Imagination in the philosophy of mathematics and its implication for mathematics education

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Abstract: *Teachers of Mathematics and researchers in Mathematics Education are recommended to be engaged in a creative endeavor. However, creativity cannot be thought of independent of imagination. This article analyzes the relation between imagination and the philosophy of mathematics. The later concept is inclusively concerned with arguments on whether or not: (i) mathematics is invented or discovered; (ii) the existence of mathematical entities and objects; and (iii) the truthness of mathematical discourse. On the other hand, the philosophy of Mathematics Education covers the understanding of, and the meaning we attribute to, mathematics and its nature. In this article, I argue that "imagination" is a vital asset in Mathematics Education.*

Keywords: imagination, mathematics, mathematics education, philosophy

Introduction

I am a co-founder of *Mathematics Forum* in Ethiopia. Once up on a time, a 12th grade student posted a question via a channel in the social media seeking reflections on the importance of learning mathematics. Some members of the *Forum* reacted to his query. The discussion is reported below using pseudo names.

Noah:- why do we need learning Maths since it's not useful in our life? Please, I need answer.

Gawo:- Noah has raised interesting question that we all need to answer.

Tole:- If you found it useless in your life, you better stop learning mathematics.

Noah:- … but, I love math than all subjects; if u know, why don't u tell me…

Tole:- If you say "it's not useful in our life", you have already concluded that it is useless. I do not have to waste my time. kkkkkkk....

Noah:- I apologize for my [poor] English [skill]; I'm grade 12 & I have a plan to learn applied math… When students ask me the importance of math, I lose to answer…

Tole:- It is okay. You are fluent speaker. I am not laughing at your English bro[ther]. Anyways, let us meet in the battlefield of mathematics. If you have spare time, attend Applied Mathematics courses.

Adem:- There are two options, either you are using it unknowingly or you are being used by it unknowingly!!!

Baye:- ... the best strategy is just relate what you have learnt before with the daily activities. You may observe some applications in other subjects. Moreover, you may read books that promote Mathematics. Can you imagine this world with mathematics?

Zena:- I don't want to talk about the day to day activities that requires basic mathematical skills (numeracy skills) but let me make a few statement about mathematics:

1. *Mathematics provides a powerful universal language and [an] intellectual toolkit for abstraction, generalization and synthesis.*
2. *It is the language of science and technology*
3. *Mathematical training disciplines the mind; [it] develops logical and critical reasoning and develops analytical and problem-solving skills to a highest degree.*
4. *...*

Baye:- ... useful to your life….

The student had sought thoughts on the nature of Mathematics. He reported that he loves the subject; yet, he was not clear about the purposes of learning it. Some replies were presented from participants of the Forum. I found the conversation worth to reflect on. Nevertheless, I am not in a position to judge who is right and who is wrong. The implication is that explaining mathematics to a student demands a creative teaching. This is an issue of Mathematics education.

There are three narratives about mathematics education: mathematics provides a true foundation for human knowledge and progress; it serves functions of oppression; and it has critical potentials and might serve the development of social justice (Skovsmose, 2020). Such a comparison of "narratives about mathematics education" with the bearing of mathematics might pose a question. In fact, there have been doubts on the distinctions and connections among *Mathematics* versus *Mathematics education* versus *Mathematics education research* (Fried, 2014). Accordingly, tensions among *Mathematician* versus *Mathematics educator* versus *Mathematics education researcher* might be expected. We may see mathematics education as a site for socio-political struggles.

The field Mathematics education is included in the disciplines Mathematics and Education and at the same time encompasses themselves (Degu, 2019). Thus, a practice and research in mathematics education will uncover the understanding, and the meaning making we attribute to, mathematics and its nature. Nowadays, large parts of the philosophy of mathematics are with descriptions and analyses of mathematical practices (Degu, 2015), the relationship between mathematics and human beings, and socio-philosophical reflections on the role of mathematics for society (Prediger, 2007). Since Mathematics has plenty of applications, it may imply for generating new interdisciplinary approaches, instruments, and models to adopt a more holistic approach. Thus, if such a study of applied mathematics is sought, a new subfield, applied mathematics education, would be resulting in.

The Philosophy of Mathematics

In a discipline as wide-ranging as contemporary mathematics; Mathematical output can be estimated at several million documents (Parrochia, 2018). Then, it would be impossible to catch up the theoretical foundations of them. Generally, the present-day philosophy of mathematics (Ravn & Skovsmose, 2019) addresses the following concerns:

- ✓ *Where is mathematics?*
- ✓ *How certain is mathematics?*
- ✓ *How social is mathematics?*
- ✓ *How good is mathematics?*
- ✓ *Sociological issues*
- ✓ *Ethical issues*
- ✓ *How beautiful is mathematics?*
- ✓ *How religious is mathematics?*
- ✓ *How political is mathematics?*

Yet, some of the above concerns are extensions of others. For instance, the last question could be assumed as an extension of the social dimension of mathematics. On the other hand, a *socio-political dimension* might be thought of. As a result, considering a few particular viewpoints of mathematics could not be exhaustive. More generally, I consider the philosophy of Mathematics as it accounts for the study of the fundamental nature of knowledge, reality, and existence.

Ontological Concerns

I think the aforementioned questions could fall in any of these three categories. For instance, the question "how religious is mathematics?" might be in the domain of ontological. Nevertheless, how far such a specific concern is up to the status of a dimension? Ontological consideration deals fundamentally with what exists (Ravn & Skovsmose, 2019). Questions like "do numbers exist?" and "in what sense do they exist?" are ontological.

Broadly speaking, the ontological domain would enable us considering the nature of mathematics and the existence of objects or entities. For instance, the irrational number π with approximation of $\frac{22}{7}$ is actually a different number $3.1428571428571\cdots$ where the decimal goes on forever without repeating. Besides, the introduction of $\sqrt{-1}$ or i brought the set of complex numbers. So, a complex number is written in the form of $a + bi$ where $i = \sqrt{-1}$; i is imaginary number. Mathematics is concerned with abstract features of actual or possible objects or systems of objects (Linnebo, 2017).

On the other hand, idealization is common in geometry. For instance, in a circle, each point is *exactly* the same distance from its center. Another characteristic of mathematics is computation. It goes to the algorithmic operations on syntactic signs or other systems of representations. The signs, on which we compute, can be used to represent objects and their properties. So, computation plays an essential role in connection with term formalism, introduced by the 20th-century German mathematician David Hilbert's finitism. It holds that all mathematics can be reduced to rules for manipulating formulas without any reference to the meanings of the formulas. Formalists contend that it is the mathematical symbols themselves, and not any meaning that might be ascribed to them, that are the basic objects of mathematical thought. For every numeral, we extrapolate and start to reason about the entire sequence of numerals.

Epistemological Issues

Epistemological consideration deals with what we can know and how we may obtain knowledge (Ravn & Skovsmose, 2019). The other basic question, "how certain is mathematics", is in line with knowledge and hence falls in epistemological dimension. It is becoming increasingly clear that mathematical thinking is essential to understanding the world around us (Krantz, 2018). While it would be unreasonable to impose a causal requirement on the knowledge of mathematical truths, we still want an informative answer to the question (Linnebo, 2017). The social constructivist's philosophy of mathematics (Ernest, 1998) is about the processes of constructing mathematical knowledge; and hence, it goes to the epistemological sub-domain of philosophy.

There are many reasons that necessitate understanding and meaning making of mathematics. First, Mathematics is characterized by a combination of natural language, symbols (e.g. $\cup, \cap, \emptyset, \Delta, \nabla, \forall, \geq, \div$, etc), formulae, models (e.g. Cartesian plane, 3-dimensional space, Poincare model, etc), and visual displays for expressing ideas or concepts. Consequently, students must learn the ways of thinking and communicating them. Second, though there are arguments, mathematics is a human activity, a social phenomenon, part of human culture, and intelligible in a social context (Ernest, 1991). So, one concern would be the awareness of the mathematical way of thinking at the world.

Metaphysical Concerns

In a most concentrated version, epistemology is about knowledge while ontology is about being. Connecting epistemology and ontology leads to the question: How may we acquire knowledge on a particular domain? Lastly, the question, "where is mathematics" belongs to the metaphysical dimension of philosophy. A similar question "is mathematics invented or discovered?" has occupied the interest of philosophers and mathematicians for centuries. This will be addressed if we understood the nature and role of imagination in mathematics.

For instance, in Mathematical Platonism, it is assumed that there exist abstract objects that are non-spatiotemporal, non-physical, and non-mental. Again, there are true mathematical sentences that provide descriptions of such objects. If, for Platonists, numbers exist outside time and space, how much is true their existence? On the other hand, *Nominalism* denies the real existence of any entity. Mathematical objects such as numbers do not really exist; there are such things in people's heads. A common version of Nominalism is *Fictionalism* which claims a mathematical discourse is false though useful. Hence, it is not the truthness of mathematical claims. What kind of imaginations are those stands? Therefore, doing pure mathematics (not merely doing computations) is an exercise in imagination (Saiber & Turner, 2009).

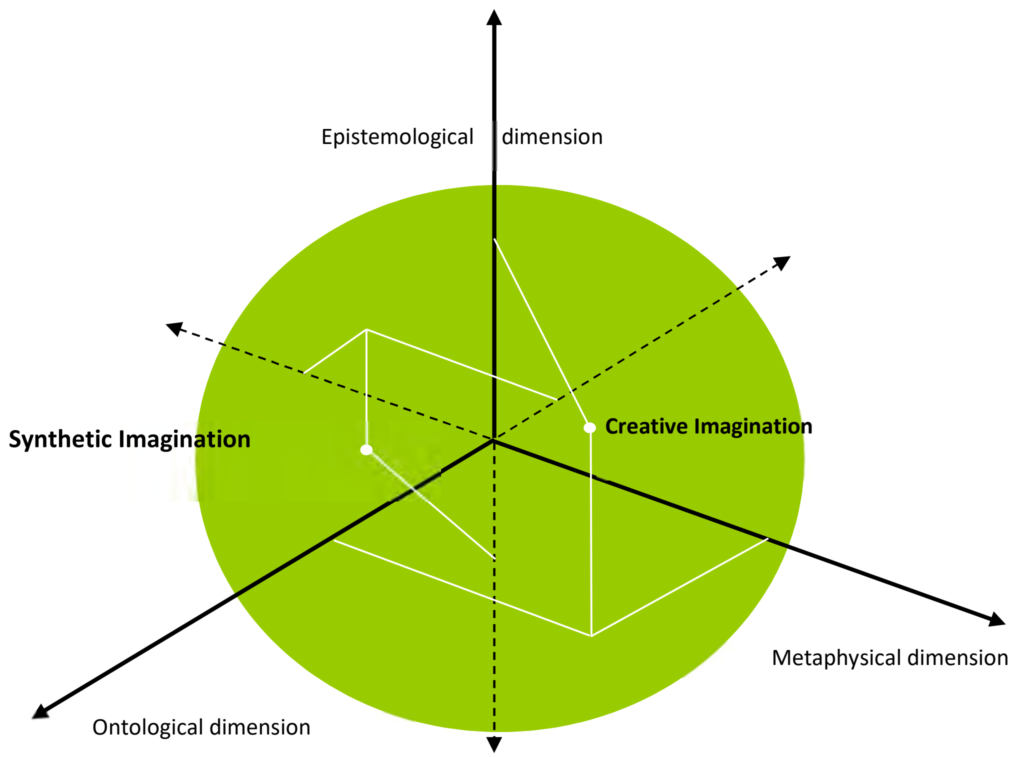

Figure 1: Imagination in the Philosophy of Mathematics

The dimensions and perspectives could entail us that Mathematics is about multiple realities, relative truths, complexities, and ambiguities. It is possible to take a position in any of the eight octants. The first octant might be assumed as *Creative Imagination* and the eighth one might be considered *synthetic imagination.* These extremes are twofold usages of imagination (Liang, Chang, Chang, & Lin, 2012). Then, by taking straight or curved lines, we can think of eclectic positions. Let me support my assumptions with some sorts of arguments.

In the beginning of 20th century, Théodule A. Ribot and Lev S. Vygotsky concurrently introduced the concept of creative imagination (Karwowski, Jankowska, & Szwajkowski, 2017). As research and theories developed, attention was drawn to the complexity of creative imagination (Smolucha & Smolucha, 1986). For instance, Karwowski et al (2017) noted the concept refers to the ability of creating and transforming mental representations based on the material of past observations, but significantly transcending them. A brain activity includes re-experiencing of images. C**reative imagination is** *recombining experiences* in the creation of new images directed at a *specific goal or aiding in the solution* of problems ((Beaney, 2010). Such a recombination of experiences seems to be an issue of Formalism. On the other hand, the *goal*-oriented temptation is in line with Fictionalism. As long as, the founder is Vygotsky, and engaging in creative imagination demands the social dimension, then, Ernest's Social Constructivism is an alternative view in mathematics. Thus, creative imagination is a faculty of invention and linked with genius, particularly in mathematics.

Synthetic imagination, as the ability to retain a simple impression of objects, implies the discovery aspect in mathematics. It demands consideration of a rather different set of issues. It is clear that mathematical thinking is essential to understanding the world around us (Krantz, 2018). It would be unreasonable to impose a causal requirement on the knowledge of mathematical truths; we still want an informative answer to the question (Linnebo, 2017). Certainly, there is room for intuition in mathematics, and even room for guessing (Krantz, 2018). But, a mathematical situation and/or solve a problem by being very logical; logic makes the process dependable and reproducible. It shows that what we are producing is a verifiable truth. As a result, it seems that synthetic imagination matches with Platonism, Logicism and Absolutism. In these schools of thought, mathematics might be discovered and reproduced.

Philosophy of Mathematics Education

The philosophy of mathematics education is rooted from the nature of mathematics itself (Bendegem & François, 2007; Ernest, 1991) and education (Ernest, 2016). It tackles questions about (1) the purposes and meanings of teaching and learning mathematics; (2) our understanding of, and the meaning we attribute to, mathematics and its nature; and (3) the relationship between mathematics and society (Radford, 2018). Research and theories in mathematics education are analyzed according to the branches of philosophy they draw upon, including metaphysics and ontology, epistemology, social and political philosophy, ethics, methodology, and aesthetics (Ernest, 2018). So, it is beyond studying about the purposes of teaching and learning mathematics. It addresses the following concerns: Mathematics, learning, pedagogy and applying philosophy into Mathematics Education. Since philosophy is concerned with making explicit the nature and significance of concepts, the philosophy of Mathematics Educations is concerned about the status of Mathematics Education as a field of knowledge. Basically, its position is to analyze, question, challenge, and critique the claims of mathematics education practice, policy and research (Ernest, 2018). Hence, it is vital to be clear what it means.

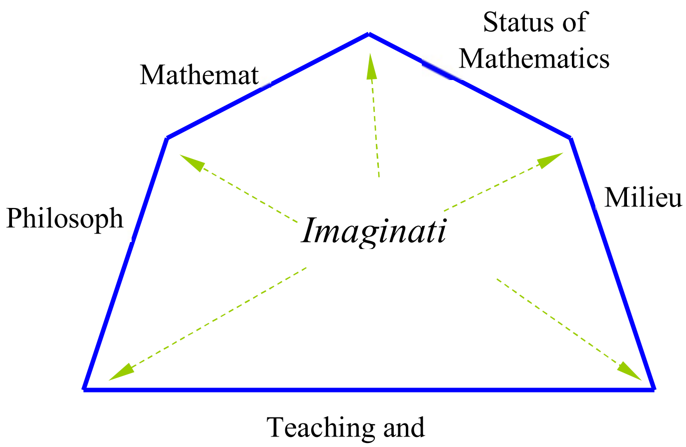

Figure 2: Imagination in the Philosophy of Mathematics Education

In the coming paragraphs, I discuss the main components of the Philosophy of Mathematics Education and then I would go to reflect on the integration of imagination within these categories.

Understanding of Mathematics and its Nature

I assume the field Mathematics education is rooted from and connected to Mathematics and Education. In other words, it is included in the disciplines and at the same time encompasses them. Thus, a practice and research in mathematics education will uncover the understanding, and the meaning making we attribute to, mathematics and its nature. For instance, an instruction or research on the notion of limit will be concerned with definitions, symbols, practical examples, conjectures, and proofs in relation to the concept. In many mathematics curricula, the notion of limit is introduced three times: the limit of a sequence, the limit of a function at a point and the limit of a function at infinity (Fern´andez-Plaza & Simpson, 2016). On the other hand, the concept of derivative is introduced as the limit of slopes of secants of smaller widths. Likewise, Riemann integral is the limit of Riemann sums of finer meshes. It can be said that Limit is fundamental to the standard formal foundations of many aspects of Calculus. Besides, students' understanding of limit concerns the use of dynamic imagery (Fern´andez-Plaza & Simpson, 2016) by considering the pattern of numbers or functional values.

Purposes of and Meanings in the Learning-Teaching of Mathematics

In the classroom, the basic event is instructional process thereby curriculum, teacher and students are key players for learning to happen. The implementation of a curriculum would come up based on teacher's perception and how the student is experiencing it. Then, the organization of tasks, mathematical objects, actions, teacher's and students' activities determine the overall outcome of the instructional process. The purposes of and meaning attributed in the teaching and learning of mathematics is another concern in the philosophy of mathematics education. If philosophical inquiry is applied to classroom mathematics, it may aid in the opening of a "wider horizon of interpretations" that includes a potential expansion of students' mathematical experience, and promises to provide bridges for establishing richer, critical, and more meaningful connections and interactions between students' personal experience and the broader culture (Kennedy, 2018). An approach to understanding learning from a sociocultural and interactional perspective implies a focus on development and appropriation of knowledge (Langman & Hansen-Thomas, 2017). Thus, in the teaching and learning of mathematics, there would be a demand for activities that involve explanations and justifications of arguments. These, in turn, require sophisticated use of language linked to mathematics-specific content area learning.

The Mathematics We Live By: Mathematics and Society

Much of the mathematics we are using today was developed as a result of modelling real world situations. Mathematics shapes the wo;2rld in which we live; the world in turn shapes the discipline of mathematics (Greenwald & Thomley, 2012). As the societal problems evolve, new mathematical solutions would be created. A situational perspective focuses on idealization of some aspects of the world; Mathematics is part of the physical world (Poythress, 2015). Thus, a mathematical knowledge is socially and culturally situated. It is a cultural phenomenon; a set of ideas, connections, and relationships that we can use to make sense of the world (Boaler, 2016). In this regard, socio-cultural and situational perspectives are relevant to study the "mathematics" being practiced at a certain milieu.

Engaging in Imagination

The concept *imagination* is an abstract and philosophical construct (Kind, 2016). As it was discussed in the preceding sections, the philosophy of Mathematics education focuses on the epistemology and ways of coming to know (Ravn & Skovsmose, 2019). So, what is the subject to be known? Mathematics! There are five characteristics of mathematics (Linnebo, 2017): abstraction, idealization, computation, extrapolation and infinity, and proof. These issues are elements of imagination to. The epistemological perspective is about how humans can achieve knowledge of their surroundings, and thus centers on the human being as the starting point of any insight into the order of the world (Ravn & Skovsmose, 2019). The "ways of coming to know" are sciences of learning. So, learning Mathematics is the development of a particular type of imagination (Nemirovsky & Ferrara, 2009). Thus, taking imagination is a critical part of the educational process (Gallas, 2003). An Imaginative Curriculum would include the following features for the students (Jackson, Oliver, Shaw, & Wisdom, 2006).

1) Being imaginative (using imagination to think in ways that move us beyond the obvious, the known into the unknown, that see the world in different ways or from different perspectives, that take us outside the boxes we normally inhabit and lead to the generation of new ideas and novel interpretations).

2) Being original (making a contribution that adds to what already exists) by inventing, innovating, transferring and adapting.

3) Exploring for the purpose of discovery (experimenting and taking risks, openness to new ideas and experiences typically linked to problem working).

4) Using and combining thinking skills (for example critical thinking to aid evaluation, synthesis and intuition to interpret and gain new insights and understandings).

5) Communication – this is integral to the creative process (for example, storytelling as a means of communicating meaning within the discipline).

The first feature of an imaginative curriculum is inclusive of the others. The second feature which talks about "being original" is in line with the creative aspect of imagination and thus would enable students invent a 'Mathematics". The last three would go to the reproductive version of imagination. Hence, they would contribute for certain mathematical discoveries. As the students develop personal confidence and feel comfortable on the subject, they would be motivated to address their material to groups and to express themselves and their ideas with strong conviction. We can extract some features that qualify the curriculum as imaginative. For instance, if students are guided in such a manner, they will be equipped with communication skill and could explore for the purpose of discovery.

Conclusion and Implications

Learning Mathematics requires a chain of thoughts to be built and followed (Czarnocha, Baker, Dias, & Prabhu, 2016). The mathematical development of students posits issues of teaching, learning the mathematical contents, and school environment (Degu, 2015). Hence, the teachers of Mathematics and researchers in Mathematics Education are recommended to be engaged in creative enterprise. However, creativity cannot be thought of independent of imagination (Ayalew & Areaya, 2019). This article questioned about the relation between imagination and the philosophy of mathematics, if any. Although there is no single conception of the term imagination (Kind, 2016), the various thoughts in philosophical discussions can be grouped in to synthetic imagination and creative imagination. Then, the view of mathematics as discovered is related to synthetic imagination where, as the consideration of mathematics as invented, goes to the creative imagination. On the other hand, the sides of the philosophy of mathematics education have imaginative features.

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