MATHEMATICS TEACHING RESEARCH JOURNAL
Special Issue on Philosophy of Mathematics Education
Summer 2020 Vol 12 no 2

Philosophy, rigor and axiomatics in mathematics: intimately related or imposed

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Abstract: *One of the main tendencies of mathematical development in 19th and 20th centuries seem to be on rigor and formalization. Rigor and formalization took place on axiomatic basis leading to more abstraction. Euclidean type of an axiomatic model became a model of mathematics even for constructively developed analysis. Even though rigor and axiomatic method are different, and it is not necessary for rigor to be based on axiomatic method, in practice, rigor and axiomatics have been the requirement for valid mathematics. Different interpretations seem to be made about it. Some explain it as mathematical necessity and some relate it more as the result of philosophical underpinnings, especially that of formalism and foundationalism. Such situation motivated me to examine how philosophy, rigor and axiomatic are related. It seems that philosophy has distant but determining impression on the nature of mathematical knowledge, but rigor and axiomatics seems to be more internal to mathematics. Since such issues have more been associated to western mathematical traditions, it seems that such issues also need to be examined going beyond non-European traditions, such as, Hindu mathematical traditions which has significant contributions to mathematical development but which seems to have absence of axiomatic proof as well as philosophical presumption of absolute certainty of mathematical knowledge.*

Keywords: Rigor, axiomatic, proof, philosophy, formalism, foundationalism, certainty, metamathematics, Platonism, Hindu mathematics, Upapatti.

INTRODUCTION

Most mathematicians being less interested in the philosophy of mathematics indicates the foreign nature of philosophy to mathematics. Ever since the ancient development of mathematics, mathematicians seem to be less concern to the philosophy, as such situation is reflected in Socratic Dialogue (Reny, 1997), in which the ancient Greek philosopher Socrates mentions that the leading mathematicians of Athens did not know what their subject about; that is, about philosophy of mathematics. This implies that mathematicians do not know what their subject is about if they do not know about the philosophy of the subject. Plato, being so much devoted to the philosophy in general and to the philosophy of mathematics, seems to be much motivated by Socrates. The contribution of Plato as the Platonic thinking of mathematics (Platonism) descended through centuries with the contributions of the great scholars as the only basis of the philosophy of mathematics until recently. Through centuries mathematicians developed vast structures of mathematics which became most useful in technical and scientific endeavors, but the explanation/interpretation of nature of the truths of mathematics has mainly been the job of philosophers even though some well-known mathematicians have made philosophical contributions. Such situation yields a complex relationship between mathematics and its philosophy. Looking through such lens, it seems that mathematics is related to philosophy at least from educational point of view as interpreted recently, such as, by Paul Ernest (1991, 1997). An attempt is made to analyze such situation in the following sections of this writing.

On the other hand, The Greek concept of deductive/axiomatic method culminated in Euclid's Elements became a paradigm of mathematical certainty until the recent past. Euclid erected a magnificent axiomatic and logical system in his Elements which was taken as a sole model of establishing mathematical certainty until the end of 19th century. Perhaps the most evident modern feature of the Elements is axiomatic method, which is taken as the core of modern mathematics (Mueller, 1969). Rigor has been another requirement in formal mathematics. It has been common to speak of *rigor* in higher mathematics (university mathematics). But this term "Rigor" seems to be mostly accepted without any further discussions (as I feel). As I feel, the concept of *rigor* seems to be less apparent than that of *axiomatic.* So, in this article, an attempt has been made to examine the terms: rigor and axiomatic in some detail. How much rigor and axiomatization should be made seems to be dependent more on how mathematics is seen. And how mathematics is seen is more a philosophical question. This is why the paper seeks to examine the philosophical reflection on the rigor and axiomatic in mathematics. For it, we begin by considering some "Know-How?" as to *rigor* and *axiomatics* in mathematics so as to make its *philosophical reflection* meaningful. Hence, the rest of the portion deals with:

- Rigor,
- Axiomatic, and
- Philosophical Reflection (on them)

in sequence so as to examine how philosophy, rigor and axiomatic are related. In doing so, some pedagogical concerns are also being taken to clarify from educational point of view.

RIGOR IN MATHEMATICAL ARGUMENT

While looking up the precise meaning of the rigor in mathematical argument, I came to the article of Philip Kitcher (1981) which deals with the concept of the rigor in conventional way. Kitcher mentions that *central to the idea of rigorous reasoning is that it should contain no gaps, that it should proceed by means of elementary steps*. As mentioned, the argument of reasoning involves a set of premises, a finite sequence of statements and a conclusion. He mentions that the argument is rigorous iff the sequence of statement leads to conclusion and every statement is either a premise or a statement obtainable from previous statements by means of an elementary logical inference. Most possibly, formal system of Euclidean geometry which is a part of school mathematics curriculum might be one most familiar example of such structure. It is also well-known structure of formal and higher mathematics prevailing up to the day. It shows how the concept of rigor is associated to axiom (axiom/postulate) in formal mathematical derivation. (Later, consideration is made on how it is intimately related to fulfillment of philosophical purpose of absolute certainty of mathematical knowledge). One of the central ideas of rigor is that it should contain no gaps, which begs more clarification. Because mathematical argument is a kind of convincing argument required to convince the reader as to the truth of the argument, rigor in mathematics is aimed at establishing secure and consistent results through the sequence of logical argument which draws upon first principles (such as, axiom, definition and logic). More recently, wider basis of convincing arguments has been taken into consideration in which *cultural* and *psychological* basis in addition to *cognitive* basis are included. It depends on cognitive, psychological and cultural basis of the reader (Joseph, 1994:194) although standard textbooks writers on pure mathematics seem to be mainly based on conventional mode of cognitive basis. Unless and otherwise stated, rigor in mathematical derivations is considered in the line of conventional mode of thinking as has been used through long times. In the following para, one illustration is given to so as to clarify the situation.

To illustrate how formal deductive proof employs rigor in mathematical derivation, let me take one example to prove 1+1=2. To show **1 +1 =2** might be one example to show how the rigorous proof is basically based on assumption of axiom/postulate (along with the definition) and logical rules of inference (Ernest, 1991: 4-6) even to establish simple mathematical result. What became very noticeable to note to the students is that the proof needed to set up a set of rules in advance in the name of definitions, axioms and the rule of logical inferences as the rule of the game. And then the proof is developed step by step by using these assumptions as the rule forming 10 steps (Ernest, 1991: 5). The question may arise here: What does rigor indicate here and what advantages do we have by doing so tedious proof although most steps are in a sense familiar to students. Pedagogically, as I feel, it is a problem to convince many students to conceive and commit its ridiculous rigor. As I see, the rigor implies here is the development of consecutive steps without the lapses of reasoning in the formation of the argument. Alternatively speaking, the sequence of the steps in the argument should be capable of connecting statements to form the integrity of the proof. As Kitcher mentions there should be no gaps. But, the concept of rigor like other many concepts is a relative concept and may dependent on many factors. Ernest (1991) has developed 1o steps to show that **1 + 1 = 2**, but Reuben Hersh (1999) has used only three steps prove **2 + 2 = 4** (p. 254) which shows that rigor of proof also depends on the situation of consideration and other factors. But the traditional view of ideal mathematical knowledge can explain why such rigor is needed; according to which, by constructing rigorous proofs of known truths we improve our knowledge of these truths either by coming to know them as priory as certain, or , at least, making our knowledge more certain than it was before (Kitcher, 1981). Kitcher calls this type of mathematical thinking "Deductivism" which consists of two claims that all mathematical knowledge can be obtained by deduction from first principles and this is the optimal route to mathematical knowledge (which is less vulnerable to empirical knowledge). Kitcher mentions that although we can use deductive / axiomatic proof in mathematics, it lacks epistemological basis which the deductivists' theses attribute to the first principles. Lack of epistemological basis means that it cannot tell why the first principles are true and on which basis. Leaving aside to the first principles, a rigorous reasoning may be characterized not only as the description of the argument, it is rather a logical explanation which shows why the reasoning leads to true conclusion. In a sense, reasoning in deductivism (as mentioned by Kitcher, 1981) seems to be a narrow, linear and strict that based on first principles (basic assumptions) and proceed on logically so as to establish unique results. One may observe that so much constraints are imposed in mathematics, specially, in developing the proofs of special nature yielding unique results in almost linear consequences. Most probably, this is why mathematics has been so much useful in computer programming and formatting power of mathematics in a package (Skovsmose, 2009). Generally, deductivists' way is not the way how mathematicians produce mathematics; but mostly, it seems that it is the way to organize already discovered mathematical facts.

Traditionally, rigor in mathematics seems to be more dependent on Euclidean paradigm which laid a magnificent deductive/axiomatic structure which until the end of 19th century represented a sole model of mathematical thinking. The discovery of non-Euclidean geometries based on the logical consequences of Euclid fifth postulate laid a strong foundation for the development of deductive/axiomatic rigor in mathematics. Although, the development of non-Euclidean geometry was ingenious reconstruction in mathematics which opened new horizon in the field of mathematics as well as in philosophy of mathematics, it was limited to the field of geometry as the twin brother of Euclidean geometry. The great advantage of it was that it signified the sound basis of an axiomatic system and the basis of rigorous development in mathematical systems. The most important aspect of the rigor here indicates to seek for strict logical consequences of Euclid's fifth postulate (whose Playfair's version is that through the given point not on the given line, one line can be drawn parallel to the given line) and its two alternative postulates as the postulates of Non-Euclidean geometries. This may be one of the many reasons why rigor in mathematics got dependent on axiom/postulate. Probably, the great contribution of the axiomatic method goes to David Hilbert, the great mathematician of the first half of 20^{th} century as well as the great formalist philosopher, who formalized the Euclidean geometry. What is interesting to note is that the Euclid's Element defines basic geometric objects, such as, point and line, but Hilbert does not. Hilbert does not define the line, rather he axiomatize it: Two distinct points determine a line. Hilbert efficiency lies on recognizing that a line cannot be defined satisfactorily, rather it can be axiomatized. What is interesting to note is that the modern geometers realized that a line being so simple geometric figure cannot be given satisfactory definition. It is done so in an attempt of rigorization of Euclidean geometry. Here, the rigorization refers to the attempt to make Euclidean geometry free from logical weakness. Rigorization, in this sense, seems to be most dependent on axiomatic and logical inferences.

Recently, rigor in mathematics seems to be considered differently depending on the characteristics of the development of different areas of mathematics. Philip Kitcher (1981) critically examines on the non-deductivist consequences of mathematical knowledge, although he admits that even in non-deductivist theory of mathematical knowledge, deductive inferences are going to play important part. One of the key areas in the development of mathematics is calculus (which is developed as analysis) is taken as the study of limit and continuity (James and James, 1988). In the western mathematical development, Newton and Leibniz have been credited to introduce the subject of calculus. They introduced the calculus as a technique for answering questions on tangents to curves, maxima and minima, lengths of the curved arcs, area enclosed by curvilinear figures and so forth. Although, their predecessors had offered methods of answering some of the questions at least in special cases (such as, by Descartes and Fermat), they not only were able to treat a more extensive class of curves, but also they managed to give a unified treatment of problems which had previously been tackled by separate methods (Kitcher, 1981). To find the slope of the tangent at particular point is done by differentiating the function and evaluating it at the point. It gives the correct solution which is well known. As the process, the slope of the tangent is identified as the limit of $[f(x+d)-f(x)]/d$ as d goes to zero. What is then a problem as mentioned by Kitcher? The problem comes as to the value of d. As mentioned, d is small, or d goes to zero. If d is small, $[f(x+d)-f(x)]/d$ represents a cord rather than a tangent; and if d goes to zero, the function $[f(x+d)-f(x)]/d$ is ill formed. Referring to such cases, Kitcher mentions:

"Here we find a situation in which a demand for rigor may legitimately arise. … tells us that a certain type of reasoning will lead to true conclusions, but when we try to find a rigorous argument which shows us why that reasoning leads to true conclusions, we fail." (page: 10).

Kitcher shows how the attempts of rigorization of the concept of limit in the process of evaluating tangent to the function [such as, y = f(x)] created problems and failures in the original contributions of Newton and Leibniz. What is intended here to mention is that "rigor" here basically deals with strict reasoning; and looking through such reasoning, Newton and Leibniz fail to explain their problem of finding tangent although the result was correct. To give strict reasoning attempts were made throughout eighteenth century thinking that there must be an explanation, but Kitcher mentions that an appropriate explanation did not occur. Different interpretations were made including concept of non- zero infinitesimal. Later, Newton proposed a different strategy: he introduced the notion of limit and suggested that the tangent to a curve at a point should be interpreted as the limit of a sequence of chords, connecting the point to successively closer points. Strictly speaking, such interpretation is taken as more complicated than earlier because such approach adopts a kinematic view of geometry (where curves are regarded as swept out by the motions of points and the tangent is regarded as the line in which a generating point would travel if the instantaneous velocity were held constant). Newton's interpretation of the tangent as the sequence of cords connecting the point to successively closer points gives clear conception of the limit, but the problem is that it is a kinematic view of geometry. It is different area of cognition which originally departs from Newton's geometrical character. In rigorous explanation, such interpretations are not most welcomed because one need to cross beyond the area of consideration. Such kind of interpretations of the rigor is in a sense is a narrow band of restricted thinking because it impedes connection among different areas of mathematics. But what is also important to note is that the restriction or constraint involved in mathematical problems give rise to the development of mathematics as the history of mathematics tells. The restrictions on the solution of classical problems of mathematics (such as, trisecting an angle, doubling the cube and quadrature of circle) are such examples which engaged mathematicians to long time and gave rise to many mathematical developments. The development of analysis during the last two hundred years is said to make up it more rigorous by identifying and defining many terms and developing more satisfactory development of mathematical derivations and problems. Ultimately, constructively developed analysis has been put into deductive structures. Most mathematics have been transformed into deductive structures so as to make it more formal and rigor. Such representation of mathematics has made it more robust and rigor in the line of Bourbaki, but it seems to have been detached from its position for which it was developed and to which it was used. In the respect of rigor in mathematics, E.T. Bell (1934) made the following remark in his article " The place of rigor in mathematic":

“No mathematical purist can dispute that the place of rigor in mathematics is in mathematics, for this assertion is tautological, and therefore, according to Wittgenstein, it must be of the same stuff that pure mathematical truths are made.”

The statement clearly reveals that the rigor of mathematics for mathematical purist is inherited in mathematics, and it is the same stuff that pure mathematics is made of; that is, it is the same core element that the truths of mathematics are made of. (It is to be noted that Wittgenstein does not regard pure mathematical truths as the truth of absolute certainty, rather he considers it as the rule of the language game). Bell does not seem to be satisfied with the rigorous development of mathematics text for graduate students as the following dramatic scenario asserts:

“The present plight of mathematical learning-instruction and research- in regard to the whole question of rigor is strangely reminiscent of Robert Browning's beautiful but somewhat dumb little heroine Pippa in the dramatic poem Pippa Passes.”

The para reveals that mathematical rigor is beautiful like the heroine Pippa in the drama but not in real field of mathematics education. He further mentions:

“…to convince the student that the exponential limit is what it is by an argument that proves nothing but the author's mathematical ignorance and incompetence, reputable authors now state explicitly that a proof is beyond the capacity of the student at his present level.”

It is revealed while making review of college mathematics texts by mathematicians so earlier, but the situation does not seem to be much encouraging up to now. Many scholars and mathematics educators (of 20th century) have shown their concern on the difficulties in teaching and learning mathematics due to excessive emphasis on formalism which draws upon axiomatic and rigor. one who contributed more to make mathematics as an educational task is Hans Fruedenthal whose contributions (such as, mathematics as an educational task, 1973) critically examines the role of mathematics for educational purpose. The intuitionists, Richard Courant and Herbert Robins, in their epoch-making book “What is mathematic?” (1941) emphasis on intuitive and constructive nature of mathematics. Jersome Bruner, an American learning theorist and a well-known scholar expressed his anxiety over the frustrating situation in mathematics education bought by formalism. Bruner, in writing preface to Dienes book “An experimental study of mathematical learning” (1964), mentions:

"… I comment on the difference between Dr. Dienes and some of the others of us who have tried our hand at revising mathematical teaching. Perhaps it can be summed up by saying that DR. Dienes is much more distrustful of 'formalism' than some of the rest of us."

Such situation reveals that the more formal and rigorous development of mathematics has long been regarded as the hindrance in the development of mathematics education. As we realize that the future of mathematics depends to the large extent on its educational role besides its use in science and technology. The consideration of rigor in mathematics and mathematical proofs get its real problem when we come to mathematics education, more specially, to higher mathematics teaching at college and university classes. For many students, the concept of rigorous proof in mathematics seem to be less clear and vague as I feel from my experience of teaching courses in mathematics education to graduate and undergraduate students. Many students seem to have difficulties in understanding the essence of rigor in mathematical arguments involved in the proof. It seems that, specially, in the proof of common truths, such as, to show **1 + 1 = 2,** it becomes more confused. Almost students seem unclear or confused to develop the very understanding of the proof (the core of proof for the justification the truth) and Very few students are seen capable to give justification of the proof only after aided with prompts and hints (Shrestha, 2019). There might be many reasons behind it, but the main reason seems to lie on the difficulties of understanding caused by excessive rigorization (that consists of essence of proof) as demanded by mathematical formalization. Pedagogically, such rigorous proof for common mathematical facts have been seen less motivating and confusing for many students towards the formation of conception of the proof. Mathematics educators, since long ago, have suggested to avoid such proofs and to use constructive proofs so as to motivate students towards the proof in geometry. Kitcher also mention that we might adopt the hypothesis that the demand of rigorous proofs of known truths are just confused (Kitcher, 1981). But what is also noticeable is that proving known truths are also important from the purpose of developing mathematical rigor. The development of non-Euclidean geometry would not possibly have been made in 19th century if the Euclid's fifth postulate were not questioned even the truth of the postulate holds true for anyone in the flat plane.

Recently, it is recognized that a proof in mathematics is a convincing argument and it is based on cultural and psychological basis in addition to cognitive basis (Joseph, 1994: 194). Social constructivism as the philosophy of mathematics (Ernest, 1991,97) has given novel interpretation as to the genesis of mathematical knowledge. Lakatos' Proof and Refutation (1976) has been the source resource for the genesis of mathematical knowledge, which explains how mathematics develops into well-arranged formal rigorous forms. Such interpretations have given risen to alternative and constructive interpretations of the rigor in the formalization of mathematics. Such recent views of mathematics draw upon its history and philosophy in addition to ontology and epistemology. Looking through such lenses, the answer to the questions "what is rigor?" and "how much rigor is needed in mathematics ?" is not any already fixed/given discourse, rather it is dynamic discourse determined by the society of mathematicians for the purpose of the validity of mathematical knowledge in the existing situations.

As already mentioned, axiomatic has been one of the key characters of rigorous formal mathematics. As mentioned by Kitcher, even in nondeductivist mathematics, deduction has important role. The sequential nature of mathematical development and its consequential relations in addition to the nature of reality of mathematical truths to the large extent might be one reason why mathematics is based on rigid basis of axiomatic. As I feel, no one should only think that so much vast majority of mathematicians followed the trend in orthodox and blind way. They might have been motivated in different ways which have been the matter of examination more recently. It is necessary to examine the situation from different points of views. Looking through different points of view has been an emerging character of the recent times thinking which has been more inclusive thinking. More light has been shed on the nature of mathematics including axiomatic. Such considerations have highlighted the importance of philosophical perceptions needed to examine the issue of axiomatic. This is why the next consideration has been the axiomatic foundation of mathematics.

AXIOMATIC FOUNDATION OF MATHEMATICS

Let us begin with the dictionary meaning of axiom. The mathematics dictionary (James and James, 1988) states that the axioms of mathematical system are the basic propositions from which all the propositions can be derived. Axioms are independent primitive statements in the sense that it is not possible to deduce one axiom from the other. Historically, it is Euclid's scholarity that he gave five and only five postulates in addition to five common notions and about two dozen of definitions to begin to develop proof of the theorems. Mathematicians/geometers through many centuries suspected to the last fifth postulate and attempted to prove it. But what is interesting is that only in the 19^{th} century, they came to the conclusion that the fifth was a postulate independent of the other four and hence it could not be proved using others. Rather they discovered that alternative to fifth postulate, other postulate(s) could be stated which leads to other systems of geometries (such as, hyperbolic and elliptical geometries). Perhaps, the fifth postulate provided one great example of independent role of an axiom/postulate in mathematical structure. Since the conclusion of the proof of a theorem is logical implication of the truth of the axioms, such model of deriving truth is called axiomatic model; and both in the Elements and in a representative modern work like Hilbert's *Grundlagen der Geometrie* where one finds some sentences postulated as starting points and the rest derived from them (Mueller,1969). Ever since the importance to the development of mathematical theory, much has been written on the axiomatic method (Wilder,1967). Highlighting the excessive use of axiomatic in the 20th century, he mentions that except for the work of pioneers in the early part of the century, such as, Skolem and Tarski, most of the writings has been done so during the past 20 years, roughly speaking. Bourbaki is the pen name of the young French formalist mathematicians most active in the middle of the 20th century. Their version (1953) has been quoted by Weintraub (1998) in his article:

"From the axiomatic point of view, mathematics appears as a storehouse of abstract forms- the mathematical structures; and so, it happens without our knowledge how that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation."

The views of Bourbaki clearly mentions that the axiomatic basis of mathematical structures represent certain aspect of empirical reality. Such interpretation basically draws upon Platonic view of mathematics that mathematical truths are propagation of already existed reality. Let me quote the version of one of the grandmasters of the axiomatic mathematics, David Hilbert:

"I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms ... we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science." (As cited by Waitraub, 1998).

As mentioned by Hilbert, axiomatic method is not merely a means of establishing scientific reasoning, but by gaining dipper insight of scientific thought in form of axiomatic thinking, mathematics is summoned to leading role in science. Axiomatic method has also been used in social science, more specially, in economics in the assumption that the relationship between rigor and truth require an association of rigor with axiomatic development of economic theories since axiomatization was seen as the path to discovery of new scientific truths. The general form of axiomatic method as used in sociology applies to a set of propositions summarizing our knowledge in a given field and for finding further knowledge deductively (as cited by Costner and Leik, 1964). Such illustrations are mentioned here to show the influence of axiomatic formalization beyond the field of mathematics not only as the method of establishing truths but also as the path toward discovery. Since axiomatic method is intimately tied with the development of Euclidean, non-Euclidean and Hilbertian approaches, its consideration seems to be more meaningful when we compare them. As I think, the comparison of Euclidean approach (traditional approach) with that of Hilbert modern approach could shed more light on axiomatics. Some says they both are fundamentally the axiomatic formal structure while for some other it is not so because saying so would obscure the fundamental character of logical axiomatic development of modern mathematics. Some discussion is made as to it to clarify the situation. in this regard, scholars can also be divided into two groups.one who regards the ancient Euclidean geometry as the model of axiomatic reasoning though it possesses some lapses and shortcomings; and the other who see it as the model of intuitive thinking guided by the purpose of systematization without logical basis. Mueller, in his article" Euclid's Elements and his Axiomatic Method" has focused about it although he seems to belong to the later position. Referring to Szabo's position (1960), Mueller argued that Parmendes had a central position in the history of mathematics and the change from *empirical* to *pure mathematics* is closely connected with the idealistic, anti-empirical character of Eleatic and Platonic philosophy. This view clearly reveals that the thinking involved in the Elements was anti-empirical character of Eleatic and Platonic philosophy. But Mueller says such hypotheses are unnecessary and Greek mathematics in developed form cannot be justified as anti-empirical. He says although both in the Elements and modern work of Hilbert one find some sentences starting as postulate and the rest is derived from them, they differ in deep: Modern mathematics, such as Hilbert geometry is a formal-hypothetical character of modern axiomatic, but the Greek mathematics was not a hypothetical science in this sense; for Greeks mathematical assertions were true and of interest only because they were true. This seems to be one of the fundamental differences. This is why because the Euclidean axioms/common notions are taken as self-evident truths. They are not as the assumptions of modern axiomatic which are held for mere logical existence. Rather, they might have been representative of the basic truths they perceived physically existent. Mueller has considered in detail throughout his article to show how Euclid's concept of first principles (common notions, postulates and definitions) differ fundamentally with that of their modern counterparts. As an example, he compares Euclid's postulate with Hilbert axiom for line. He takes Euclid's postulates about line:

"Let it be postulated to draw a straight line from any point to any point" and

"To extend a limited straight line continuously in a straight line."

He then compares with modern counterpart:

“For any two points (distinct pints) there exists exactly one straight line on which they both lie.”

What he mentions is that the Euclidean postulates are not existence assertions like that of modern counterpart. Rather, they might be called licenses to perform to perform geometric operation (such as, performing operation of drawing a line). He mentions that such kind of operation have been used in the proof of some proposition (e.g., On a given straight line to construct an equilateral triangle). Commenting on the proof of such proposition, he writes:

“A Euclidean derivation, then, is a thought experiment of a certain kind-an experiment intended to show either that a certain operation can be performed or that a certain kind of object has a certain property. Thus, Euclidean derivations are quite different from Hilbertian ones, which are usually said to involve no use of spatial intuition.”

If Euclid’s Elements is not comparable to modern formal axiomatic development of mathematics, how could we explain Euclidean scholarity of recognizing the existence of five and only five postulates in geometry so earlier when so many scholars through long time suspected on the existence of the fifth postulate?

As I feel, some discussion on the above query might be helpful to examine the status and role of the Elements in mathematical development. Euclid’s fifth postulate attracted so many scholars since so long time so as to examine whether it was a theorem to prove or it was an independent postulate. Mathematicians ultimately became able to conclude that it was postulate independent from the other four postulates and the result was the discovery of non-Euclidean geometries in addition to Euclidean geometry. Then the subsequent query arises: How could Euclid/Euclideans see the necessity or the sufficiency of the five postulates so much remote even to the modern mathematicians? As I think, among many factors, the two considerations might explain something more. One, in the line of Mueller view (as mentioned above) as intuitive thinking aided with rational generalization. The next one, I want to mention here relate to seeking valid and consistent foundation so as to set mathematics as most trustful and worthy discipline. R. L. Wilder’s article (1967) “The role of the Axiomatic Method” has been useful here for such purpose which focus on hereditary stresses (internal needs of mathematics). As to the hereditary basis of the development of axiomatic method, Wilder write:

“In studying the evolution of mathematical concepts, I have found it convenient to distinguish between those cultural stresses which influence the development of concepts, according to their ‘hereditary’ and their environmental aspects. For example, the inception of most early mathematics, such as arithmetic, and geometry in its primitive forms, was due to environmental forces; but the axiomatic method occupies a unique position in that it appears to have owed its inception chiefly or even entirely to hereditary stresses. In other words, the factors which forced its creation were based on internal rather than external needs."

The hereditary stresses as mentioned by Wilder indicates that the axiomatic development of ancient Greek mathematics is most probably based on to cope with paradoxes (as those of Zeno) which compelled the formulation of a basic set of principles upon which to erect the geometrical edifice in form of the Elements. In such sense, it is said that the chief role played by axiomatic in Greek mathematical development lies on providing consistent foundation. What seems to be noted is that the search of consistent foundation needed a logical methodological aspect which ultimately made mathematics subordinate to logic. Such necessity might have paved the way toward the development of first principles on which to erect formal system. Since the first principles were thought to be the minimum principles of independent nature on which the rest derivations needed to be drawn, after much exercises, Euclideans might have ultimately been able to select the five postulates.

The history of mathematics reveals that the inception of early mathematics to the large extent is driven by the need to cope with the physical environment and hence the inception of most early mathematics, such as, arithmetic and geometry, was due to environmental forces. Wilder mentions that Greek geometry even after it was formulated as an axiomatic system, was apparently considered as science of physical space. At the same point, Mueller's view reveals that this is why Euclid's axiomatic development of mathematics basically draws upon empirical or intuitive phenomena not based on logic although logic might be used here. Considering the nature of Euclid's first principles (postulates, assumed assertion or common notions, and definitions), he insists that they are taken as things agreed upon for the sake of an unconfused development of mathematics. Mueller does not say that Euclid's mathematics does not possess logic rather he insists that it was not logically based.

More or less, just reverse to Mueller's position, Wilder points out that the Greek mathematics performed the role of providing foundation as well as consistency. Highlighting the importance of axiomatic method in the development of critical movement and abstraction, Wilder writes:

" ….. we find the axiomatic method playing a new role, namely that of introducing increasing abstraction into mathematics. E. T. Bell [1; p. 239] dates the period from 1879 to 1920, placing the emphasis upon the theory of groups which during this period received a thorough axiomatic treatment."

Wilder says the works of Grassman, Pasch, Peano, Padoa, Hilbert, and Russell, are a few of those involved who contributed to the evolution; and by the early part of the present century (20th century) the notion was certainly recognized if not completely assimilated as evidenced in the works as those of Frachet on abstract spaces, E. H. More in general analysis, formulation of theory of groups, and Housedroff's axiom in a general topological space. Such a view to the large extent falls on foundationalism which believes on firm foundation of mathematics. But both hold the view that the development of Euclidean axiomatic is motivated by internal necessity. H. F. Ferh's (1973) view as quoted by Clements and Battista (1992) mentions how geometry was developed from empirical processes into theoretical structure:

"Arising out of practical activity and man's need to describe his surrounding, geometric forms were slowly conceptualized until they took on abstract meaning of their own; thus from the practical theory of earth measures they developed a growing set of relations or theorems that culminated in Euclid's Elements, the collection, synthesis and elaboration of all these knowledge."

The above paragraph, Ferh's view summarize very shortly how the science of geometry came into being through a long historical development. What seems to be is that we make distinct levels/category of thinking hierarchy mostly for our convenience. But the nature of development of thinking, such as, informal and formal, seems to be of more continuous than distinct. The modern formal approach to mathematics certainly has nature of distinct feature in comparison to its early counterpart. The modern mathematics approach of formal mathematics (such as, Hilbert's approach to geometry, category theory etc.) is different from Greek approach of the Euclid's Elements. As already mentioned, Euclid's postulates seem to be self-evident truths as determined by the convenience of truth perceived from empirical activities (or intuition based on perceptions) while modern axiomatic, as those of Hilbert axiom (e.g., two distinct points determines only one line) is based on assumption. But, one question may arise: How could Hilbert state such axiom? There might be many guesses for it. What is commonly assumed is that a mathematician, generally, does not come to conclusion at once in vacuum. He/she makes hunches, guesses, trials, and see its compatibility to already established theories; and he/she goes back and forth to see whether a new concept/axiom/ theorem leads to acceptable consequences as perceived in Lakatos' proof and refutation (1976). Such perception which is most active recently relate how theory draws upon empirical or intuitive phenomena in recursive ways. Similar situation might hold in the development of Hilbert perceptions. Seeing the logical shortcomings and lapses in Euclidean geometry (as seen from strict logical point of view) and keeping in mind different spaces, he might have framed his structure. The Hilbert axiom for Euclidean geometry regarding the line seems to be given in the respect of the plane as that of flat surface. Today, mathematicians can construct hypothetical structures, such as, structures in N-dimensions and the requirement of any hypothetical structure need not necessarily be isomorphic to any physical structure. After the discovery of non-Euclidean geometries, mathematicians could say that they could develop as many geometries/ mathematical systems (consistent system) as they want by laying down required sets of axioms/postulates. But what is also remarkable is that the development of the models (such as, models for non-Euclidean geometries: Beltrami model) was so much important for its significance. What is intended to mention here is that Euclidean sense of axiomatic/deductive mathematics is different from modern approach of axiomatic/deductive mathematics. Euclidean derivations are said to be quite different from the modern approach in the sense that Euclid's axioms and derivations involve intuitive and spatial intuition whereas modern one (such as, Hilbertian approach) involve no use of spatial intuition. Modern formal mathematical derivations are intended to show off their representation without spatial intuition. Pedagogically, this is what mathematics has become less appealing to many learners. Mathematicians/geometers can work without intuition (after having enough intuition and observation which they have already had) or they might think so and students could be trained to do so. But, at least, pedagogically, it would be harmful to the learner to prevent them not to involve in the use of spatial intuition or physical representation for both axiom and axiomatic conceptualization of mathematics. Axiomatic and rigorous development of mathematics in the line of establishing absolute certainty of mathematical knowledge that took place in the past two centuries separated mathematics from which it was developed and to whom it was developed. Against excessive emphasis on rigor, axiomatic and formalism, new thinking has emerged which guides us to look back to historical development to dig out its meaning and making mathematics as dynamic interplay among its counterparts. In the upcoming section, rigorization and axiomatization of the mathematical practices are seen from philosophical point of views.

Philosophical Reflection Philosophy is considered in different ways and it is also taken to be a projection/reflection on what is mathematics about. The philosophy of mathematics generally does not treat specific mathematical questions, rather it attempts to present thoughts which come through reflection on what mathematics is, what the mathematician does, and to state the present state of affairs in mathematics (Lorenzen, 1960). It is not mathematics rather it is about mathematics as mentioned by Reny in Socratic Dialogue (1997). As mentioned in "A Socratic Dialogue in Mathematics", Socrates, in course of dialogue with his friend Hippocrates mention that the mathematicians of the Athens (of that time) did not know what their subject is about. The dialogues in essence reveal that the philosophy of mathematics is about mathematics. It is about reflection on what is mathematics about when it is looked from outside or from distant view. It might be one of the reasons why Rueben Hersh in the preface of his book “What is mathematics really?” (1999) says that Richard Courant and Herbert Robbins in their book “What is mathematics?” (1947; revised by Ian Stewart, 1996) do not answer to the question as raised by the title of the book. This is the book praised by Albert Einstein as the lucid presentation of the fundamental concepts and methods of the whole field of mathematics. Herman Weyl praised it as the work of high perfection and astonishing to what extent what mathematics is. What I am intending to mention here is that although the book has been very helpful/useful to understand many fundamental concepts of mathematics, it does not explicitly deals with the question “what is mathematics?” because it is basically a question that deals about the nature of mathematics; a question of philosophy. The task of philosophy is to reflects on and accounts for the nature of mathematics (Ernest, 1991:3). The main task of this section is to consider philosophical perception with regard to axiomatic method and rigor in mathematics. Specially, it is intended to examine the impression of philosophy in rigorization and axiomatization of mathematics. For that purpose, Mueller and Wilder's perceptions (as mentioned by them in their articles) in addition to other references are being taken into consideration from philosophical point of views.

In studying the evolution of mathematical concepts, R. L. Wilder(1967) distinguished two type of influences: one cultural influences which he referred as “Hereditary” , and the next as “Environmental” in which the development of most early mathematics, such as, arithmetic and geometry in its primitive forms was due to environmental aspects whereas the axiomatic development (as those of Euclid's Elements) was incepted chiefly due to hereditary stresses. Here, the hereditary aspect indicates that the arrangement of mathematical concepts in form of structures (such as, Euclid's Elements), was mostly forced by their internal needs to cope with paradoxes (such as, Zeno's paradoxes) and the problems such as, the problem of incommensurability of the side and diagonal of a rectangle/square. Such types of problems and paradoxes could not be solved empirically because there is no incommensurability between the side and diagonal of a rectangle empirically. Such situation is said to compel the development of systematic reasoning based on self-evident truths. Wilder writes “most historians seem to agree that crises, attendant upon the attempts to cope with paradoxes such as those of Zeno, compelled the formulation of a basic set of principles upon which to erect the geometrical edifice”. Such consideration indicates that the development of axiomatic system is chiefly determined by its own necessity. The axiomatic method is, without doubt, the single most important contribution of ancient Greece to mathematics which deals with abstractions and which recognizes that proof by deductive reasoning offers a foundation for mathematical reasoning (Kleiner, 1991). He also mentions that various reasons-both *internal and external* to mathematics have been advanced for the emergence of the deductive method in ancient Greece. Both Wilder and Kleiner attribute the development of axiomatic method to the necessity of mathematics. Mueller's position is somewhat different from that of Wilder and Kleiner's, but they all seem to be internalist in the sense that they all share the position that the development of Greek's axiomatics is basically oriented by the need of order and systematization.

Mueller, throughout his article critically counters Szabo's position that the change from empirical to pure mathematics is closely connected with the idealistic, anti-empirical character of Eleatic and Platonic philosophy. He instead insists that a Euclidean derivation, is a thought experiment of a certain kind, an experiment intended to show either that a certain operation can be performed or that a certain kind of object has a certain property, and hence Euclidean derivations are quite different from Hilbertian ones, which are usually said to involve no use of spatial intuition. Unlike Szabo, he differentiates between Euclidean and Hilbertian mathematics. He explains that the development of modern mathematics which is based on logic and independent of spatial intuition or thought experiment is related to the Platonic philosophy. He mentions that the evolution of the axiomatic method is explicable solely in terms of the desire for clarity and order in geometry while the philosophical conceptions of mathematics, such as those of Plato and Aristotle, were more probably the result of philosophically colored reflection on mathematical practice than causes of that practice. Keeping these views in mind together with his view that the development of modern axiomatic is motivated by the philosophy of mathematics, it seems that it represents a line of thinking which sits between Wilder's view and that of Szabo's view that the development of axiomatic method is closely connected to the philosophy. Such situations indicate that there does not seem much debate that the development mathematics is mostly guided by its necessity of systematization and validation, but there is much debate as to the role of philosophical thinking on the causes of mathematical practices.

When we come to the development of the three schools of foundationalist philosophy where the well-known mathematicians were involved to repair the foundation, philosophy of mathematics got more closure to mathematics. The 20th century development of the three schools (logicism, formalism and intuitionism/constructivism) were mainly guided by the purpose of establishing firm foundation of mathematics for absolute certainty. Paul Ernest (1997) has termed them as mathematicians working to lay firm foundation of mathematics although they could not do so even after a great deal of attempts. Godel's Incompleteness Theorem checkmated foundationalists program, specially the Hilbert program (Hersh, 1999: 138). If the philosophers other than mathematicians were involved in the development of philosophy of mathematics, the case could have been different. In other sense, it can be said that the involvement of well-known mathematicians in the establishment of the foundation of the philosophy of mathematics created a new situation in the relationship between mathematics and philosophy. The maverick philosopher, Reuben Hersh writes about the Hilbert's motive toward the philosophy of mathematics. For that he quotes Hilbert version in his book "What is mathematics really?" (1999):

"I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. … Having constructed an elephant upon which the mathematical world would rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling" (p.151).

Such version reveals motive behind Hilbert's formalistic philosophy as the means of achieving certainty in mathematics as well as his unsuccess in laying a firm foundation for it. The last sentence clearly states that the axiomatic foundation of mathematics is not fulfilled. Such motive might be indicative of intention rather than internal necessity. This is why the connection of axiomatic method with that of philosophy of mathematics is under debate. In a sense, axiomatic method seems to be intimately related to mathematics as of hereditary stresses as demanded for its own sake. In other sense, it is not genuinely inherited in mathematics because it is not the way to construct mathematics, but a rather a method to organize already discovered facts. One can think that the necessity of organizing already discovered facts by using an axiom/postulate and deductive logic is a natural way of arranging mathematics. This is natural in the sense that it is driven by its necessity. Up to now, axiomatic method has been taken into consideration. Onward now, rigor is being taken up.

Let us begin with the Kitcher's characterization of the rigor. Kitcher (1981) writes "central to the idea of rigorous reasoning is that it should contain no gaps, that it should proceed by means of elementary steps." Further clarifying the situation, Kitcher writes:

"Conceive of an argument as a triple, consisting of a set of premises, a conclusion, and a finite sequence of statements. The argument is rigorous if and only -if the sequence of statements has the conclusion as its last member, and every statement which occurs in it is either a premise or a statement obtainable from previous statements by means of an elementary logical inference."

Kitcher's criterion is normative in the sense that it gives criterion to what the rigorous argument should be. The statement reveals that the rigorous argument should proceed by means of elementary steps which are ultimately based on premises/first principles and with reasoning of logical *coherence*. Looking with this criterion, rigor is based on axioms/postulates in addition to other requirements, such as, logical *coherence*. This shows how rigor and axiomatic have been integral components in the convention of mathematical argument. The utmost emphasis was laid on axiomatic and rigor so as to develop proofs. So, if the development of the three schools and more specially the formalism and logicism are considered, rigorous development of mathematics has been the main function of the philosophy. Mathematicians, like David Hilbert and L. E. J. Brouwer and philosophers like Bertrand Russell and Whitehead devoted themselves in philosophy to lay firm foundation and to give satisfactory explanation of mathematical knowledge. Rigor in mathematics is made possible to mathematical argument with the use of first principles and logical sequences containing finite steps leading to conclusion. It shows how the concept of rigor is associated to axiom (axiom/postulate) in formal mathematical derivation and how it is intimately related to fulfillment of philosophical purpose of absolute certainty of mathematical knowledge. Hilbert formalism represented the excessive rigorization of mathematics where for the purpose of consistency of mathematical structures, mathematics was given meaningless interpretation. Hilbert definition of mathematics as the game played with meaningless marks and following the simple rules, is an example. A great mathematician of the first half of the 20th century seems to do so only for the purpose of establishing consistency in mathematical reasoning. As mentioned by Hersh, he wanted certainty in mathematics in the kind of way in which people want religious faith. In a sense, he did so for the rigorous development of mathematics. L.E. J. Brouwer being foundationalist contrasted with Hilbert mainly due to such position of Hilbert. Brouwer followed Kant's intuitionist theory that mathematics is founded on intuitive truths. Since he focused on construction of mathematical knowledge, it became constructivists' basis of mathematical derivation under foundationalism. Such situation reveals that the rigor in mathematical development based on formal axiomatic seems to be more external requirement or condition imposed on mathematics for the purpose of absolute certainty of mathematical knowledge.

In the above discussion formalism, rigor and axiomatic seem to be as characterized by Kitcher. It seems that Kitcher did so following the normative basis of rigor. Rigor is not only dependent on axiomatic as illustrated by Kitcher in the interpretation of derivative by Newton and Leibniz, the founder of the calculus. Kitcher shows the lack of an appropriate interpretation of the derivative as the rigor (as consider under the section “rigor”). What is intended to mention here is that rigor might be considered differently in calculus and analysis. The arithmetical interpretation of the notion of integration independent from geometrical intuition might be considered rigorous in the sense that it further clarifies the concept. The development of calculus in 19th century seems to make many concepts clear by formulating concepts through definitions. The intuitive concepts were explicitly formulated that helped to conceptualize them precisely resulting the further conceptualization of the subject. Such cases might represent rigor in non-axiomatic setting although the analysis today has taken the axiomatic form where the subject begins with definitions and axioms/postulates. If we consider rigor for its clear and precise meaning as considered in calculus/analysis, it might seem more internal to mathematics and independent from philosophy. But, when it comes to base it on first principles, it is motivated philosophically too.

Although, mathematics as an absolute body of knowledge has been more controversial, mathematics as an objective social reality has been more accepted. In this respect N. G. Goodman (1979) writes that a philosophy of mathematics is closely analogous to a view about the nature of material objects. Such considerations indicate that philosophy of mathematics basically deals with the nature of mathematics and reflection on mathematical practice. For many mathematicians, the axiomatic method is used for orderly and unconfused development of mathematics which in turn requires rigor. Let me cite some more examples to clarify the situation. Peano's postulates were drawn in an attempt to define the natural numbers. What is notable is that the well-known set of counting numbers needed to be axiomatically presented. Euclid gave definition to a line, but Hilbert axiomatized it by saying " Two distinct points determine a line". Hilbert's efficiency lies on the recognition that the line cannot be defined satisfactorily as did by Euclid, rather it can be axiomatized. From modern perception, Euclid's definition is not satisfactory. This is because the geometrical objects, such as, point and line are conceived as undefined terms. As I think, there might be interpretations which show that they cannot be defined satisfactorily. If we define the point as an entity which has no length, breadth and thickness, but only position (as did by Euclid), then the question might rise "What is that object which has position but not the dimensions?" If so, how could we interpret a line which contains an infinite number of points. In modern Euclidean geometry, a space is a set of points. If Euclidean definition of point is taken, then it does not lead to conceptualize the space which contains a set of points. A straight line as perceived by Euclid, is a line that is formed as that of stretched form of very thin thread. What did mathematicians recognized is that the straight line seems to be simple to visualize but difficult to define. Then they might have decided to axiomatize it as did by Hilbert that for any two points there exists exactly one straight line on which they both lie. The axiom states the existence of the line rather than saying what line is about. Euclid's definition of a line as breadthless length seems to be an intuitive idea for pedagogical nature, but it does not define line.

Many references imply that the development of Greek axiomatic method is closely connected with the dialectical method in Greek philosophy. Socrates claim that truth can be reached only if one searches on the basis of accepted hypotheses although they are accepted temporarily. Referring to the Posterior Analytics, Mueller quotes Aristotle's view that insists that the assumptions of a science be not merely true, but also primary, immediate, and more known than, prior to, and causes of the conclusion drawn from them. Such views indicate the influence of philosophy on axiomatic method and on the development of rigor in modern mathematics. But the question arises: Is mathematics intimately related to philosophy? I think, the question of intimacy is relative one and it is an issue to be under debate. Much study might be needed to through light on it. But what seems to be more probable is that philosophy cannot be put entirely apart from mathematics because mathematics alone cannot tell its story as to how and why it came to us what we have today. The recent studies on philosophy have given more comprehensive interpretations as to the relationship between mathematics and philosophy. One most important interpretation is that mathematics is an intellectual cultural product. Such cultural interpretations have lent to the new recognition to the mathematics and its contribution of non- European mathematical developments. The result is that mathematics of non-European civilizations, such as, Hindu mathematics (commonly represented as Indian mathematics) has got more importance and validity. This is why because Hindu mathematics though being rich and significantly contributing to the development of mathematics, it lacked formal axiomatic proof in mathematics. What is interesting to note is that mathematics can flourish without the use of axiomatic rigor. Some para has been devoted about it so as to examine the nature of relationship of the philosophy with axiomatic and rigor in mathematics. Lack of proof has been a common charge against Indian mathematics as mentioned by C. B. Boyer in his book "History of Mathematic" (1986). The neglect of Indian mathematics can be seen:

"Although in Hindu trigonometry there is evidence of Greek influence, the Indians seem to have had no occasion to borrow Greek geometry, concerned as they were with simple mensurational rules" (P.238).

Indian historian of mathematics has remarked that the great charge against Indian geometry, in particular, and mathematics in general, is the lack of proof (deductive/axiomatic) that was so much beloved to the ancient Greeks (Amma, 2007: 3). Sarasvati Amma further mentions that the splendid achievement of Greek geometry was ignored by Indian scholars until 18^{th} century by saying: "It was only in the 18th century, nearly 2000 years after the active contact of the Indians with the Greeks that Euclid's elements were translated it to Sanskrit." (p.4) Recently, such questions have been critically examined to examine the status of proofs in Indian mathematical traditions. To clarify the situation in this respect, the following version mentioned in the Ganita-Yukti-Bhasa (2008) seems to be useful:

"Many of the results and algorithms discovered by the Indian mathematics have been discovered in some detail. But little attention has been paid to the methodology and foundations of Indian mathematics. There is hardly any discussion of the processes by which Indian mathematicians arrive and justify their results and procedures. And, almost no attention is paid to the philosophical foundations of Indian mathematics, and the Indian understanding of the nature of mathematical objects, and validation of mathematical results and procedures" (P. 267).

The above version clearly indicates that in several books on the history of mathematics, much space has been given for the results and algorithms discovered by the Indian mathematicians, but there is hardly any discussion of processes, and almost no attention is paid to the philosophical foundation.

There are many aspects of mathematics developed in Indian subcontinent (south Asia). To make very short review as to is philosophical position on mathematics and rigor, we focus on reasoning involved. In the history of mathematics in the Indian subcontinent, much attention has been given to very large numbers. Ernest (2009: 200) speculates that much attention on the large number (which might have been possible with the advantage decimal place value system) with the decimal fractions might had helped to conceptualize a large number of series expansion in Kerala, India and contributed much of the basis for the calculus, which is traditionally attributed to 17^{th} 18^{th} century European mathematicians. As mentioned by Sarasvati Amma, deductive-axiomatic proof has been taken as the great lapses. The questions arise then: If so, does not Indian mathematical development contain valid and rigorous proofs? and then What is philosophical basis behind them? To examine such situation in few lines, we begin by reviewing reasoning involved for justification which is called "Upapatti": kind of convincing argument.

Upapatti seems to be developed as a means of convincing arguments to intelligent students of the validity of the theorem so that visual demonstration was quite an acceptable form of proof in geometry (Amma, 1999: 3). This is compatible with the recent views of thinking and teaching mathematics in which proofs are recognized as convincing arguments in constructive ways and not only in the classical ways of formal deductive structures which is educationally unsound. The great teachers of Hindu mathematics wrote commentaries of original texts which included upapattis for the results and procedures enunciated in the text and the explanation of the rational were left to oral instruction. The oral instruction might have played significant role in transmitting knowledge and the loss of such tradition (pedagogy) brought about by foreign invasions might have resulted the loss of transmission of knowledge through generations. It seems that it was one of the reasons of Europeanization and ultimately toward the direction of Eurocentrism in mathematical thinking.

Upappati as a means of establishing validity of mathematical truths and removing doubts: The Indian mathematicians are clear that results in mathematics cannot be accepted valid unless they are supported by Yukti(scheme) or upapatti (Ramasubramanium et. al., 2008: 288). In this respect as a commentator, Nrsimha Daivajńa explains that Phala (object) of upapatti is Pánditya (scholarship) and also removal of doubts which would one to reject wrong interpretations made by other due to Bhranti (confusion) or otherwise (as cited by Ramasubramanium et. al; 2008: 286). Similar view is expressed by Ganeśa Daivajńa in his preface to the commentary on Bhaskaracharya's Lilavati that:

"Whatever is discussed in the vyakta or avyakta branches of mathematics without upaptti it will not be nirbhranta (i.e., free from misunderstanding). It will not acquire any standing in any assembly of scholars' mathematicians. The upapatti is directly perceivable, like looking in a handy mirror. It is therefore, to elevate the intellect (buddhi vriddhi) that I proceed to enunciate the upapattis" (as cited by Joseph, 1994: 200; Ramasubramanium et.al., 2008: 286).

The above version of Ganesa reveals that the Indian epistemological position on the validation of mathematical knowledge is unique. Establishing the validity of mathematical knowledge by general agreement among the Indian mathematicians (agreement as to what an upapatti is) seems to be quite unique in Hindu mathematical development. Such view of mathematical thinking seems to be similar to some extent with quasi-empiricist view of mathematics which states mathematics is a dialogue between people tackling mathematical problems (1976). The quasi-empirical nature of Indian mathematics (to some extent) lends itself analogues to natural sciences. Such situation might have occurred due to the development of mathematics for astronomical and sacrificial needs.

The method of proof by contradiction is used only to show the non-existence of certain entities (Joseph, 1994: 201). But unlike tradition, there are no upapattis which support to establish existence of mathematics object merely on the basis of reason alone (Ramasubramanium et. al., 2008: 289). As an illustration, the upapatti to show that a negative number has no square-root as given by Krishna Daivajna (16th century AD) had used such proof style (Joseph, 1994: 201). Problem solving seems to be the major focus rather than proving assertions. Method of contradiction and indirect proof were not encouraged. Even among 20th century foundationist, intuitionist/constructivist did not accepted method of contradiction. The type of convincing arguments used by Medieval Hindu mathematics for the validation of mathematical truths seems to be somewhat similar as has been advocated by social constructivists.

Philosophical perception in Hindu mathematics does not assume the concept of absolute certainty, but mathematics is regarded as supreme of all the secular sciences. It seems to me that ancient Hindu scholars did not attributed absolute certainty to any secular sciences since for them there was something above it to which it deserved. Such situations seem to reveal that mathematics has objective certainty and its development is dependent on own historical and cultural basis. Since mathematical methods (such as, axiomatic /heuristic method) and rigor are social and intellectual products dependent on objective basis, they are associated to philosophy of mathematics which throws light over them.

CONCLUSION

Rigor and axiomatic seem to be central to mathematical proofs. Since it is guided by the intention of flawless derivation of mathematical truths based on premises (such as, axioms/postulates), they seem to be more internal features of mathematics. In a sense, the development of both axiomatics and rigor seem to be guided by the purpose of the development of valid mathematical systems as in the line of pioneering work of the Euclid's Elements. In other sense, the unique type of the axiomatic based model of rigorous development of mathematical system seems to be the result of ancient Greeks' intellectual and cultural tradition which is motivated by the philosophical thinking of Plato and Aristotle. Looking in such ways, it can be seen to be two lines of thinking (as considered here): one, in the line, such as, that of Mueller which insists that the evolution of the axiomatic method is explicable solely in terms of the desire for clarity and order in geometry. Such view implies that the axiomatic based rigor is the outcome for clarity and order in mathematics, and the philosophical conceptions of mathematics, such as those of Plato and Aristotle, were more probably the result of philosophically colored reflection on mathematical practice than causes of that practice. In other way, the development of the Greek axiomatic method is closely connected with the development of the dialectical method in Greek philosophy (Szabo’s view). What seems to be commonly accepted is that the modern formal axiomatic development of mathematics is the new edition of axiomatic guided by purpose of establishing consistently true mathematical truths, which in effect, demanded rigor based on axiomatic and formal logic. It is used in mathematics as the sole method of formal valid deduction, and hence, it became an integral component in the arguments of higher mathematics. Although, 20th century foundationalists like Hilbert, Russell and Brouwer could not lay the foundation for the absolute certainty of mathematical knowledge, the higher mathematics today has been mainly guided by it.

Most professional mathematicians working in their fields seem to be no more concern of philosophical debates. To explain its cause, Sal Restivo (1994:216) writes, the grater the level of cultural growth, the greater the distance between material ground and its symbolic representation and the boundaries separating mathematics from mathematics worlds from each other and from the social worlds thicken and become increasingly impenetrable. Such interpretation explains why pure mathematics seem to be isolated from social world. Similar reason might apply to its relation to philosophy because philosophy is also an outcome of cultural growth. Such situations suggest that philosophy exerts a distant view on mathematics which seems to be remote but powerful and which provides norm for the rigor and axiomatic development of mathematics. Frege, Hilbert and Russell's views, in essence, are intimately related to Platonism (Hersh,1997). Hilbert seem to purposively impose his theory of metamathematics only for laying rigorous foundation for absolute mathematical certainty. But, the remarkable development of Hindu mathematics (Almeda and Joseph, 2009) without axiomatic rigor and without any well-formed philosophical presumptions might tell a different story of the mathematical development. Lack of axiomatic rigor did not impede the mathematical development in south Asian region (commonly known as Hindu mathematics). Without the use of axiomatic method, Hindu mathematicians (including others) contributed significantly to the development of mathematics. Even though Hindu mathematics did not have proofs based on axiomatic basis, it developed reasoning for the clarification and validation of mathematical truths in the name of *Upapatti*, which can be interpreted as convincing arguments. Recently, high recognition is given to non-European mathematical developments and it is due to the rise of new thinking which is mostly motivated by socio-historical and socio-cultural interpretation of mathematics. Such situations indicate that philosophy, rigor and axiomatic cannot be independent to each other, but how and how much related they are (intimately or to some extent) might depend on socio-historical development of mathematics. Since, this is controversial issue which needs to be further clarified on the basis of more extensive and intensive studies, this writing is made just to raise some queries.

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