On the networking of Husserlian phenomenology and didactics of mathematics

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Abstract: *In the spirit of the networking of didactical theories, it is advocated in this presentation in favor of a mixed networking of philosophical theories, namely Husserlian phenomenology and hermeneutics, and didactical theories to produce a fertile interplay between philosophy and mathematics education. The cross-analysis of students' work on a problem in Abstract Algebra leads to linking the notion of horizon of expectation developed by Jauss with Brousseau's didactic contract from the theory of didactical situations. It also opens up further perspectives of interrelations between dimensions of the horizon in the sense of Husserl and didactical constructs.*

INTRODUCTION

Ernest (2018) highlights in his synthesis three distinct directions of interplay between philosophy and mathematics education: philosophy applied to or of mathematics education; philosophy of mathematics applied to mathematics education; philosophy of education applied to mathematics education. In this study, we are concerned with phenomenology and hermeneutics as two branches of philosophy that provide theoretical tools, namely the notion of horizon and some of its variations, to be applied in the context of mathematics education. This research thus fits in the first case described by Ernest. Yet, it will also be argued, conversely, that didactical contexts and theoretical constructs may enrich philosophical accounts through such an interaction of both fields.

More precisely, the interplay between philosophy and mathematics education presented here – I will rather say "didactics of mathematics" since the main theory considered in the sequel takes its origin in the French tradition of the field – will be envisaged as a form of networking of theoretical frameworks (Bikner-Ahsbahs & Prediger, 2010). Indeed, the same educational phenomenon, namely how students solve a given problem in Abstract Algebra, is analyzed from the perspectives of both Husserlian phenomenology and hermeneutics, and the didactics of mathematics. In the case of networking of didactical theories, such a research practice allowed the combination of

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complementary insights but also led to the linking of theories at different levels (by comparison, contrast, synthesis, local integration, ...). We will take as a methodological hypothesis that it may also be applied fruitfully in the case of a mixed networking of philosophical and didactical theories.

The choice of a problem in Abstract Algebra – the so-called "theory of banquets" – originates from the research conducted by the author on the teaching and learning of algebraic structures (Hausberger, 2017). The idea of involving in mathematical contexts the notion of horizon of expectation, introduced by Jauss (1970-72) to account for the reception of a piece of literature, is due to Patras (2013) who applied it to discuss the historical significance of a mathematical text and methodological issues in the history of mathematics as a field of research. This research thus developed as a joint work and brought as a first result a relation between the didactic contract (Brousseau, 1997) and the horizon of expectation (Hausberger & Patras, 2019). The present state of this joint work will be presented synthetically in this communication, together with a reflexive point of view on the type of networking that was achieved : I will sketch the data that was submitted to the cross-analysis (the theory of banquets) and the philosophical tools that are used, then provide a synthesis of the ideas that emerged as fruits of the networking strategy, and finally engage in a meta-analysis of the research itself using a meta-tool: the "scale of networking stategies" (Bikner-Ahsbahs and Prediger 2010, p. 492; see also below).

THE THEORY OF BANQUETS: A DIDACTIC ENGINEERING

As a piece of didactic engineering (Artigue, 2009), the theory of banquets was built on the basis of an epistemological analysis of mathematical structuralism (Hausberger, 2017). The structure of banquets is therefore an invented structure, which bears some similarities with Group Theory but is much simpler and therefore allows an in-class discussion of the structuralist methodology through reflexive thinking on the assigned tasks. It must be taught after a course in Group Theory, so that students have already developed techniques to classify finite groups of small orders up to isomorphism, techniques which are to be thematized in the context of banquets.

A banquet is a set E endowed with a binary relation R which satisfies the following axioms: A1. No element of E satisfies xRx; A2. If xRy and xRz then $y = z$; A3. If yRx and zRx then $y = z$; A4. For all x, there exists at least one y such that xRy. In part 1 of the worksheet, students investigate the coherence and the independence of axioms, then they are requested to classify all banquets of small order ($n \leq 4$). The next sections are dedicated to the further development of the theory: notions of sub-banquet, irreducible banquet, structure theorem (a banquet is the disjoint union of tables) which corresponds to the well-known theorem of canonical cycle-decomposition of a permutation.

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The theory of banquets carries several phenomenological aspects, starting with its very name that brings an intuitive background and draws on the mental image of guests sitting around tables for a meal. This approach thus meets Freudenthal's (1983) point of view that mathematical structures organize phenomena and should be developed together with mental images and representations. It also aims at responding to Patras' (2001) critique, in the tradition of Husserl, of the gap between axiomatic presentations of mathematical theories in modern papers (and most textbooks on Abstract Algebra) and their underlying intuitive contents, which results in a loss of meaning in contexts of communicating, teaching or learning mathematics. Those phenomenological aspects are discussed extensively in Hausberger (2017, section 3).

The educational purpose of the theory of banquets is to facilitate the access to structuralist thinking. It has been experimented in the classroom but also with pairs of students in a research protocol. Several types of analyses may be performed: a semiotic analysis to gain insight how students make sense of an abstract mathematical structure, the role played by the mental image, the dialectic between syntax and semantics, that is how models are used in students' reasoning on the axioms to investigate and prove mathematical statements (Hausberger, 2016, 2017).

TOOLS FROM HUSSERLIAN PHENOMENOLOGY AND HERMENEUTICS

The notion of *horizon* was introduced in phenomenology by Husserl to account for the fact that multiplicity is inherent to intentionality, among others because synthesis always drives the unity of consciousness (Husserl et al., 1950, Sect. 18). According to Husserl, "this multiplicity is not exhausted by the description of actual *cogitata*", since each actual *cogitatum* has its own potentialities that, "far from being undetermined are, as far as their content is concerned, intentionally pre-traced in the current state itself" (ibid.). In other words, each state of consciousness has a horizon that accounts for the potentialities of consciousness. For instance, the expected ability of the students to solve an exercise is connected to and could not be understood without the existence of a horizon of their understanding of the content of the questions they have to solve. More generally, in mathematics, these phenomena relate to the fact that, besides being directed towards problems, objects, proofs, our consciousness is also shaped implicitly by the structural properties of the horizon in which they happen to be embedded.

The notion of *horizon of expectation* builds on the general idea of horizon by putting forward some specific features, particularly relevant when it comes to analyze aesthetic and cognitive phenomena. It has been developed largely in the hermeneutical context. The idea of linking phenomenology with hermeneutics owes much to Gadamer (1960), one of the most prominent theorists of philosophical hermeneutics who, as a student of Heidegger, added ontological features

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to the Husserlian phenomenological idea of horizon. Our interest will however focus on another theorist of hermeneutics, Jauss (1970-72). The work of Jauss and of the Constance School to which he belongs, contributed to put forward the idea that literature cannot be understood without taking into account the point of view of the reader. In other terms, the reader contributes to define the meaning of a poetry, a novel or an essay. To indicate how Jauss' ideas may be transported in the didactical context, analogical statements that refer to mathematical education are added inside brackets to the following quote (translated into English from the French edition).

> Even when it appears, a literary work [a mathematical exercise] does not present itself as an absolute novelty emerging out of a desert of information; there is a full game of announces, signals -patent or latent-, of implicit references, of familiar characteristics, that predispose its public to a certain mode of reception [...]. At this first stage of the aesthetical [didactical] experience, the psychological process of reception of a text does not reduce itself to the contingent succession of simple subjective impressions; this is a guided perception that proceeds according to a well-determined indicative scheme [...]" (Jauss, 1970-72, Sect. VII, French ed., 1978).

NETWORKING OF THEORIES: IN PRACTICE

The strategy is to analyze with the previous tools the didactical experience and reception, by pairs of students, of the worksheet on banquets. The data collected from the work of two pairs will be briefly presented first, in the form of the transcript of dialogues between students and a short account of semiotic representations introduced and used by one pair of students (Figure 1). Full transcripts may be found in (Hausberger, 2017).

Student A : Classical, the structure is specified by means of relations, that's it

Student B : Anti-symmetry [About axiom A1]

A : Not exactly, this is non-reflexivity; there is a single guy on the right and a single one on the left, that's the idea [laughing]; nobody is sitting alone on a table.

B : Elements are people? And they are related when together on a table?

A : Yes, that's it. The relation is to sit at the right (or the left). But you cannot have more than one guy on your right, and also on your left, there is at least someone on your right.

By contrast, students C and D tried to make sense of the axioms by searching for a form ("So, what does this structure look like?" asks C; "The ordering on the real numbers looks like this… the fact that **R** is archimedean… no, it's not" replies D). Their semiotic production (semiosis) to interpret the axioms soon turned in the direction of diagrams ("Globally, we have a point x that leads to y and to z, by necessity we have an equality"; left part of Figure 1). This process led to the drawing at the right of Figure 1 as a representation of a banquet of order 3.

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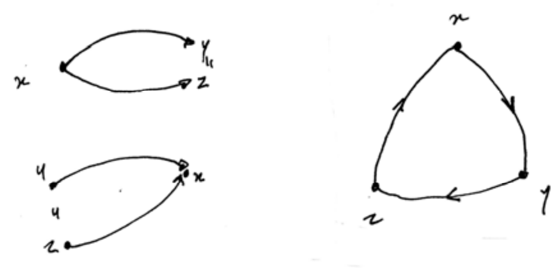

Figure 1: Semiotic representations produced by students C and D to make sense of a banquet

At this point, the interviewer chose to intervene in order to clarify the status of this diagram. It was found out that students saw it as an "idea of a model" rather than a real model, in the sense of Model Theory, constructed in the language of Graph Theory.

Teacher: What is, for you, the status of these drawings?

D: These two aim to make relations more explicit, I mean axioms A2 and A3, and this one (pointing to the drawing on the right) is a means for us to get an idea of a model that would resemble to this (pointing now at the axiomatic of banquets).

C: In the 3-case, rather.

Teacher: Do you know any mathematical domain in which similar representations are used?

C: Graphs

Teacher: Can we consider that this graph is a model of banquet constructed inside graph theory?

C: I don't see why it shouldn't be one.

D: a priori yes.

C: In the 3-case, yes.

D: Let's look at the case of 4.

In phenomenological terms, the (philosophical part of) the methodology of data analysis amounts to describing the components of the Husserlian horizon in which students work. The transcript of the dialogue between students A and B given above suggests several dimensions to account for: a *theoretical horizon* structured by previous knowledge (e.g. anti-symmetry, non-reflexivity) but also a *semantic horizon* connected to the idea of model (the mental image of guests sitting around tables). These ideas will be developed in a future work (in progress) with Patras, and connected to didactical analyses. In fact, our most elaborate results of mixed (philosophy and mathematics education) networking concern a third dimension of the horizon that may be called *didactical horizon* and consists in the horizon of expectation (in the sense of Jauss) shaped by the didactic contract (Hausberger and Patras, 2019).

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The didactic contract designates the "system of reciprocal obligation" that determines "explicitly to some extent, but mainly implicitly - what each partner, the teacher and the student, will have the possibility for managing and, in some way or another, be responsible to the other person for" (Brousseau, 1997, p. 31). According to Brousseau, the student's reception of an exercise is driven by his preconceptions on what he believes the teacher to expect. In hermeneutical and didactical terms, the teacher partly shapes the horizon of expectation of the student by the negotiation of the didactic contract in a phase called *devolution* of the problem. This doesn't mean that the contract will remain stable: "It is in fact the breaking of the contract that is important [...] Knowledge will be exactly the thing that will solve the crisis caused by such breakdowns" (ibid., p32). The didactic contract thus possesses shared feature with the Husserlian horizon: its under-determination, objectivity and its dynamical structure.

Indeed, in Phenomenology, the horizon of an intentional act is constantly changing and evolving. According to the Cartesian Meditations, it is an essential feature of consciousness that it can transform itself into new modes of consciousness and be however always directed towards the same intentional object. In such a situation, the object remains the same, but the horizon of the intentional act is evolving, and this evolution can be analyzed since implicit components of the initial horizon can be grasped in the new one. In Hermeneutics, this dynamical feature may be further described as follows:

> The relationship of an isolated text to the paradigm, to the series of prior texts that constitute a literary genre, is also established according to a permanent creation and modification process of an horizon of expectation. The new text evokes to the reader (or the listener) a whole set of expectation and rules of the game to which he has been familiarized by prior texts and that can be, along the reading, modulated, corrected, modified, or simply reproduced (Jauss 1970-72, p. 56).

In mathematics education research, accounting for the process of finding a contract may serve to model and explain the observations. For instance, graphical representations in standard mathematical didactic contracts are often not granted the status of genuine mathematical objects. This seems to apply to students C and D: the didactic contract specific to Graph Theory is not considered until the intervention of the interviewer. This is a phase of partial *institutionalization* (Brousseau, 1997) that allows to renegotiate the didactic contract, structure further the horizon of expectation and facilitate the development in the direction of a specific theoretical horizon. It is worth noting that students have firsthand adopted a scientific yet somewhat doubtful attitude ("don't see", "shouldn't", "a priori"). The success of the intervention can only be asserted when they engage further in the classification task by explicitly using the graphs' repertoire.

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NETWORKING OF THEORIES AND PHILOSOPHY OF MATHEMATICS EDUCATION: META-ANALYSES

The purpose of this section is to reflect on the type of networking that was achieved, using a meta-tool: *the scale of networking strategies* that will be briefly presented below. These meta-analyses also aim at facilitating fruitful interactions between mathematics education research and philosophy of mathematics, in view of future studies.

As explained in (Bikner-Ahsbahs & Prediger, 2008), this meta-tool was constructed to show the diversity of strategies that can be developed to establish connections among theories in mathematics education. The scale distinguishes eight strategies, organized into pairs and ordered (according to the degree of integration) between two extreme positions: "ignoring other theories" and "unifying globally", both considered not desirable. Precise definitions are provided for these different strategies: for instance, *coordinating* means that "a conceptual framework is built by well-fitting elements from different theories" (assuming that the theoretical approaches involved complement each other), while *combining* means that "the theoretical approaches are only juxtaposed according to a specific aspect". The combining strategy can thus involve theories with some conflicting basic assumptions. At a higher level of integration, *integrating locally* and *synthesizing* refer to strategies focusing "on the development of theories by putting together a small number of theories or theoretical approaches into a new framework", the distinction within the pair being based on the asymmetry/symmetry of the theories involved (for instance, in terms of scope).

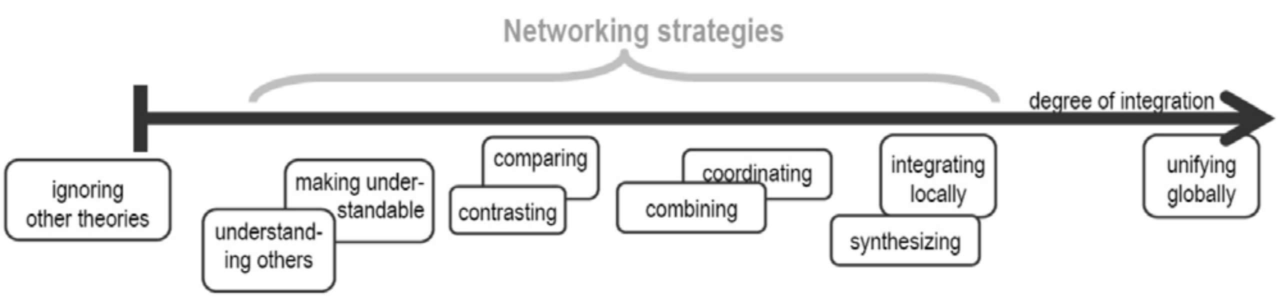

Figure 2: Networking scale (Bikner-Ahsbahs and Prediger 2010, p. 492)

The networking approach is seen as crucial in the context of rapid expansion of theoretical frameworks and constructs that challenge the integrity of mathematics education as a field of research (Artigue, 2020). In our context of interrelationship between mathematics education and philosophy of mathematics, Ernest (2018, p. 14) points out that "Philosophy of Mathematics education has emerged as a loosely defined area of research, primarily concerned with the philosophical aspects of mathematics education". He thus explored the field both in terms of

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(philosophical) questions raised by practices from mathematics education, including mathematics education research (bottom-up perspective), and in terms of the application of branches of philosophy (ontology and metaphysics, aesthetics, epistemology, ethics, etc.) to mathematics education (top-down perspective). The role played by Philosophy thus goes beyond providing theoretical tools that complement or may be combined with didactical tools to analyze teaching-learning phenomena: it also offers meta-tools to question mathematics education research practices. These aspects are not considered in this paper. By contrast, the (mixed) networking that is proposed here aims at coordinating theories from Philosophy and Mathematics education research with an attempt to giving each field a more symmetrical position for mutual enrichment.

In our example, the joint analysis (first stage: comparing/contrasting) of the didactical experience and reception by students of the mathematical theory of banquets led to the identification of common features (second stage: combining/coordinating) between Brousseau's didactic contract and the horizon of expectation in Hermeneutics: under-determination, objectivity and dynamical structure. Referring to the networking scale, the words *synthetizing* and *integrating locally* may be used whenever theoretical development is aimed at. Such a stage has not yet been reached, but arguments may be given both regarding the possibility of such an integration and the fruitfulness of such an approach. Firstly, the Theory of Didactical Situations to which the didactic contract belongs may be envisaged as an epistemic approach in mathematics education while Phenomenology and Hermeneutics are cognitive-enrooted philosophical trends (for instance, Husserl initially looked for psychological foundations for the concept of natural numbers). Secondly, this study on the reception of the theory of banquets shows evidence of complementary and mutually enriching point of views. For instance, there is a tendency, particularly in interventionist studies, to push for the explication of the didactic contract, which is misinterpreted as a set of didactical rules, thus conventions (Sarrazy 1995, p. 94). By contrast, the hermeneutical point of view gives new tools to focus on what is left implicit - on purpose - and needs to be transformed through its journey in the horizon of intentionality of students. Conversely, didactics appears to be a quite natural field of investigation for hermeneutics and phenomenology. It is precisely a direct scope of the teacher to shape and engineer the horizon of expectation of a given assignment to the students. We face therefore a situation where horizons are not a mere abstract view on intentionality and cognitive processes, but (although implicitly) a key component of a theoretical and practical endeavor. In terms of the networking scale, the coordinating stage has thus been fully reached.

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CONCLUDING REMARKS

The aim of this communication is to provide evidence for the fruitfulness of mixed networking of theoretical ideas from Husserlian phenomenology, hermeneutics and didactics of mathematics. It is argued that phenomenological insights and techniques may enrich didactical analyses and that, conversely, didactics of mathematics may offer a rich context of investigation for hermeneutics and phenomenology. A first example in this direction consists in coordinating (in the sense of Bikner-Ahsbahs and Prediger, 2010) Brousseau's didactic contract with Jauss' horizon of expectation (the hermeneutical contract). This study tries to show that such relationships can give rise to a research program at the interface of didactics and philosophy, that would consist in adapting various fundamental concepts and techniques of hermeneutics to the didactical context in an attempt to augment Brousseau's theory with new epistemological insights, besides creating a possibly fruitful dialog between didactics and a central piece of contemporary theories in aesthetics. In terms of the networking scale, this first study calls for synthetizing between the Theory of Didactical Situations, Phenomenology and Hermeneutics as a fruitful networking strategy to develop the (sub)field of Philosophy and/of Mathematics education.

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