materially incompatible with possession of the other. **Square** and **green** are merely or indifferently different, in that though they are distinct properties, possession of the one does not preclude possession of the other" (Brandom, 2019, p. 56). The last material relationship we need is material consequence, or incompatibility entailment, which is also pre-logical and is defined as "p incompatibility-entails q just in case everything incompatible with q is incompatible with p" (Brandom, 2008, p. 121). By pre-logical, material incompatibility and consequence produce what Brandom calls the "nonpsychological understanding of conceptual contentfulness" (2019, p. 539).

Analytic Pragmatism

Analytic pragmatism is explicated in Brandom's (2008) work Between Saying and Doing, which he describes as orthogonal to the work he was doing in inferentialism (i.e. Brandom, 1994, 2000a). It deals with language use at the syntactic and semantic level, and so can do some of the fine-grain work necessary in articulating a mathematics. The basic idea is that language comes to mean what it means by way of how it is used. This is pragmatism. What makes analytic pragmatism unique is the division of language into vocabularies, which are essentially sets of words in a specific language like English, and practices-or-abilities, which are the repeatable algorithmic expressions that govern the use of a specific vocabulary. The key theme Brandom explores is whether or not the deontic-normative vocabulary of commitments and entitlements is a pragmatic metavocabulary for alethic modal vocabulary of necessity, and possibility. This allows for a kind of grounding of alethic modality in the normative. In the absence of such a ground there is a risk that my treatment of analytic pragmatism could result in stacks of vocabularies, practices-or-abilities, and pragmatic metavocabularies 'all the way down' (i.e. an infinite regress). I am going to ground my use of analytic pragmatism in judgment, but otherwise leave the notion of vocabularies and practices-or-abilities open to many other varieties besides the modal logical vocabularies and practices-or-abilities. Brandom's purposes are not subject to the same possible problem because he is dealing with modal logical vocabularies, while I am dealing with the vocabularies of everyday mathematical reasoning.

In prior work (Savich, Jacobson, Bharaj, & Eker, 2019), colleagues and I related some elements of analytic pragmatism for a mathematics education audience. Brandom's (2008) goal is essentially philosophical, concerned with specific modal categories of logical vocabularies. My purpose here is more in using analytic pragmatism to rationally recollect how relatively sophisticated mathematical vocabularies, like arithmetic, can be produced through algorithmic elaboration from relatively basic vocabularies like object collections. To torture the term a bit, this

is like doing mathematical history, in the sense that we can describe how epistemologically earlier vocabularies come to be transformed into new vocabularies.

Deontic-normative vocabularies articulate both subjective experience and normative inferential proprieties (Brandom, 2008; 2019). The deep principle here is that subjectivity is essentially, not just accidentally, intersubjective. The important elements of this vocabulary for the present study are {commitment, entitlement, incompatibility}, but one might also add terms like {believes, feels, perceives, should, ought}. The second part is related to the notion of objectivity and is articulated in an alethic modal vocabulary, using terms such as {necessity, possibility, incompatibility}. I shall attempt to use these terms in a relatively regimented manner in order to use this work as a building block for future explications of the relations between these two modalities (Brandom, 2019, p. 650). But the point of doing mathematics in the deontic normative modality is to express a mathematics that is less rigid than axiomatic mathematics, so it is important that the rules articulated below are understood as inferential proprieties, not simple statements of objective fact. Different normative (i.e. cultural) horizons both limit and enable the production of different expressions that can be understood as material inferences.

Brandom's (2008) thesis is that what one must do in order to count as saying something in the alethic modality can be expressed in the deontic normative vocabulary. That is, the deontic normative vocabulary is a pragmatic metavocabulary for the alethic modal vocabulary and is thus pragmatically prior to the alethic modality in terms of an order of explanation. In this way we can articulate objective 'facts' as being instituted through the deontic-normative vocabulary and its practices-or-abilities.

The other principal use I put analytic pragmatism to in this paper is the idea that the practices-or-abilities that govern how a particular vocabulary is to be deployed can be algorithmically elaborated into a different set of practices-or-abilities that then come to determine how a different vocabulary is deployed. That is, we can rationally reconstruct how one vocabulary is transformed into another vocabulary through the notion of algorithmic elaboration. This is a way to inject a very minimal concept of history into a purely mathematical discourse. Through the concept of algorithmic elaboration, we can track how talk about object collections can be transformed into talk about numbers, which can then be algorithmically elaborated into basic arithmetic and so on.

Substitution

The following discussion of substitution is derived from chapter 4 of (Brandom, 2000a). We begin the discussion of substitution with the controversial claim (Wolf, 2019) that assertions, or Kantian judgments, are the fundamental unit of awareness to which Frege would say pragmatic force can attach and that Wittgenstein would claim thereby count as making a move in a language game

(Brandom, 2000, pp. 124-125). Assertions can be decomposed using Fregean substitution into elements that have symmetric substitution licenses (singular terms) and asymmetric substitution licenses (predicates). Singular terms are terms that "purport to refer to just one object" (Brandom, 2000, p. 124), but invoking the notion of "object" in an explication of the act of counting merely replicates dogma. The dogma undermined in this paper is the pre-Kantian ontology of separate substances, the objective side being the sort of thing that we are incapable of taking responsibility for and the subjective side the side of consciousness. The symmetry of substitutions that singular terms are caught up in allows for "objects" to be discussed within a philosophy of language.

In the sentences α: "Isaac Newton wrote about physics" and β: "The author of the Principia wrote about optics," the term "Isaac Newton" can be substituted with "The author of the Principia" in any circumstance where interlocutors endorse the Simple Material Substitution Inferential Commitment (SMSIC) that asserts the intersubstitionality of "Isaac Newton" with "the author of the Principia." That is, commitment to "Isaac Newton wrote about physics" and the SMSIC just mentioned has as a material consequence the assertion γ: "The author of the Principia wrote about physics." It is not necessary to actually utter γ, it is just that this sentence inherits its propriety from the prior sentence.

Substitution allows for assertions to be carved up into three structural roles. Terms can be substituted-in, as how "The author of the Principia" is substituted-in α to form γ. Terms can be substituted-for, as how in α "Isaac Newton" is substituted-for "The author of the Principia" to form γ. Lastly, and most importantly for the purposes of this paper, substitution allows for the discernment of a sentence frame, a kind of substitutional remainder or byproduct of the substitution process. In α, the sentence frame that singular term substitution allows one to discern is " wrote about physics." The sentence frame itself does not have pragmatic potential, and so I shall avoid evoking frames without singular terms.

In allowing the substitution machinery to run wide open, individuals come to discern two fundamental classes of terms: those that produce symmetric substitution licenses and those that produce asymmetric substitution licenses. The terms {Isaac Newton, the author of the Principia} form an equivalence class of singular terms that are symmetrically intersubstitutable with one another. The predicates {wrote about physics, wrote about optics} are involved in asymmetric substitution inferences. That is, I can discern individuals who have written about physics but who have not written about optics, but I cannot meaningfully discern individuals who wrote about optics without writing about physics. Optics is a subcategory of physics. Predicates/universals/concepts are structured hierarchically, where one predicate is inferentially stronger than the other and so implies the other when it is deployed.

Brandom's (2000) thesis is that any language that makes use of conditionals will have some terms that are only involved in symmetric substitution inferences (singular terms) and other terms that are necessarily involved in some asymmetric substitution inferences (predicates). In the everyday account of mathematics, I am providing here, we can substitute object for singular term and concept/universal for predicate, and thus acquire a grounding for those thorny terms based solely on their roles in substitution. This allows us to stay within the domain of language when I turn later to object collections. This might seem like pure technicality, but it is important to recall that freedom is responsibility. We can be responsible for how we deploy language in ways we cannot be responsible for the object domain in the pre-Kantian ontology. In a deep sense, Brandom's discussion of singular terms is emancipatory, at least insofar as one buys into the identity between responsibility and freedom.

Anaphora

When I utter a sentence like "That pan is hot," the demonstrative "that" seems to unambiguously point out what pan I am talking about. I could use a deictic gesture and be assured that whoever was in the room with me would understand what pan I was referring to. But I could also go on to say, "That pan is Teflon," "That pan is cool," and "That pan is cast-iron." The demonstrative "that" is not actually fixing a referent on its own and can be used to produce incompatible assertions. Similarly, indexicals like "now" feel unambiguous: the "now" is this very moment. However, as Hegel discusses, "now is night" loses its truth when uttered during the day but regains that truth at some later now. What actually fixes the content of demonstratives and indexicals is their ability to be picked up by anaphoric terms. Anaphoric terms are most commonly understood as pronouns like {it, he, she, they, we}, though I will expand that usage to include numbers in the context of object collections later in this paper.

For example, in uttering "That pan is hot," a next possible utterance is "it has been in the broiler drawer." The term "it" is a co-referent with "that pan," pointing towards the very pan that is hot and has been in the broiler drawer. Brandom writes "the use of indexical and demonstrative vocabulary presupposes the use of anaphoric vocabulary. An utterance qualifies as cognitively significant and semantically contentful only if it can serve as a premise in inferences. For that reason, securing reference requires making possible non-accidental co-reference" (Brandom, 2008, p. 59). Without the ability to refer to one thing in different ways, it is impossible to know what thing is being referred to.

When I say, "now is night," I can refer anaphorically to the now in a next possible expression "it is far past my bedtime." The "now" is fixed through co-reference via the anaphoric "it." What allows for the apparent immediacy in demonstrative and indexical assertions to be used as premises

in inference that one might conclude something from, remember, or use again "is the possibility of picking up that content and making it repeatable, by treating it as initiating an anaphoric chain: 'This chalk is white. It is also cylindrical, and if it were to be rubbed on the board, it would make a mark'…The chain 'This chalk' … 'It' … 'it' … 'it' is a repeatability structure that makes the content of the original demonstration repeatably available" (Brandom, 2019, p. 129). In the context of well-formedness, anaphoric terms are co-referents with singular terms but generally fall under different predicates. That is, "That pan is hot, it is hot" is well-formed but the second phrase does not contribute additional content. "That pan is hot, that pan it" is not well-formed. I turn next to how we can understand those judgments in terms of repeatability structures.

The Judgment

There are many possible accounts of judgment. In the one I put forward here, I am focusing on the inner dynamics between the two basic existential needs articulated above: the need to be recognized as a normative subject and the need to be recognized as infinite. In this story about judgment, the existential needs are self-propagating, like how electromagnetic radiation consists of a magnetic field that produces an orthogonal electric field, which produces another magnetic field orthogonal to the electric field and so on. The need to be recognized as a normative subject induces the condition of needing to be recognized as infinite, which induces the recollective act with reference to the need to be recognized as a normative subject. It will be helpful to recall that the {I} in both cases is the negative, which I will express here as difference. Rödl (2018) describes the threefold character of judgment to consist of power, power-act, act. By power Rödl means "the concept, or nature, of something, considered as explaining it. A nature, or concept explains what bears this nature, or realizes this concept, therein explaining the latter's conformity to its nature, or concept" (2018; p. 96). I borrow this structure, though much else of what Rödl says is absent.

First Moment

Take the perceptual judgment "That pan is hot." This judgment was produced after an experience of error, articulated above, where there was a difference between appearance and reality. The {I} here is the empirical consciousness that responds to particular circumstances, presupposing some interest, desire, or intention. In this case, my friend uttered "that pan is hot" with the intention of communicating why they did not pick it up and to communicate that it is dangerous. From the initial distinction between appearance and reality, consciousness must produce a well-formed assertion. "That pan that pan" simply will not do. The criteria for well-formedness includes the ability to substitutionally discern particular (singular term) from universal (predicate). An additional criterion for the first power is that the judger must have the ability to refer to singular terms anaphorically in next possible judgments. That is, the judger must be able to say something

like "That pan is hot, it has been in the broiler." I take the first power to be the power to produce well-formed assertions with respect to the difference between appearance and reality with the additional ability to refer to the singular term anaphorically.

Second Moment

I take it that the second moment of power-act is the power to claim universal positions on the content of that well-formed assertion by integrating it into a system of commitments. This is the power to bind particular to universal and understand oneself as a normative subject and so be taken as responsible to the norms that form the recognitive communities of which one is an element. The act part of the power-act can be understood in terms of synthesizing the assertion into a constellation of commitments that follows the norms of systematicity (Brandom, 2019). Systematicity is understood as a set of responsibilities:

- One's critical integrative-synthetic task responsibility is to reject commitments that are materially incompatible with other commitments one has acknowledged.
- One's ampliative integrative-synthetic task responsibility is to acknowledge commitments that are material consequences of other commitments one has acknowledged.
- One's justificatory integrative-synthetic task responsibility is to be able to provide reasons for the commitments one has acknowledged, by citing other commitments one acknowledges of which they are material consequences. (Brandom, 2019, p. 69)

The act of the power-act is actually attending to these integrative responsibilities by integrating the predicate of the assertion as a material consequence of some more general predicate. This is the moment of claimed universality. Here, the {I} is a collection of commitments. The intrusion of a new particular bound to a universal, referred by the first power, perturbs the collection of universals. This is because binding a particular to a universal changes the universal it is bound to. If I were to say, "This banana is overripe," the particular banana is added to the concept of overripe. The concept of overripe exists as a particular to some other universal like ripeness, which could change a universal like things-that-are-good-to-eat. Every particular instantiation lends content to the property under which it falls in judgment, which in turn perturbs the whole system of conceptual contents. The relevant constellation of commitments is interconnected. The whole system of conceptual contents is here the synthetic unity of apperception. It is the {I} as a creature of norms, whose existential need is to be recognized as a good person. Because this {I} is sensitive to the perturbation induced by its own integrative activity, the intrusion of particulars, it knows itself to be incomplete. This self-consciousness of its own incompleteness is explicated as the transcendental unity of apperception, or the {I} as infinite.

Said another way, in binding particular to universal, the constellation of commitments that constitute the normative self-changes. That is, if I bind "that pan" to the universal "hot," the universal "hot" which had previously not included the particular pan now has new content. This changes other commitments. With the pan now categorized as "hot" it likewise falls under the universal "dangerous." In changing the content of the universal "dangerous," I likewise alter the content of related concepts like "safe" or "matte metal pans" to now be "potentially dangerous." Every time a particular is bound to a new universal, the texture of whatever commitments are integrated with that universal are likewise altered.

Consequently, the identity of the self as a normative subject is altered with every judgment that is not already a recollection of one's prior commitments. That is, every time I say "that pan is hot" in the future my identity is not altered, but my identity in this first judging was changed. So far, what has changed is my normative identity for-myself. It is the interior facing aspect of identity that is moved in this moment of the judgment. I have diagrammed this progression in Figure 1. This moment where an assertion is integrated into the normative self is, and feels like, an expansion of self. The general desire/intention/anticipation that judgment begins with is satisfied in the integrative moment. That is, the successful integration is a moment of recognition of the telos of the anticipation. The empirical self-experiences the ending of the second moment of judgment as the recognition of being in ways that cannot be expressed in strictly conceptual terms. When the conceptual self is altered in the second moment, and recollected in the third, the movement from treating the present as a past allows for the feeling that things are necessarily so to be in reference to some past rather than the tremulous or abstract present. The whole body experiences a feeling that is being/knowing, the I-feeling (Carspecken, 1999). We might describe the recognition of being as a proprioceptive expansion of the {I} that accompanies the movement towards explicitness. It is somewhat hard to describe this, not only because it is a feeling (the feeling of being and knowing), but also because of an inescapable temporal gap. The feeling of being and knowing is always as a just-was, and so must be quite beyond description. Description is an element within that process. That being said, I take it that we have all experienced the feeling of understanding. Sometimes this experience of understanding is the conceptual "click," which is a kind of ephemeral expansion of the I-feeling. I say this because the "click" is something that I associate more with the resolution of conceptual incompatibilities which can feel a bit like a puzzle piece fitting into place that is external to oneself. In this, the mathematics in question can still be externalized as a game, rather than a facet of being that is identical with self. Instead, the feeling I think we are after as educators is the feeling of understanding with certainty, the this-is-so. The actual certainty is not in the content that seems to "click" into place but is rather certainty of one's self as infinite. Because this feeling is associated with universal indexicality, the global

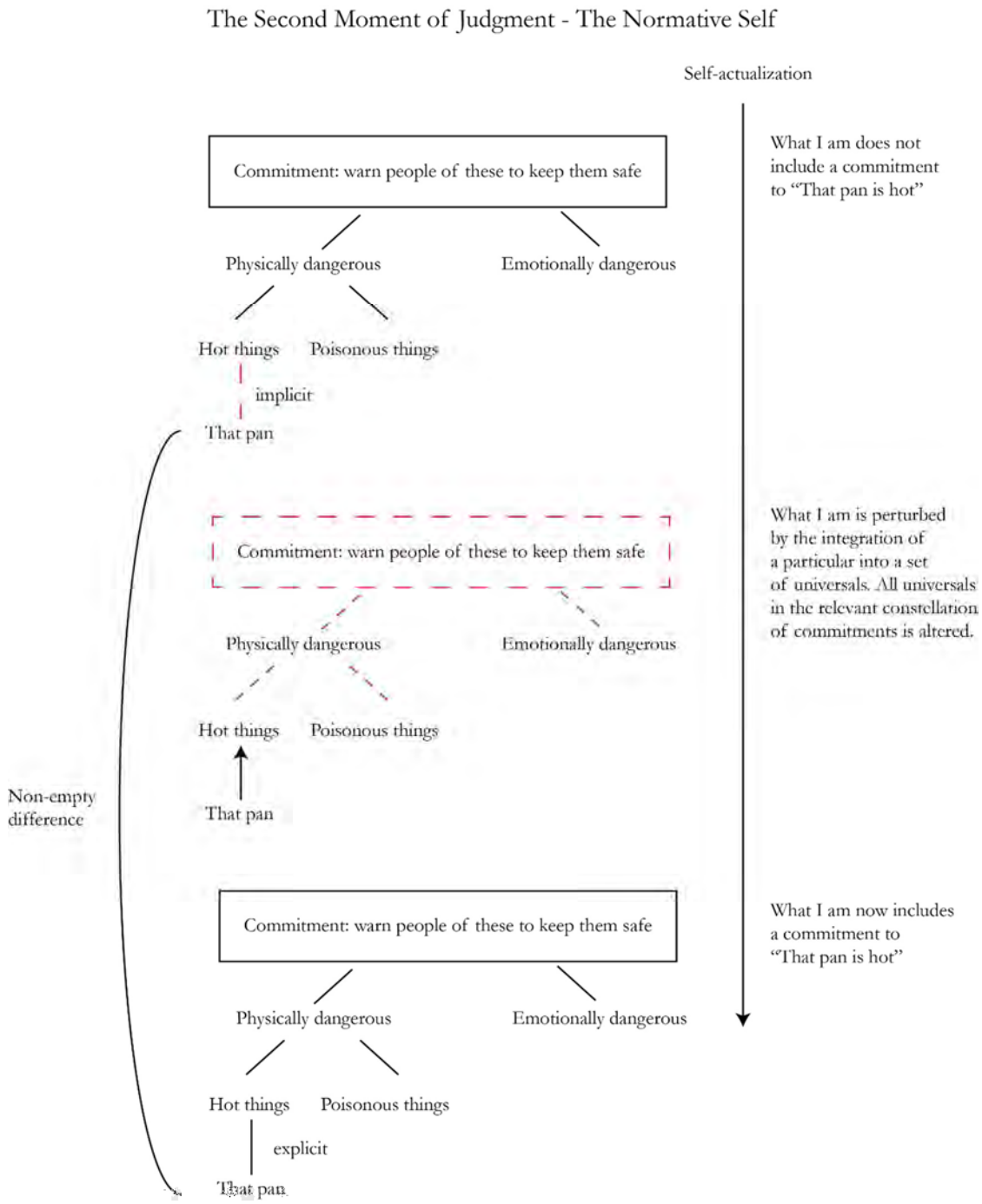

Figure 1. More of one's being is actualized through the explication of implicit knowledge. The integration of a new particular changes the universals in the relevant constellation of commitments. This diagram is only partially successful because it only includes the self as a collection of commitments and does not include the self-consciousness of that self which is the thought of its own incompleteness.

isomorphism of I, here, this and now (the negative), it is not extendible into all possible future judgments. It exists purely within the now. Consequently, as we have new experiences and fuse new constellations of commitments with old ones, new horizons with our given horizons, this feeling only lasts as long as the experience lasts. I tend to feel its evaporation as soon as the commitment is recollected, vocalized, or when pen leaves paper, though sometimes the excitement that arises lingers for quite a while. Certainty in the incompleteness of a constellation of commitments demands that any such experience of certainty that rests on judgment with contrary — as opposed to the absolute knowledge of philosophy (Rödl, 2018) — and so is consequently impermanent. That does not mean that it is somehow inconsequential. Rather, it drives the whole process of knowing, when such knowing is understood in the absence of pure instrumentalism. That is, we might answer "why do I need to learn this stuff" with "because it will help you get a job," "develop cleaner burning jet fuel" or whatever. But we might also answer that same question with "to understand." To understand oneself, each other, and the world in one moment. This is communicative action.

In altering my identity, how might my identity change for-others? There is now a difference between how I understand and recognize myself and how others understand and recognize me because what I am for-myself has been invisibly altered. The conditions to fulfill the existential need to be recognized as infinite (i.e. one's infinity) can be referred to as an actuality because the difference between what the normative subject as a constellation of commitments is for-itself and what it is for-others is non-empty simply by virtue of the activity of judging. This feels like a self-affirmation of one's certainty as infinite, because of the non-emptiness of the difference between how one now recognizes oneself and how one is recognized by another. This does not fulfill the need for recognition aspect; rather, it just necessitates the next moment in which recognition can occur.

Third Moment

There is now a difference between what consciousness is for-itself as an altered collection of commitments and what consciousness is for-another as an unaltered collection of commitments. In the third moment of judgment, I orient myself to next possible actions that could satisfy the need to be taken as a normative subject by articulating the assertion that produced the alteration in commitments. In doing so, I take the attitude of the other and consider the judgment from their perspective and thereby fix the assertion from the first moment by recollecting it from within this anticipating moment. What this produces, is a recollection of the first moment, the assertion "that pan is hot," with respect to some next possible action. The next possible action could be a simple doing like putting on a potholder, or it could be the actual utterance of the assertion in anticipation of another person grasping the pan with their unprotected hand. In either case, what is referred to

is the need to be taken as a normative subject, someone who lives up to their commitments in their now altered form. Perhaps I am cooking with that same friend and I know they know that the pan could be hot and so I suppress the utterance out of respect for their cognizance.

Said another way, the last act of judgment is the activity of the transcendental unity of apperception that fixes the content of the judgment. It does so in anticipation of the next moment of experience. This is the general function of anaphora but is also recollection. It is odd to think of recollection within judgment, but the self-consciousness of the infinite {I} is its temporal incompleteness. When the {I} anticipates the next possible act, the {now} in the previous act becomes a past with respect to some future. There is always some future that judgments point to, so the activity of recollection, within a moment of the now, is in anticipation of some future.

In shifting temporal perspectives to be oriented to the future, the self-certainty of oneself as infinite is projected from self-certainty to certainty of the recollected assertion. It is this moment that produces the feeling of this-is-so. This moment alienates the self-certainty, transforming it into a certainty of the content of one's judgment, rather than the certainty of oneself as infinite. In doing so, the certainty of oneself as infinite – which is what we are actually certain of – is transformed into a knowledge claim which is contingent.

For example, suppose Subject A says, "This banana is overripe"; in so saying they arrest the normative self in anticipation of the next act, which might be something like a disgusted "so I will not eat it." When this happens in dialog, the actual utterance "This banana is overripe" contains within it the {I} in both understandings in the sense that it licenses the other participant to consider what one is saying about the self as a collection of commitments or engage with next act. That is, Subject B could respond "that banana is not overripe, it is perfect, you are a fussy eater." In this case, they are challenging whether or not the particular falls under the universal, perhaps only engaging with the anticipated "so I will not eat it" by implicitly communicating "you should eat it." If Subject A had internalized Subject B's judgment of their moral character as deficient, the next possible move would be different. For example, the act of fixing the content in recollection, and the self-consciousness of that act in anticipation of some future act could result in Subject A suppressing their actual speech and just sit there eating the old banana. If Subject A were instead in dialog with Subject C, who has been internalized as someone more sympathetic to Subject A's disgust, Subject A might vocalize the original judgment in anticipation that Subject C would respond with "oh, gross, I really dislike old bananas too, you should throw it away." There is more to be said here about how Subjects B and C become internalized as a generalized other for Subject A but that is a story for another time.

I include this not only to explain how judgments fix their contents through recollection, but also to partially explain the ringing silence in many mathematics classrooms as indications of internalized expressions of moral deficiency. That is, the self-consciousness of the last act of judgment which is the anticipation of the next act is often suppressive in mathematics classrooms. Compliant students might choke down the slimy old banana of mathematics they do not recognize themselves within. Non-compliant students might boycott participation altogether. Students who recognize themselves within the mathematical content are presumably unsure what the fuss is about a perfectly fine banana.

In mathematics, the next possible act is often written in through anthropomorphizing the symbol set, which can really be quite confusing. For example, teachers will often say things with regards to an expression like "$7x = 14$" that the x's want to be by themselves. This does not make sense. Symbols do not have desires in any meaningful sense. We, the ones who utter such judgments, do have particular desires, but those desires are with respect to some context. The naked assertion "$7x = 14$" does not have any obvious context. If we encountered it on a worksheet or something, our training might now anticipate that x should be solved for, and so we might go ahead and render the necessary judgments. That is, I might anticipate the result "x = ?" Because I know how to do some algebraic manipulation, I can produce a next well-formed sentence "$x = 2$." This can be integrated into my system of commitments, and I have brought the implicit into the explicit within my normative identity. This produces a moment of understanding, which can be felt as a kind of "click" when anticipation has been satisfied. This is the moment of proprioceptive expansion. But it fades quickly upon recollection in the third moment of the judgment, where self-certainty as infinite is transformed into certainty-of "$x = 2$." When I actually write the answer down, I am no longer experiencing self-certainty, but am instead feeling certain-of some fact, which I am now communicating to others so that I might be recognized as a normative subject who performs their duties. The need to be recognized as infinite is not expressed, which produces a dissatisfaction in this mode of discourse that can be satisfied by moving horizontally in this domain by doing more problems or whatever. This seems to be what teachers want in general, because the other activity that students engage in seeks affirmation of their infinite nature through apparently unproductive behaviors. An individual might know perfectly well that $x = 2$ but suppress that assertion because it does not satisfy their need to be recognized as infinite. Alternatively, students might become truculent about other matters in the absence of recognition.

Of course, students might also never reach the moment where an assertion is successfully integrated as an identity claim and so never experience the proprioceptive feeling of identity expansion. This is really very common in mathematics classrooms. Student might have no idea that they are even supposed to be treating mathematical assertions as commitments that form an

important part of their identity. If symbols are anthropomorphized, or if mathematics is reduced to the behavior of symbolic manipulation rather than commitment, there is basically no reason for the self to actually get involved. When we treat students instrumentally, they treat mathematics instrumentally, and there is no real learning where being/knowing is moved from implicitness to explicitness happening because there is no change in the student's normative identity. There may be a kind of machine learning where we learn to smoosh some symbols together, but this is learning without understanding

This has implications for how we understand the teaching and learning of mathematics. When the feeling of understanding that is the expansion of recognitive being/knowing is completely absent from discourses about what counts as mathematical knowing, we are dooming mathematics itself to pure vacuity. We are structurally insisting that the subject itself is pure alienation when we insist on stopping with the power-act so that judgments are just well-formed and integrated into a systematic constellation of commitments. Instead, we need the moment of identification, when concepts and being reciprocally recognize one another. This, I take it, is one of the reasons why people literally hate and fear mathematics. Mathematics is not them.

Mathematics

Von Neumann ordinals are a common and useful way to represent numbers and so I will use them as a normative standard of assessment for representations of number. Von Neumann ordinals are sets of the empty set. They require either the axiom of the empty set (for Kripke-Platek set theory) or at least the construction of the empty set, as is possible within the Zermelo-Frankel axioms for set theory. This construction amounts essentially to the definition of "the set of elements not equal to themselves." This is not mysterious when contemplated as a positive "thing in-itself"; it is merely impossible in the sense that our normative capacity to give and ask for reasons dictates that objects cannot hold incompatible properties (Brandom, 2008, p. 191). Consequently, the empty set is either an axiom or the set of impossible things. The idea is that one is the set that contains zero, two is the set that contains zero and one, three is the set that contains zero, one and two etc. This is summarized in Table 1. My point in the deduction that follows will be to discern von Neumann ordinals as implicit within a judgment that consists of singular terms and predicates.

Number	Successor representation	Representation as empty sets
0	{ }	∅
1	{0}	{∅}
2	{0, 1}	{∅, {∅}}
3	{0, 1, 2}	{∅, {∅}, {∅, {∅}}}

Table 1. von Neumann ordinals understood as sets of the empty set.

To do this, recall from the discussion of substitution that there are three structural roles, one of which is the sentence frame. Substitution allows us to explicate why some judgments are well-formed ("that pan is hot") and so thinkable, while others are not ("is hot is hot"). Singular terms are substitutable with singular terms to produce well-formed (if not necessarily correct) sentences. Predicates can be substituted with predicates to produces well-formed sentences (again, not necessarily correct sentences). This criterion for well-formedness allows us to discern a sentence frame like " is hot." I shall also introduce a convention that the symbols "" shall be intersubstitutable with {}. That is, when I am specifically engaged in the practice of recollection, of which quotation is a species, I will use {} to notate that recollection.

Mathematical Judgment

I take it that the first mathematical judgment involves Particular Instantiations (PINs; Priest, 2018) of properties/concepts/universals and has the form of something like:

PIN_A of property P is distinct from PIN_B of property P.

That is, the first judgment I would call mathematical involves taking distinct particulars as falling under the same universal. Rather than sticking with the generic language of "particular instantiations," I am going to switch to simpler language that feels more like a beginning. Indulge me in an assertion:

Pokey and Buddy are distinct dogs. (1)

Implicit in (1) is the judgment:

Pokey is a dog. (2)

Pokey is actually my friend who is a dog, so I judge (2) based on recognition that may not be shared by readers. I am assuming a universalized audience that could, in principle, agree with my assertion. In my lifeworld, "dog" is a kind of thing that people talk about and understand, and if I could print a picture or if you could come to my home you might find her here and agree: "ah, yes. Pokey is a dog." Moreover, I am assuming an audience that would recognize that (2) is well-formed and would take issue with a sentence like "is a dog is a dog."

Judgments, in general, are issued in the middle of things. A kind of first principle to understanding the mathematics I am trying to articulate is a presupposition of intersubjectivity. Intersubjectivity has a lot of interpretations, and so it is easy to misunderstand this. What I mean is that, in judging/asserting/thinking, an individual unavoidably assumes an audience that would agree with the judging/asserting/thinking. We presuppose a consensus as members of a "we," a recognitive community, that would agree with our judgment. In general, this consensus is implicit, part of our

lifeworld that churns in the background. It may be backgrounded in part because as soon as we start explicating it, we find that the strength of the intersubjective field, the assumed consensus, can be quite weak, but we take communicative actions to reach understanding and strengthen that assumed consensus. On the other hand, this assumed consensus can be so common and so strong that breaking out of it in order to explicate it (even if one fundamentally accepts its tenets) can be jarring. Sometimes this "we" is an idealized future recognitive community that we imagine when we say things like "no one understands what I'm talking about, but I'll eventually be understood." These moments are not generally "churning in the background."

Part of the what is assumed before a judgment is a whole set of next possible actions. I am judging (2) with the anticipation of drawing out a numerical structure within (2). My judgment is shaped by the purposes, intentions, or interests that may only be implicit in the speech act and may be pre-conscious. In the case of (2) I have the interest in emancipation as I explicate the judgment, and so this interest is not pre-conscious but rather is explicit. Other interests are backgrounded but can be drawn out through conversation. We often leave these interests implicit, but in doing so it becomes challenging to locate where subject is object in the speech act. What makes speech action, rather than mere wind whistling through one's lips or ink stains on a page, is that some subject must be doing the action. The first-person is implied in every utterance that is to be taken as a speech act. It is only through action that subject and object can be unified, and that unity is expressed in Habermas' (1971) terms as interest and Brandom's terms as the intentional nexus (i.e. Brandom, 2019).

Apperception, Judgment, and Self-Consciousness

After the notion that being and knowing are identical, the notion that thought/judgment/assertion is self-conscious is the most challenging Hegelian insight that I need in order to explicate a critical mathematics. I have already discussed apperception in a section above, arguing that the {I} is a unity of the synthetic and transcendental unities. Here we will take another look at the notion of apperception, distinguishing between three relevant interpretations.

The first is taking apperception to be an integrative task responsibility that normative subjects answer to in order to be taken as normative subjects which Brandom (2019) describes as systematicity. This is the interpretation of apperception I named as synthetic in an earlier section. In this interpretation, in order to successfully judge as in (2), one must attend to critical, justificatory, and ampliative integrative task responsibilities (Brandom, 2019). The critical responsibility is to judge without contradiction. So in judging (2) I am excluding the contrary statement "Pokey is not a dog." The ampliative responsibility is to integrate (2) with other commitments. That is, from (2) I must accept the material consequence that "Pokey is a mammal,"

insofar as the recognitive community I wish to be taken as a normative subject within acknowledges that material consequence. I also must attend to the justificatory responsibility which is the responsibility to provide reasons for my judgment. For example, "Pokey is a dog because she is a furry animal who barks and bites."

This leads to the second interpretation, which is apperception as the self-consciousness of judgment. When these criteria of systematicity are met, we are integrating a judgment like (2) into a constellation of commitments. In this case, the justificatory task responsibility is obviously underfunded when compared to the kinds of biological science that are relevant. In marking the inadequacy of this justification, we may take the last stage of judgment to be an acknowledgment of the incompleteness of the constellation of commitments as articulated. We might say "oh, but I haven't talked about genetics, domestication, canids…" This drives the movement of the judgment from less adequate forms (representations) to more adequate forms and essentially understands apperception as the self-consciousness of judgment. It is important to understand that the recognitive communities one has internalized as a "we" are what determine the adequacy of judgment from within the first-person experience of judgment. These internalized subject positions can be inadequate when faced with expanded recognitive communities, and so the "no" of another person contributes to the fundamental incompleteness of judgment. Any talk about the "conformity of a judgment to its inner measure of perfection" (Rödl, 2018) necessarily includes acknowledging the incompleteness of judgment that drives its movement. This is an element of the internal necessity of judgment that understands apperception more along the lines of judgment's self-consciousness. The self-consciousness of judgment is that judgment is identical with the reasons for why the judgment is so.

The theoretical thought of any content cannot be understood as the momentary or punctuated grasp of a solitary item. The thought of the content is also, is identical with, the thought of whatever reasons there are to delimit a concept in such a way and not some other, for example, the thought of discreteness in its contrast with continuity, or the thought of essence in its contrast with appearances. We are not thinking of discreteness if we cannot think of what such a notion excludes, presupposes, requires, if we have no idea how such discrete magnitudes could form a continuum" (Pippin, 2019, p. 109).

That is, in judging as (2), I am simultaneously asserting its well-formedness and my responsibility to sets of consequences articulated through the critical, ampliative, and justificatory responsibilities. To judge is to judge that this is so, and for such a claim to be taken up as though it were claimed by a normative subject means that the claim includes reasons for its own utterance. Rödl (2018) does not really talk about apperception, but I take it that his discussion of science and self-consciousness points in a similar direction to Brandom's notion of systematicity.

Last, apperception is negation, or rather perceiving with negation. When I observe in a perceptual field and pluck out an object like a counting cube, I am creating a unity in that object that is the negation of all otherness. Similarly, the {I} "enters into representational frameworks as the negative" (Carspecken, 2009), as the negation of all that is perceived, or all else that is. This exclusion of otherness is how unities are formed in general. To judge as I have in (2) is to exclude or negate all contrary judgments. There is still a fair amount of work to do in order to articulate a coherent notion of what, precisely, apperception is, but these three articulations can be taken in concert to understand what is entailed in judgment.

The I-think

One might notice that the {I} is doing a lot in the previous paragraphs. In fact, the "I think" is inseparable from any judgment. Consequently, the representational form of (2) is incomplete, and it may be better to mark it with an "I think," to remark on the activity of the {I} as essential in judgment. This could be done by simply adding "I think" to "Pokey is a dog." But the problem with this is twofold. First, its ubiquity makes it apparently vacuous so there are reasons to avoid typing it out. However, it is enormously significant and inescapable, so let us find some way to keep it in. Second, it does not separate the fact that any set of ink stains on a page can be taken as just that. Merely adding "I think" does nothing to dissuade from that interpretation.

Instead, we could follow Frege and put the judgment stroke (⊢) as a preamble to the judgment and produce (3):

⊢ Pokey is a dog. (3)

This is, in effect, saying that the "I think" can accompany any assertion, something that Kant noticed, and so is somehow empty in its ubiquity. For my purposes it is important to understand the "I think" as something internal to the judgment. Rödl (2018) writes about the problem of the judgment stroke saying "if our notation confuses us, suggesting as it does that I think is added to a p that is free from it, we may devise one that makes I think internal to p, we may form the letter p by writing, in the shape of p, the words I think" (p. 9). To think, assert, or judge, carries with it knowledge of itself. Thought is self-conscious in the sense that thought knows what it is to think. It does not know intrinsically about brains or the details of pragmatism modeled after a language game, but it carries within it its own criteria of adequacy – it knows what it means to be a thought simply by virtue of being thought. I shall have to adjust (3).

Recall from the discussion of substitution that there are three structural roles, one of which is the sentence frame. Substitution allows us to explicate why (2) is well-formed, while "is a dog is a dog" and "Is a dog Pokey" are not. Singular terms are substitutable with singular terms to produce well-formed (if not necessarily correct) sentences. Predicates can be substituted with predicates to

produces well-formed sentences (again, not necessarily correct sentences). In particular, substitution allows us to discern a sentence frame for (2) which is “ is a dog.”

Because a statement like (2) is already a flattening of experience into a representation and that representation has the characteristics of apperceptive judgment, we can clarify this representation. That is, the text on the page for (2) does not yet reflect that it is a judgment, intended to make a move in a language game, not just ink on the page. I tried to remedy that in (3) by including the judgment stroke, but that suggests that judging consciousness is outside of the judged contents in ways that Rödl would object to. What needs to be expressed is that (2) is already a rational recollection of a process. It is a kind of quotation of experience that flattens experience. The self-conscious and apperceptive character of judgment cannot really be represented, but we can represent this failure of representation. A rational recollection, of which quotation is a type usually symbolized as “” will instead be symbolized with {}. So (2) becomes (4):

{Pokey is a dog}. (4)

Moreover, my active role in judgment as excluding the contrary of the contents of the assertion and the self-consciousness of judgment are not really representable, so their representation is null (∅) but both occur within judgement. I shall locate this at the union of particular and universal, the location of the in a sentence frame. That is, where singular term is joined to predicate in the sentence frame I shall symbolize the representational nullity of self-conscious activity. So (4) becomes (5):

{Pokey ∅ is a dog}. (5)

Looking over our shoulders at the normative assessment of the correctness of numerical representations that are epitomized by von Neumann ordinals, our next step is to understand (1) in terms of (5). That is, I shall unfold (1) according to its internal necessity.

Referring back to (1), to hold that Particular Instantiations of the same property are distinct is to implicitly refer to a different property that is not materially incompatible with the first property but is rather “indifferently different” (Brandom, 2019) like square is to green. That is, within (1) there is an implicit assertion that there is a different property that one of the dogs but not the other one conforms to. In this case, let me assert that Pokey is alive, but Buddy is dead. So (1) can become (6):

{Pokey and Buddy ∅ are dogs, {Pokey and not-Buddy ∅ are alive}}. (6)

This has the structure of a recollection of a judgment within a judgment. It is a representation of the process of holding distinct, which can be further flattened and returned to its original form but

with the additional structured moments of the I, or apperceptive judgment, likewise flattened into numerical form.

Pokey and Buddy are {∅, {∅}} distinct dogs. (7)

Pokey and Buddy are 2 distinct dogs. (8)

While this reconstruction produces a numerical term, the term in question is only proto-numerical. Its sense depends on particular objects, but its reference is to the thinking conscious that is not representable. Consequently, the numeral in (8) depends on an object collection. The difference between an object collection and a number is what I turn to in the section "Are numbers singular terms or anaphoric terms?"

This reconstruction of the act of holding distinct is still quite imprecise, and I assume other rational reconstructions of the same act would produce tighter and more general results. But I offer this as a start for how we can understand that judgements that produce number internally depend on the recollection of other judgments for their sense. There is more to say about what ∅ is doing here that makes its traditional name of the empty set inadequate. In particular, I have called it representational nullity, but it is not at all empty. Rather, it is the {I think}, not the emptiness of impossibility or the axiomatic assertion of emptiness, but the thing that we are all most intimately familiar.

The Exercise

It is always challenging to know when a story ought to begin. In this paper, I began with some introductory philosophical ideas, but the better place to begin is the true middle of things. I shall re-start with the {I}, specifically the I-feeling (Carspecken, 1999) because it is integral to the mathematics I have begun to explicate. I shall speak of this feeling through the hopeful thought that readers will attempt to follow this deduction of feeling. It is based primarily on Carspecken's (1999, pp. 169-172) exercise for approaching the body-feeling from proprioception. "Approaching" is important because the feeling has a contradiction where it is in the past when noticed. When it is fully realized as the "I am," it is no longer an ephemeral feeling but something deep and abiding.

Imagine that you have taken some time to lie down and try to enter a state of deep relaxation. You run your attention over your body, starting at the toes and sweeping towards the top of your head. Perhaps some of these regions do not have much contrast and so your attention hiccups over those regions to encounter different regions that feel like they are under some tension. For me, this is often in the shoulders and the back of my knees, but we all carry tension and trauma in different places with varying degrees of opacity. As you encounter those regions with the intention of

relaxing, you might let go of the tension (avoid the trauma for this exercise). A feeling of panic or excitement might happen before letting go, especially if, like me, you worry that what is holding you together as a singular entity is the tension you carry. But for the sake of relaxation, you let go anyway. Perhaps you feel a flood of relief as the region under tension fades into the undifferentiated proprioceptive body-feeling. This feeling evaporates when noticed. For a moment, the relaxed region is integrated into the general body feeling. Images are inadequate here, but it is something like perspectival shift from perceiving a jigsaw puzzle of the Mona Lisa as a thousand distinct pieces to perceiving the whole painting, recognizing a strange smile that was there all along but somehow broken.

Consciousness is always on the move, and so the moment of body-feeling is broken by the next act, which is judgment. Perhaps you say to yourself "that tension was in my left shoulder." In so saying, you realize there are a whole host of rules associated with words like "left" and "in". You have just deployed several different vocabularies implicitly: orientations, containers, and bounded regions. These shall have to be explicated, but now is not the time. Instead, you relax focus to return to the exercise.

You live in the space of undifferentiated body feeling for a moment, feeling the breath go in and out in smooth motions. But then you notice the feeling and in so noticing it evaporates and you return back to judgment and think, "I am not sure that the body is really like a container at all." The idea of a proprioceptive limit ending at the skin feels wrong; you can feel the breath moving in and out. You can feel the breath moving in and out and so the limit of the skin seems wrong, or perhaps you feel vibrations from your neighbor mowing their lawn that enter the body-feeling through the soundscape. Each of these feelings evaporates when noticed and flattened into judgment. Perhaps it occurs to you that all physical matter is connected. Maybe you get a physicists' feeling that all matter is just energy with a wave function that contributes to the vast wave function of the whole universe – all is one. This is an interesting and relevant thought, but it is not quite the body-feeling. It is more a response to the inadequacy of judging the body as an object of matter, with the supplying of a new judgment to fill its place, the body as an energy field connected to the world of being through this shared physical property of matter as energy.

If you stop here, you might obtain a comforting feeling of connectedness or you might feel trapped in the vicious jaws of deterministic physicalism. Matter abides by certain rules, and if the only means of our connection is matter, then the feeling of unity can be binding in ways that do not feel quite right since we take ourselves and each other as free as a necessary feature of communication. From either of these feelings might arise a new judgment, which is the question of whether or not the you who is doing the judging is within the unity which is the thought of the matter of the universe as energy – or are you outside?

From this question of the unity of matter arises the thought that unity is always created through negation. What makes one thing what it is and not something else is its being not something else. This seems tautological. But when you try to pick out an individual object in the perceptual field, what gives this thing its individuality? How then is the materialist universe or whatever unity is at stake so unified? It is unified by the {I}. The {I} is negation, as is the this, here, and now. None of these indexicals are what they are by virtue of being a thing but mean what they mean by being what they are not.

If you continue on, relaxing, dropping, and undifferentiating, perhaps learn to respond in the same way to judgments as they arise, you might begin to feel a primordial rhythm between the unbounded feeling, punctuated by judgment that flattens the temporal experience of unboundedness into some statement about the feeling you just experienced.

What does not disappear in all this is the 'I' as universal, whose seeing is neither a seeing of the tree nor of this house, but is a simple seeing which, though mediated by the negation of this house, etc., is all the same simple and indifferent to whatever happens in it, to the house, the tree, etc. The 'I' is merely universal like 'Now', 'Here', or 'This' in general' I do indeed mean a single 'I', but I can no more say what I mean in the case of 'I' than I can in the case of 'Now' and 'Here'. When I say, 'this Here', 'this Now', or a 'single item', I am saying all Thises, Heres, Nows, all single items. Similarly, when I say 'I', this singular 'I', I say in general all 'Is'; everyone is what I say, everyone in 'I', this singular 'I'. (Hegel, 1977, p. 62)

The nature, moments and movement of this knowing have, then, shown themselves to be such that this knowing is a pure being-for-self of self-consciousness; it is 'I', that is this and no other 'I', and which is no less immediately a mediated or superseded universal 'I'. It has a content which it differentiates from itself; for it is pure negativity or the dividing of itself, it is consciousness. This content is, in its difference, itself the 'I', for it is the movement of superseding itself, or in the same pure negativity that the 'I' is. In it, as differentiated, the 'I' is regflected into itself; it is only when the 'I' communes with itself in its otherness that the content is comprehended [I.e. in terms of the Notion]. Stated more specifically, this content is nothing else than the very movement just spoken of; for the content is Spirit that traverses its own self and does two for itself as Spirit by the fact that it has the 'shape' of the Notion in its objectivity" (Hegel, 1977, p. 486).

The feeling of a negative unity of pure difference and change as a shadow over the positive unity of matter, the feeling of the {I} as a universal also in the {this, here, now}, is not at all the feeling of being determined by being, but is a kind of universal freedom. It is the "I am." Each feeling had a just-was-but-now-gone quality, fleeting and ephemeral. "I am" is all I can say about this next moment, it is not ephemeral. It is the {now, this, I, here} without temporal distinctions.

Perhaps you feel, at this point, like you are quite far away from your body in considering the universe or whatever, but before you exit, you might feel the universality of the identity of {I, here, this, now}. The universal {I} in some deep sense is the {now}, in the sense that both are motion or negativity. The universal {I} is the negativity that forms unities. Empathy is being like another, but this is more like the recognition of unification with the other. This is not unification in the sense of somehow knowing the thoughts of another or knowing how the sunrise looks on Saturn. There is no positive content to this unity; it is just unity in the sense that within the now everything is not. This recognition is the logically simple "I am." Perhaps to sustain it requires a great deal of practice, but it belongs to no tradition in particular or to any individual along some prescriptive pathway of self-formation, because we have this feeling within us constantly, suppressed and at risk of losing the grammar for how we might talk about it, so much so that even writing this little exercise out makes me feel quite jumpy and vulnerable. What if the infinite has no place in mathematics? But then again, who else is as conceptually equipped to deal with infinity besides the mathematical community?

Algorithmic Elaboration: from body-feelings to concepts

In this section, I will take judgments that resulted from proprioceptive experience in the Exercise and algorithmically elaborate the notion of categories that I have been already using as concepts/predicates/universals. During the exercise, I mentioned the host of rules associated with words like "left" and "in." These are involved in important mathematical vocabularies and practices-or-abilities regarding orientations, containers, and bounded regions. In fact, the discussions of universals and particulars that have occurred so far throughout the text have really depended on some specific body-feelings. In particular, we might notice that the proprioceptive feeling seems to stop around the skin. It may be extended through further acts of undifferentiation, but in the initial moments of the Exercise, the skin exists as a kind of boundary, that forms an inside or interior of the body in which the proprioceptive feeling seems to be limited, and an outside or exterior of the body which is not included in the body-feeling.

There are rules that govern how the terms {inside, outside, interior, exterior, boundary} are deployed. The moment we judge, we subject ourselves to a normative assessment for the correctness of how these terms are deployed. These are rules like "there is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside" (Lakoff & Núñez, 2000, p. 31).

The set of material inferences regarding bounded regions are as follows:

1. If you're in a bounded region, you are not out of that bounded region

2. If you're out of a bounded region, you are not in that bounded region
3. If you are deep in a bounded region, you are far from being out of that bounded region
4. If you are on the edge of a bounded region, you are close to being in that bounded region.

These can be written out as an algorithm as in Figure 2.

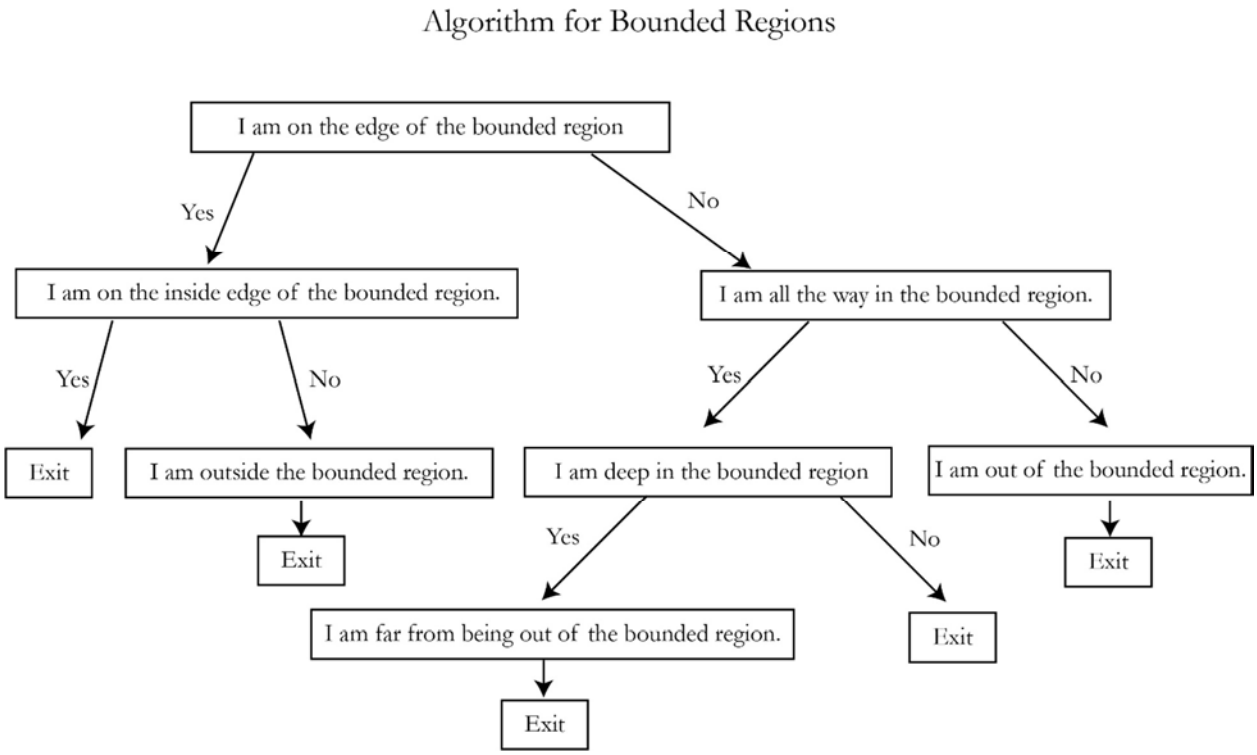

Figure 2. An algorithmic expression of the practices-or-abilities associated with the vocabulary of bounded regions.

This vocabulary can now be algorithmically elaborated into a vocabulary for categories (Lakoff & Núñez, 2000, p. 43) or what I have been calling concepts/predicates/universals. This algorithm is a set of simple substitutions.

1.) Substitute "Categories" for "Bounded regions in space."
2.) Substitute "Category members" for "Objects inside the bounded regions."
3.) Substitute "A subcategory of a larger category" for "One bounded region inside another."

What this allows is for the body-feeling to ground our notions of bounded regions, which in turn ground our vocabulary of categories. That is, we can flow from one vocabulary to another while maintaining a relatively robust notion of correctness. We can determine at which point along the elaboration something has gone awry in our speech or others. I take it that this is what tutors actually do when tutoring. We trace back from one more complicated set of vocabulary to more foundational vocabularies. When teachers teach conceptually, each new vocabulary is elaborated out of some prior vocabulary to so that students' consciousnesses can flow from one type of mathematical thinking into the next.

Algorithmic elaboration is the mechanism by which one could construct a mathematical history. There are many things this notion of history would miss and so it is not a perfect analogy. For example, I suspect that there is a lot of scholarly work in the history of mathematics that would not recognize itself in algorithmic elaboration, and the point would not be to try to rework these kinds of histories that are rich in expression. Rather, the notion is that one vocabulary could flow into another and then that movement could be recollected as an algorithmic elaboration of one vocabulary into another. This seems most useful when considering research stories that may only be partially told by individual subjects. For example, Subject 1 might only utter the sentences B, D, and E and Subject 2 might only utter A and C. If a researcher suspects that the whole articulation is A, B, C, D, E and that the missing utterances were suppressed for one reason or another, they could tell that story laterally by constructing an algorithmic elaboration that includes the relevant pieces from multiple subjects. Similarly, if one were conducting a literature review, one of the principal tasks is explaining what a word that is to follow means by tracing the history of its use and the uses of its precursors in a particular domain. This could be done in a way that feels mathematical if the precision of algorithmic elaboration were deployed to accomplish that goal. Moreover, one could imagine actually programming machines to do some of this stuff to produce working models of children's thinking that could be explored by pre-service teachers in order to more fluidly disseminate research results to practitioners.

All that excitement aside, it is tedious to read these algorithms and they are not quite true to conscious experience because conscious experience can be responsive to any number of initial thinkings. To be closer to accurate would require a more sophisticated kind of algorithm that would be responsive to beginning with any of the assertions in the material inference set. Consequently, I shall avoid writing out these algorithms when possible and leave the expressions as material inferences with the understanding that those material inferences could be strung together in an algorithmic expression.

From Counting to Arithmetic Foundations

Once numerical structures are discerned within a critical framework, the notion of arithmetic can take shape by reinterpreting the cognitive scientists Lakoff & Núñez (2000) work on embodied mathematics as material inferences, expressed in the deontic-normative vocabulary of commitments and entitlements. These material inferential rules enable pre-formal arithmetic.

One of the challenges associated with working within the first-person actor's subject position to articulate mathematics in the deontic-normative modality, rather than the alethic modality, is that notions like linearity have complex algorithmic expressions. That is, 'laws' or axioms cannot be stated as extant rules in some abstract ontological realm or the risk is governance from some externality. In the alethic mode, one might say "It is necessarily the case that only one of the following sentences is true: {A < B}, {B < A}, {A = B}." Such statements require taking an observer position on a collection of possible utterances. In order to avoid succumbing to such a third-person position prematurely, I am going to express these kinds of rules as algorithms (Brandom, 2008). The idea is that consciousness takes yes/no positions on assertions individually to then produce the rule.

The algorithm I make the most use of is one I am calling the highlander algorithm in order to express the notion that "only one of the following assertions can be committed to," which entails that the others are incompatible with the one committed to. This algorithm is represented in Figure 3.

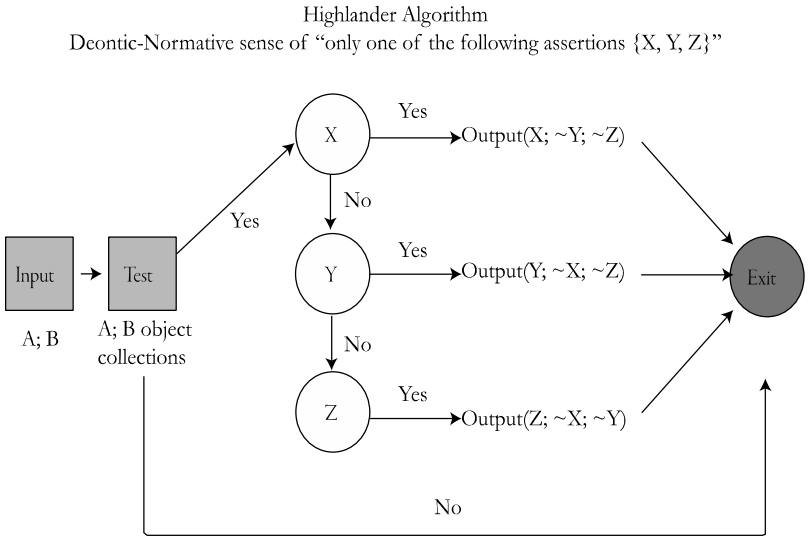

Figure 3. The highlander algorithm represents phenomenal consciousness as it moves when inferential propriety dictates that only one assertion from the set {A, B, C} can be committed to at one time. This is useful for expressing notions like linearity.

The last algorithm that I need to express is just for a specific material inference below, called the equality of result. In Lakoff & Núñez (2000), this is expressed as "you can obtain the same resulting object collection via different operations" (p. 58). This is in the alethic modality, as in "it is possible that $a = b + c$ and $a = d + e$". Expressing this in the deontic-normative modality as an algorithm, as I have done in Figure 4, misses quite a bit of this generality, as it has to be fed a specific number and then only generates pairs of equivalent numbers by subtracting 1 from the given number and adding 1 to its pair, which starts at zero. Other arithmetic 'laws' need similar treatment, but I take it that these two together form enough of an example to eventually produce sufficient conditions for arithmetic.

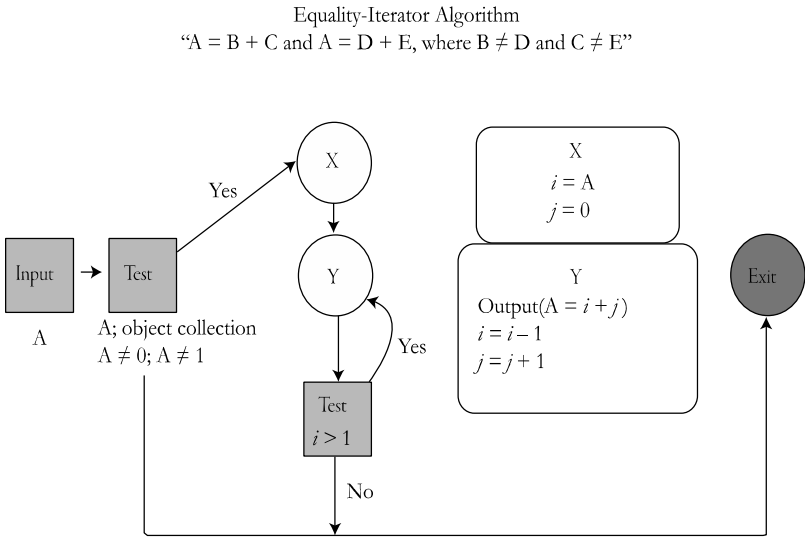

Figure 4. The equality-iterator algorithm expresses the notion of the equality of result (discussed below) in a deontic-normative modality.

In the following, I explicate the material inferences that afford the basic notion of object collections. Setting out these material inferences allows for the deployment of the vocabulary $V_{\text{object collection}}$. Should one wish to, this could be algorithmized as a branched-schedule algorithm in the given order where, should a particular inference fail to go through, then the algorithm would exit and determine that the object collections being discussed are not, in fact, object collections. This allows for the expression of the practices-or-abilities associated with object collections ($P_{\text{object collection}}$) that licenses $V_{\text{object collection}}$.

Symbol	Substituted-for

¬	Precludes entitlement to.../is incompatible with
→	entitles/has as an inferential consequent...
$\models_I$	entitles/incompatibility-entails ...
†	Highlander algorithm for "entitles only one of the following commitments"
$\rightarrow_{SI}$	Simple Material Substitution Inferential Commitment
A = B	A is symmetrically intersubstitutable with B

Table 2. Legend for symbols used below.

$V_{\text{object collections}}$: {bigger, smaller, =, added to, results in, adding, result, subtracted from, subtract, zero, one}

$P_{\text{object collection}}$:

Linearity: {A and B are object collections } †

- {A is bigger than B} $\models_I$ {B is smaller than A} → ¬{A is smaller than B}
- {B is bigger than A} $\models_I$ {A is smaller than B}) → ¬{B is smaller than A}
- {$A = B$}

Closure: {A added to B results in D} $\models_I$ {D is an object collection}

Commutativity: {Adding A to B results in C} $\models_I$ {C = the result of adding B to A}

Associativity: {D is the object collection that results from adding A to the object collection that results from adding object collection B to object collection C} $\models_I$ {D = the object collection that results in adding C to the object collection that results from adding A and B}

Transitivity: {A is bigger than B and B is bigger than C} $\models_I$ {A is bigger than C}

Addition structural roles: {A is an object collection in addition}†

- {A is what is added to}
- {A is what is added}
- {A is the result of addition}

Stuttering inference: {A} $\models_I$ {A}

Unlimited iteration for Addition: {A is an object collection } → {A added to B results in an object collection}

Limited iteration of Subtraction:

- $\{B$ is smaller than $A\} \rightarrow \{B$ subtracted from A is an object collection$\}$
- $\{B$ is the symmetrically intersubstitutable with $A\} \vDash_I \{$There are no objects left when A is subtracted from $B\}$

Zero:

- $\{A$ is symmetrically intersubstitutable with $B\} \vDash_I \{$There are no objects left when A is subtracted from $B\}$
- $\{A$ is an object collection with no objects$\} \rightarrow \{A$ is symmetrically intersubstitutable with zero$\}$

Sequential operations:

- $\{B$ is smaller than $A\} \rightarrow \{B$ subtracted from A added to C results in an object collection$\}$
- $\{B$ added to A results in an object collection larger than $C\} \rightarrow \{C$ subtracted from the result of B added to A is an object collection$\}$

Equality of result: $\{A$ is an object collection$\} \vDash_I \{$Call Equality Iterator Algorithm.$\}$

Preservation of equality:$\{B$ and C are symmetrically intersubstitutable$\} \vDash_I \{A$ added to B results in an object collection symmetrically intersubstitutable with the object collection that results from adding A to $C\}$

This is a set of substitution inferences, algorithms, and incompatibility entailments which collectively algorithmically elaborate talk about object collections into arithmetic over object collections understood as numbers. In Savich et. al. (2019) we articulate how to read meaning use diagrams. In Figure 4 I include a visualization of the algorithmic elaboration of the vocabulary of object collections can be elaborated into a vocabulary of arithmetic when all of the preceding material substitution inferences are taken collectively as an algorithm.

Discussion

Judgment refers to incomplete systems of commitments and entitlements, thought through the concept of sufficient grounds or validity claims, and thought through the concept of the object of thought. These together impel judgment through internal necessity to the next judgment. A legitimate question one might ask is whether what we normally think of as mathematical judgment can be conceptualized as being driven by internal necessity. Hegel did not think that mathematics was generally so structured, instead interpreting mathematics to be horizontal within a domain. That being said, in explications like the one just offered about Pokey and Buddy, there is an interest in emancipation through the explication of $\emptyset$, and also a kind of destination in mind with the von

Neumann ordinals serving as a normative standard of correctness. This kind of judgment feels driven by internal necessity. Alternatively, something like "2 × 3 = ?" as what one might find on a worksheet feels like an external process is driving mathematical judgment.

What makes the difference here is the which of the three knowledge interests is driving the judgment. In judging, we are always judging through the concept of the object of thought (Rödl, 2018), and so claiming a world. Habermas distinguishes between three formal worlds that are simultaneously claimed in every speech act, but which foreground different validity criteria. These formal worlds are the subjective, objective, and normative. When the interest is empirical/analytic, the objective world is most at stake. In these kinds of judgment, what drives the movement within a judgment is generally externalized. If the interest is on completing a task like filling out or writing up a worksheet in order to get an 'A' or keep one's job, then the interest is fundamentally instrumental and so external. The validity claim here can be objective, like "I got all of the answers correct," normative like "I have completed the assigned task so I have adequately attended to my responsibilities," or subjective like "I worked very hard on this." Individuals can certainly enter a state of flow while doing sets of procedural tasks that are familiar, but at the end of that period of activity if the individual's first-person experience of self is essentially unchanged, then the activity was not driven by internal necessity. A psychological model of risk/reward is not wholly inappropriate for this type of thinking.

Alternatively, if the activity is conducted with an interest in emancipation defined by self-formation, then the activity changes the actor through their action. There is no external validation here, so it is not exactly like this is a measurable validity. If at the end of a period of activity the actor can reflect on the activity and say "I am different than I was, more about the implicit world has been made explicit through this activity," then the activity was self-formative and so driven by internal necessity. In self-formation – moving the implicit into the explicit – more of oneself is actualized. In doing so, one becomes-more.

Are numbers singular terms or anaphora?

An interesting structural commonality in judgment impelled by internal necessity which is in the presence of unambiguous singular terms. In a statement like "2 × 3 = 6," the numerical terms could be singular terms, as Frege conceptualized them, or they could be anaphoric terms like pronouns.

Singular terms are defined in this paper based on whether they are involved in asymmetric or symmetric substitution inferences. In general, numbers are involved in such inferences. Whenever I say something like "2 × 3 = 6," the equals sign is essentially a symmetric substitution license between terms on the left and the terms on the right: 6 may be substituted wherever one encounters 2 × 3 and vice versa. That being said, numbers in the Fregean account and the story told here arise

from notions of object collections. Just like "he" can refer to different people, the number 2 can refer to different object collections. Numbers, like pronouns, are promiscuous.

When people talk about numbers as singular terms, they seem to be implying that numbers like "2" exist independently of thinking, perhaps in a Platonic heaven or some other strictly formal ontological realm. The general idea underlying the singular term notion of number is to take an object collection like (1) and establish a one-to-one correspondence with an object collection like (9).

Gilbert and Sullivan are distinct cats. (9)

This allows for the term "number" to be introduced via Fregean abstraction (Brandom, 1994, p. 420; Frege, 1884, p. 111). In this way, we abstract away from "dogs" and "cats" and are left with pure number, but it is unclear what that number refers to. What are we referring to when we use empty symbols besides emptiness? Is "2" a name for a special kind of no-one? How could such emptiness become an object of thought?

For these reasons, I propose that numbers are anaphoric terms that recollect first-person thought. That is, 2 is {∅, {∅}} and functions like a pronoun that fixes "I think, I think 'I think'." Or, "I think and in the course of doing that thinking I recollect a prior thinking." The symmetry of singular term substitution license is preserved by the notion that "I" is both a reference to a singular individual while simultaneously being a promiscuous pronoun. It is both singular and universal when understood as the negative, as expressed in the Exercise. This introduces a problem, because if I were to say "I think and in the course of doing that thinking I recollect a prior thinking," another might ask in an exasperated tone "What are you thinking about?!" In the case of (8) I can answer "dogs." In the case of (9), "cats."

Originally, number is tied to object collections because we do not necessarily begin our mathematical journeys as children with the logical ability to explicitly think, "I think and in the course of doing that thinking I recollect a prior thinking." Rather, we gesture, pointing at this and that, and establishing object collections. Presumably, when we abstract away from talking about object collections to considering pure thought thinking, we need a logic of thinking. Without such a logic, an exasperated "what are you thinking about?!" seems the only appropriate response. Hegel's Science of Logic and its recent expositions (e.g. Rödl, 2018; Pippin, 2019) articulates some of this kind of logic, but we do not need to pursue that movement towards pure number in order to articulate an arithmetic, as the vocabulary of object collections is a sufficient foundation. That is, to get to pure number we must be able to "think on thinking," which is a story for another time.

One might ask what goes missing from the anaphoric account of number. One of the more interesting phenomena that gifted arithmeticians describe is synesthesia, where body-feelings and sensory experiences are associated with different numbers. Additionally, some people describe numbers like personal friends, as Ramanujan famously claimed of the number 1729. These suggest a singular term notion of number because numbers are individually felt or recognized. However, the anaphoric conceptualization of number actually affords a richer explanation of these phenomena than the singular term conceptualization can.

We can fold these into the anaphoric conceptualization of number by recalling that the anaphoric terms in question here are first-person self-referential pronouns. I certainly have different body-feelings associated with different kinds of self-reference, and I imagine other people do as well. That is, different identity claims produce different body-feelings. In saying "I am a bad person because I explained anaphora badly," I feel ashamed. A self-referential anaphoric conceptualization of number could likewise explain why certain people feel differently when considering different numbers. There is no magic in thinking that identity claims are accompanied by body-feelings, while there is quite a bit of magic in thinking that formal emptiness produces sensory experience. Consequently, the anaphoric notion of number could sensibly explain synesthesia in a way that prior notions of number are inadequate to do. Likewise with Ramanujan: it makes a good deal of sense that different identity claims could produce different sensations of affinity. In saying "I am a good person because I am a good friend," I feel warmth towards myself. In keeping number close to first-person body-feelings, these two particular examples are actually somewhat easier to understand than if number was thought exist in a Platonic heaven or to not exist at all but just be structured emptiness. In short, I propose thinking of numbers as essentially self-referential pronouns that include recollective activity.

In the sections above I have explicated the movement of consciousness as inference and algorithm, I discussed judgment as unfolding from within itself according to its internal necessity, and I discussed how the vocabulary of object collections can be algorithmically elaborated into arithmetic. Where does that leave us? The point of doing this work is to articulate mathematics in the first-person subject position, honoring the primacy of first-person knowledge. Because this first-person knowledge includes a normative systematicity, it is not arbitrary. But I have not yet talked about how one knows that the mathematical knowledge we are so familiar with is true. For that, the discussion must include a better articulation of the objectivity of judgment (Rödl, 2018) and a discussion of the I-feeling (Carspecken, 1999). That being said there is enough here to begin to understand correctness even if truth remains elusive.

Moreover, this is not, as yet, an actionable philosophy of mathematics education. While I am a mathematics educator and teacher educator, this critical mathematics has been mostly implicit in

the communicative norms that I establish in my classrooms and there is no kind of measurable evidence for their efficacy. Essentially, I am not trying to say how one should teach mathematics and giving evidence for why that should is warranted. Instead, in this essay and subsequent work I am attempting to establish what the mathematics we presume to teach is. It is knowing and being, synthesized as becoming through the first-person actions of knowing subjects. In future essays it shall be important to address how the epistemologically necessary concept of history can be understood laterally within sets of data to re-cognize mathematical knowledge in a way that inextricably (if implicitly) links mathematics education research to mathematics itself. This is important not only for 'completeness', but also because the most pressing threat to this work is that it is dangerously close to collapsing under its own abstractions. That is, to replace a formalism with another formalism would simply replace one reduction in intellectual freedom for another. A lateral history of the lives of mathematical concepts is an essential "next step" towards a critical mathematics. To clarify, these histories are already present in the organon of mathematics education; what I take it that this particular work is missing in its essence is the absence of how to tell those histories in robust criticality.

Conclusion

In the sections above, I have done some things that are relatively common, but I have also produced some novel ways of talking about old ideas. I began with an experience of error that allowed me to make two existential needs explicit. We have the need to be recognized as good people by living up to our finite identity claims understood as commitments. We also have the need to be recognized as infinite, in the sense that those identity claims are inexhaustible. The framing of these needs is Phil Carspecken's thinking that I have borrowed here. These two existential needs could be explored indefinitely in mathematics education communities. We tend to be very focused on developing normative subjects rather than recognizing the infinite nature of ourselves and our students. Understanding how those needs interact with one another in judgment is presumably unfamiliar to most readers and could be a contribution to Hegelian scholarship in general.

Next, I put some linguistic structures in place that are robust enough to build a notion of quantity from within. Obtaining quantity without axioms is what might make this project relevant to critical theorists who are not Hegelian scholars. That is, the notions of substitution, singular terms, predicates, and anaphora are quite general and so could be folded into other notions of criticality that I have not explored here. Moreover, the notion of algorithmic elaboration does not feel like it belongs to one tradition or another and so could be adopted by non-Hegelians who wish to pursue critical mathematics within other frameworks. The expression of mathematical 'laws' as algorithms is presumably something that other people have worked on to varying degrees, but I assume that those uses are primarily instrumental, rather than communicative. The actual

mathematical content I have used is almost all from Lakoff & Núñez (2000), whose rich and beautiful text is somewhat marred in my rendering. I have tried to connect their work with the body-feeling, which I take to be an enrichment of embodied mathematics.

One of the glaring omissions of this piece is that student work has not been described at all, so it might be challenging to use this understanding of mathematics in an educational research context. Future papers shall address this, but the basic idea is that student work samples could be decomposed into algorithms, which are rather like schemes in constructivist thinking. In general, much more work must be done in exploring how the body-feeling is related to the various material inferences that are necessary conditions for arithmetic. Moreover, how our experiences of self-certainty relate to mathematical truth deserves finer treatment as does how this framework connects to extant research in critical mathematics education. Even with so much left to explore and explicate, my hope is that readers will take away some excitement for the possible directions in which a critical mathematics might move.

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