Understanding Proof Practices of pre-Service Mathematics Teachers in Geometry

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Abstract: *In this work, we show the results of a research with pre-service mathematics secondary teachers about their Van Hiele level regarding the proof in Geometry. We observe three different profiles whose characteristics are described. These descriptions allow us to foresee certain differences when carrying out proof teaching in secondary school. The presence of a profile with a lower level than that assumed for some high school students stands out. The other two profiles show differences regarding the presence of some advanced proof strategies.*

Keywords: proof, Van Hiele levels, pre-service teachers

INTRODUCTION AND OBJECTIVES

The teaching of proof has been a concern in the context of secondary-school teacher training due to the educational knowledge shown by pre-service teachers (Arnal-Bailera & Oller-Marcén, 2017; Dos Santos & Ortega, 2013; Makowski, 2020). In order to help pre-service teachers to develop a solid knowledge of proof, it is important that mathematics teacher educators become aware of how pre-service teachers understand proving processes (Stylianides, Stylianides & Philippou, 2007).

Several studies have also addressed the level of development of pre-service mathematics teachers'(PSMT) geometric thinking following the Van Hiele model (Lee, 2015; Mayberry, 1983; Pandiscio & Knight, 2011; Wang & Kinzel, 2014). According to Güler (2016), lack of prior knowledge, proof methods understanding, students' memorization of proofs instead of questioning them and biases against proof are the main difficulties for PSMTs in this matter. There are other studies explaining particular difficulties during proving processes such as the understanding of the meaning associated with the inductive step in a proof by induction or the logical equivalence of two affirmations (Stylianides et al., 2007).

In this study, we aim to characterize the various ways that pre-service teachers proof in Geometry, in particular their use of graphics (Mesquita, 1998), discourse (Duval, 1995) and proof schemes (Harel & Sowder, 1998). This detailed description of the proof practices would help mathematics teacher educators identify and understand the different levels shown by pre-service teachers and help them to progress from one level to the next. Moreover, our study tries to use the Van Hiele model lens to compare the levels shown by pre-service teachers with the levels required by pre-college curricula to secondary school students and, implicitly, to their teachers. In the National Council of Teachers of Mathematics [NCTM], (2000, p. 310) the importance of proof in mathematics education is clearly stated: "Students should see the power of deductive proof in establishing the validity of general results from given conditions."

We conducted a study with twenty-five students enrolled in a master's degree in secondary school teacher training at the University of Zaragoza with various profiles relating to access, prior training and age. We administered them a questionnaire with open-ended questions designed to assess their Van Hiele level of geometric reasoning. In this article we will limit ourselves to analyzing the issues relating to the proof, even though we realize that undertaking the teaching of a proof also requires knowing details about the students, curriculum, and so on. The research question we wanted to answer was: Do future secondary school teachers have the Van Hiele level needed to undertake the teaching of proof that current secondary curricula require? To answer this question, we had two research objectives:

- To identify different profiles of pre-service secondary mathematics teachers according to the geometric reasoning shown in their proof practices.
- To describe pre-service secondary mathematics teachers proof practices in Geometry.

THEORETICAL FRAMEWORK

Van Hiele model

In the 1950s, Van Hiele (1957) and Van Hiele-Geldof (1957) elaborated a model that describes the development of geometric thinking. This model establishes that there are five levels of geometric reasoning (Hoffer, 1983; Van Hiele, 1957, 1986). As these levels are also sequential and hierarchical, students pass through them in a specific order without omitting any of them throughout the geometry learning process. The strictly hierarchical nature of the levels has been questioned in the later decades. In (Gutierrez & Jaime, 1998) the authors describe that Van Hiele levels are local, meaning that people can exhibit different level of reasoning at different subtopics of geometry. Furthermore, a student can also reason at various levels on different tasks (Burguer & Saughnessy, 1986; Clements & Battista, 1992).

In this study, we will focus on the three intermediate levels: level 2 (Analysis); level 3 (Informal deduction); and level 4 (Formal deduction).

In the context of this study, focusing on proof, level 2 is characterized by proofs limited to verifying whether a certain property is fulfilled in a few particular cases. In level 3, the properties can be verified in some examples, although with informal explanations based on mathematical properties. Lastly, formal mathematical proofs are conducted in level 4 (Jaime & Gutiérrez, 1994).

Producing questionnaires that correctly measure the Van Hiele levels of geometric reasoning has been a well-studied subject for many years. Usiskin (1982) prepared a multiple-choice test comprising twenty-five questions (five for measuring each level). Burger and Shaughnessy (1986) designed an interview questionnaire comprising eight activities. These authors also described level indicators they use to place each student's answer in one level or another.

Allocating Van Hiele levels involves some difficulties. These include where to place students that show signs of being between two consecutive levels. To solve this difficulty, Gutiérrez et al. (1991) propose an alternative form of assessing Van Hiele levels by describing a way of not only obtaining the level the students are at, but also the extent to which they have acquired this level. The authors describe eight answer types (0-7) in their study. These indicate varying levels of acquisition within the same level. Consequently, on evaluating an answer, we can place it within one of these types to allocate a numeric level of acquisition in accordance with Table 1:

Type	0	1	2	3	4	5	6	7
Numerical value	0	0	20	25	50	75	80	100

Table 1. Degrees of acquisition

Authors such as De Villiers (1987), however, study Van Hiele levels by analyzing several processes or components. Jaime and Gutiérrez (1994) describe the key processes they observe for Van Hiele levels: Identification (establishing which family a certain geometrical object belongs to); definition (use and formulation of definitions of geometrical objects); classification (placing different geometrical objects into different families); and proof (statement tests). A test with eight items is presented in (Gutierrez & Jaime, 1995); it can assess each student's first four Van Hiele levels and their degree of acquisition of each level. The key assessed processes for each of the issues are also described. This test has a higher reliability than multiple-choice questionnaires and it can also be administered to larger samples than interview questions. That is why we will use this test as a tool in this study.

Pandiscio and Knight (2010) analyzed the Van Hiele levels of pre-service secondary mathematics teachers finding that they did not attain level 4. These researchers stated that level 4 should be fully acquired by pre-service teachers since secondary school students should be guided to complete the acquisition of level 3 and start the acquisition of level 4. These ideas are congruent with previous studies (Mayberry, 1983) where she stated that "The response patterns suggest that these students were not at the proper level to understand formal geometry […] any high school geometry textbook will show […] that Level III should be developed" (p.68). Note that Mayberry referred level 4 as level III. Gutierrez & Jaime (1995) analyzed different profiles of secondary students attending to their van Hiele levels noticing that 12,9% of the students in grade 12 are in transition between levels 3 and 4. Due to all these previous reasons we agree with Pandiscio and Knight (2010) that PSMTs should present certain acquisition of van Hiele level 4 in order to promote the transition between levels 3 and 4 of their students.

Proof

Harel and Sowder (1998) described a student's proof scheme as "what constitutes ascertaining and persuading for that person" (p. 244) and classifies proof schemes using three non-independent categories: externally based, empirical, and analytical. External conviction proof schemes are based on outside sources that influence students' conceptions of proofs including authoritarian (the outside source is a teacher, a textbook …), ritual (the outside source is the traditional format of the proofs) and symbolic schemes (the outside source is the blind faith in the use of symbols independently of its meaning in the situation under consideration). The second category, empirical proof schemes, includes the proving or disproving of conjectures utilizing visual perceptions or examples-based proofs. The third category, analytical schemes, includes transformational (the reasoning is oriented toward settling the conjecture in general) and axiomatic proofs (the reasoning is organized so that any result is a logical consequence of the previous ones).

In their research, Harel and Sowder (1998) found that the depth of the pre-service teachers' mathematical knowledge influenced the primary proof schemes utilized. In particular, middle school pre-service mathematics teachers used external conviction as their primary proof scheme, while teachers following a dual program (middle/secondary) showed empirical proof schemes and secondary pre-service mathematics teachers used a variety of schemes (Sears, 2019). Makowski (2020) pointed out that middle school PSMTs proofs rely mainly on inductive justifications as well as Demiray and Işiksal (2017) who found that those PSMTs think that numerical values and examples were more convincing than mathematical proofs while Weber (2010) stated that most of the mathematics majors did not accept empirical arguments as proof after receiving appropriate training. Other studies (Uğurel et al., 2015) focused in the errors showed in most of the pre-service teachers' proofs. Some of these errors were: failing to know where to getting started on a proof, showing prejudices towards construction of proof, feeling uncomfortable when constructing a proof, showing some lack of knowledge related to mathematical language and notation, method, concept and communication related problems in the proving process, and lack of content and strategy knowledge regarding the proof. This group of researchers stated that the main problems were related more to the understanding of the proving process than to the knowledge required for the proof itself.

De Villiers (1993), describes five proof functions: Verification/Conviction (establishing the truth of a statement); Explanation (proving why the demonstrated statement is true); Systematization (organizing several results into a global system); Discovery (making it possible to arrive at new results arising from the proof); and Communication (conveying mathematical knowledge). When proof is used in the classroom, the functions shown are verification and explanation (Crespo & Ponteville, 2005). Some of these functions, but not all, appear in the questionnaire administered to students. Those at level 2 cannot perform proofs in the strict sense of the term; those at level 3 can perform informal proofs with verification/communication or explanation functions, while those at level 4 can perform formal proofs that can also incorporate functions such as systematization and discovery (De Villiers, 2004). Based on this idea, and using dynamic geometry software, Lee (2015) showed how pre-service teachers at different Van Hiele levels performed proof tasks that highlighted different interpretations of proof functions. For example, level 4 pre-service teachers understood proof as explanation, discovery and deductive verification functions, while level 3 pre-service teachers only viewed it as explanation and inductive verification.

Bearing in mind that we are working with geometrical proofs, the graphic part plays an important role. When a picture accompanies the answer to a geometrical problem, there can be two cases (Mesquita, 1998); the picture can be seen as an object or as an illustration. If the picture is seen as an object, its attributes or properties can be used in the reasoning of the answer. If the picture is seen as an illustration, it is not always possible to know which theoretical object represents. The author elaborates on this idea showing a picture of a triangle that could be seen as an ideal triangle with no specific measures (object) or as an illustration of a specific triangle. In the particular case of the proof activities, this distinction can be observed in two different possibilities of the use of the pictures: information-related or perception-related use. When the use is informative, the picture shows only the information given in the statement and the picture is considered an object. When the use is perceptual, the picture shows more information than in the statement and is considered an illustration. There are studies that show the bias of the students towards to sustain their reasoning in their perception of the picture more than in the information that it actually gives (Sandoval, 2009). In this respect, Arnal-Bailera & Oller-Marcén (2020), recently brought to light related problems as taking actual measures in a picture to solve a problem or as assuming specific attributes of it when these were not explicitly stated.

The discursive part plays an important role in the proving process, to describe the development of their discourse, we consider Duval's theoretical construct about the different modes of expansion of the discourse (Duval, 1995). According to Duval, there are two different modes of expansion of the discourse: accumulation mode and substitution mode. The accumulation mode is characterized by the juxtaposition of independent propositions that could be re-ordered without losing its global meaning. The substitution mode is characterized by a logical progression of propositions that follow a non-modifiable order since one proposition is the conclusion of one of the steps of the discourse and, at the same time, the premise in the following step. In this respect, the written discourse has to progress from the accumulation mode to the substitution mode to finish the proof through a deductive process (Saorín et al., 2019).

Teaching proof

Throughout this article, we will concentrate on measuring Van Hiele levels for the process of proof, which is understood as an analytical or theoretical proof in the sense described by Gutiérrez (2005). The presence of proof in teaching has often been valued to both show the need to support mathematical knowledge and to understand the concepts involved (Mariotti, 2006). Some studies have delved into the tasks that may be involved in constructing a geometry proof in the classroom or in assessing the construction of the proof (Martin & McCrone, 2003) and how to implement them in class (filling gaps in a proof, conditional statements, local deductions, tests with hints, synthetic proofs with no help, analytical proofs with no help).

Several European countries established the importance of proof in their educative laws. The Organic Law for Improving the Quality of Education (LOMCE, Jefatura de Estado, 2013) states that learning proof during the Spanish baccalaureate (grades 11-12) is compulsory and cuts across all contents. In particular, explicit references to proof include aspects such as the teaching of several methods (reductio ad absurdum, induction, etc.), reasoning (both deductive and inductive) and proof languages (graphic, algebraic or report). In addition, the current Italian educative law, Good School (*La Buona Scuola*), establishes from 2015 onwards that in the first two years of high school (grades 9-10) students have to "understand the logical steps of proofs and construct simple proofs". The National curriculum in England for the key stage 4 (grades 10-11) asserts that pupils on this level should "look for proofs or counter-examples; begin to use algebra to support and construct arguments {and proofs}" (Department of Education, 2014).

In a non-European context, the most concrete and complete guide of mathematics teaching is the "Principles and standards for school mathematics" of the National Council of Mathematics Teachers (NCTM, 2000). In particular, concerning proof, the (NCTM, 2000) asserts that "the repertoire of proof techniques that students understand and use should expand during the high school years" (p. 345). More concretely, attending to the proof in geometry the Common Core State Standards (2010, p.76) includes the proof of geometric theorems concerning lines and angles, triangles and parallelograms, this document "allows teachers to be proficient at decision making about what students know, need to know, and how they can impart that knowledge" (Columba & Stotz, 2016). Besides, the (NCTM, 2000) describes what kind of proofs should appear in school mathematics concreting that students "should be able to produce logical arguments and present formal proofs" (p. 345). These sentence clearly reflect that formal proofs (4^{th} Van Hiele level) should be developed in school mathematics.

METHOD AND SAMPLE

The experiment was carried out with 25 students of the Masters' Degree in Secondary School Teaching at the University of Zaragoza who responded to the questionnaire for two hours. Fifteen of the students have university degrees in mathematics, six in physics, two in engineering and two in statistics. Concerning their teaching experience, five stated that they had no experience, eighteen had given private secondary classes and two had extensive experience as university tutors.

	Van Hiele levels					***Definition***			
Item	1	2	3	4	***Identification***	*Use*	*Stating*	***Classification***	***Proof***
1	•	•			•				
2	•	•			•				
3	•	•	•			•		•	
4	•	•			•	•			
5		•	•	•		•			•
6.1		•	•	•					•
6.2, 6.3		•	•						•
7		•	•	•		•			•
8			•	•		•	•		•

Table 2. Levels and processes assessed by each item of the questionnaire (Gutiérrez & Jaime, 1995).

The questionnaire we used (Gutiérrez & Jaime, 1995) covers different processes through activities involving polygon properties. It has four open-ended items –items 5, 6, 7 and 8 focusing on the proof–, each with several sub-items (see Table 2). The proof functions (De Villiers, 1993) present in the test are, essentially, verification or conviction functions. The function of the last question is systematization since it underscores the idea of equivalence between definitions. Tasks for assessing geometric reasoning are proofs with hints and unsupported proofs (Martin & McCrone, 2003).

Our study's design follows a mixed method of the explanatory type (Creswell, 2012) since the qualitative analysis follows the quantitative one. Fundamentally, the techniques used in the quantitative analysis are statistical (cluster analysis) and the answers to the questionnaire are studied in depth in the qualitative analysis.

To decide on the degree of acquisition of each reasoning level in every student, we followed the calculation methodology devised by Gutiérrez et al. (1991) based on the fact that several experts in mathematics education consider the features of mathematical reasoning shown in the questionnaire tasks to be more important than the mathematical correctness of the answers. For every student's response to a certain item, a 3-tuple is obtained indicating the degree of acquisition, as a percentage, of the Van Hiele levels that this item measures (levels 2, 3 and 4). The elements in these 3-tuples were agreed between the researchers following Gutiérrez et al. (1991) indications. In particular, we assigned a certain degree of acquisition to every single response (see Table 1), to do so we decided the most accurate descriptor from the list given in Gutiérrez et al. (1991). Finally, we found the mean of the values obtained by each student in each level to calculate his or her degree of acquisition.

The quantitative analysis includes the construction of clusters using SPSS. These clusters are obtained from the three acquisition variables. In addition, other context variables (studies and teacher experience) have been used only to describe them. To construct clusters, we apply the K-means algorithm with the squared Euclidean distance. The number of conglomerates is determined by the Hartigan criterion (Xu et al., 2016).

We conducted a qualitative analysis to characterize each of the previous clusters. In order to achieve that goal, we analyzed the answers of the students attending to the different variables described in the theoretical framework: the use of pictures, the proof scheme or the type of discourse. In addition, other emergent variables inform about specific characteristics of the proof. All these variables have been classified (Table 3) into three different groups informing about general, graphic and proof characteristics.

Groups	Variables
General characteristics	Length of the answer Completion rate
Graphic characteristics	Number of pictures Use of pictures
Proof characteristics	Proof scheme Reference to previous results Justification of the use of a previous result Argumentation grounding

	Use of properties that are consequence of the result
	Type of discourse (non-substitutive features)
	Sensibility to double implication

Table 3. Variables.

There are two general characteristics: length of the answers (number of words) and its completion rate (percentage of the students reaching to a conclusion at the end of every sub-item where unsupported proofs are asked).

There are two graphic characteristics, number and use of the pictures. Concerning the use of pictures, we follow the ideas of Mesquita (1998) and stablish several categories: exploratory examples, information and perception.

In the corresponding Tables it is shown the percentage of each use in the items where unassisted proof is asked (5.1, 6.1, 7.A and 8). For the rest of the sub-items (assisted proof tasks) the use of pictures is not considered relevant since the type of hints presented could induce it.

Categories (based on Mesquita, 1998)	Definition
Exploratory examples (E)	Pictures showing particular cases not explicitly connected to the written argumentation
Information (I)	Pictures showing only the conditions or attributes established in the statement and explicitly connected to the written argumentation
Perception (P)	Pictures showing particular cases or attributes non established in the statement that are explicitly connected to the written argumentation
Without Pictures (WP)	Answers without pictures

Table 4. Categories of the variable “use of pictures”

As we can see in Table 3, we distinguish six different proof characteristics. Based on the work of Harel and Sowder (1998) we have classified the proof schemes into empirical (E), analytic

(A) and responses in which characteristics of both types appear (E/A). There are some non-evaluable answers (NE) due to its little content or because the student left them blank.

Other variables have emerged informing about specific characteristics of the proof: in every sub-item, we studied if there were references to the mathematical results that underpin their argumentations, showing in the tables the corresponding percentages. Other specific characteristics appear only in one item. On item 5, the type of grounding: geometrical (based on arguments of geometric nature), numerical (based on the searching of the regularities in a numeric series) or mixed (characteristics of geometrical and numerical grounding appear). On item 6, we consider the (inappropriate) use of attributes that are consequence of what it is supposed to be proven. This variable refers to the percentage of tasks in which the students use as hypothesis consequences of the thesis to be proved. On item 7, the justification of the appropriateness of the use of previous results. Finally, on item 8, we found two different variables: the appearance of non-substitutive features and the sensibility to double implication. The study of the percentage of students showing non-substitutive features allow us to distinguish different types of discourse. The variable sensibility to double implication shows the awareness of proving the double implication to establish the equivalence of two definitions; this variable expresses the percentage of students making the proof by double implication.

RESULTS

In this part of our work, we present the quantitative analysis, which leads to the construction of the clusters and the qualitative analysis, which leads to their characterization.

Quantitative analysis / Clusters' construction

After marking the answers to the test, we observed that the Cronbach Alpha coefficient was higher than 0,7 (0,836), therefore we considered that it was a reliable test.

Cluster construction

We will be using 3 clusters in our analysis since, attending to the Hartigan criterion, we have obtained H(2)=15,62 and H(3)=7,10. The three clusters gather, respectively, 6, 11 and 8 individuals. In Table 5 we show the degree of achievement of each Van Hiele level (Gutiérrez et al., 1991).

	Cluster 1		Cluster 2		Cluster 3	
	Average score	*Degree of achievement*	*Average score*	*Degree of achievement*	*Average score*	*Degree of achievement*
Level 2	100	Full	100	Full	81,46	High
Level 3	100	Full	85,68	Full	32,97	Low
Level 4	74,17	High	40,80	Medium	2,50	None

Table 5. Average score and degree of achievement of each level for each cluster.

Cluster 1 (C1 from now on) gathers 21% of the individuals (6), they have achieved the four Van Hiele levels, though they only reached a high degree of achievement of level 4. In this cluster, there are 2 graduates in Physics and 4 in Mathematics. Cluster 2 (C2 from now on) includes 44% of the individuals (11), they show full achievement of levels 2 and 3 but only a medium degree of level 4. In this cluster, there are 2 graduates in Physics, 7 in Mathematics and 2 in Engineering. The third cluster (C3 from now on) gathers 32% of the individuals (8), they have a high degree of achievement of level 2, a low degree of level 3 and none of level 4. In this cluster, there are 2 graduates in Physics, 4 in Mathematics and 2 in Statistics.

Differences between clusters

Shapiro-Wilk test showed that none of the variables "degree of achievement of the level" follows a normal distribution. In addition, when doing the Shapiro-Wilk test for each level separately, we observed that there was, at least, one cluster not following a normal distribution in each level, thus we rejected the hypothesis proposing that the degree of achievement of each level follows such a distribution. As a result, we apply the Kruskal-Wallis nonparametric analysis leading to the conclusion that there is at least one cluster with significant differences regarding the degree of achievement in each level. In order to identify the statistically different means we applied the Mann-Whitney test for the pairs study concluding that there are statistically significant differences between the level of acquisition of the Van Hiele levels for every cluster and level except for level 2 in clusters 1 and 2 where both present a full level of acquisition. The influence of the variable "teaching experience" was studied too and it neither affected the composition of the clusters nor was statistically different between the diverse clusters.

Qualitative Analysis / Clusters' characterization

Item 5

Item 5 includes two sub-items (5.1 and 5.2). Sub-item 5.1 asks to deduce the formula for the number of diagonals of a polygon given the number of sides and to prove it. This sub-item offers no hints for the task. Sub-item 5.2 asks to deduce the same formula using two particular cases (n=5 and n=6) and its generalization; this sub-item asks for a justification rather than a proof. This item focuses on some of the proof functions: verification (if the formula is proved by induction), conviction (if is proved through a deductive process) and discovering (sub-item 5.2 when it is suggested to find the general formula using particular cases). These functions are to be carried out through an unsupported proof (5.1) and a proof with hints tasks (5.2). This item allows distinguishing the students at levels 2, 3 and 4.

Characteristics	Variable	C 1	C 2	C 3
General characteristics	Length of the answer (item 5)	123,8	97,3	47,7
	Completion rate (5.1)	100%	72%	50%
Graphic characteristics	Number of pictures (item 5)	3,5	4,2	6
	Uses of the pictures (5.1)	E:33,3% I:16,7% P:0% WP:50%	E:27,3% I:18,1% P:27,3% WP:27,3%	E:12,5% I:0% P:87,5% WP:0%
Proof characteristics	Reference to previous results	Not apply		
	Argumentation grounding (5.1)	G:100% Mx:0% N:0%	G:40% Mx:50% N:10%	G:0% Mx:37,5% N:62,5%

	Argumentation grounding (5.2)	G:100% Mx:0% N:0%	G:64.6% Mx:27.3% N:9.1%	G:33,3% Mx:33,3% N:33,3%
	Proof Scheme (5.1)	E:0% E/A:0% A:100% NE:0%	E:54,5% E/A:0% A:36,4% NE:9,1%	E:100% E/A:0% A:0% NE:0%

Table 6. Characteristics observed in item 5.

The answers to item 5 are longer (see Table 6) in the case of students of clusters C1 and C2 (123,8 y 97,3 words respectively) than in C3 (47,7 words). This is related, but not only, with the completion rate that is much higher in students of C1 (100%), while only 50% of the students in C3 finished the tasks. The uncompleted task was the unassisted construction of the formula while every student completed the assisted task (although some of them with errors).

With respect to the pictures used by the students in item 5, we found a significantly higher (at 90%) in C3 students' answers (6 by student on average) than in C2 (4,2 by student on average) or in C1 (3,5 by student on average). It is remarkable that 50% of the students of C1 and 27,3% of C2 did not make any picture at all when answering 5.1. In C1, the pictures were exploratory examples of the situation to study but not directly supporting the proof itself. In C2, some students (27,3%) made deductions for their argumentation based in the perception of the picture, more than in the information that this picture could actually transmit. 87,5% of the students in C3 acted the same way.

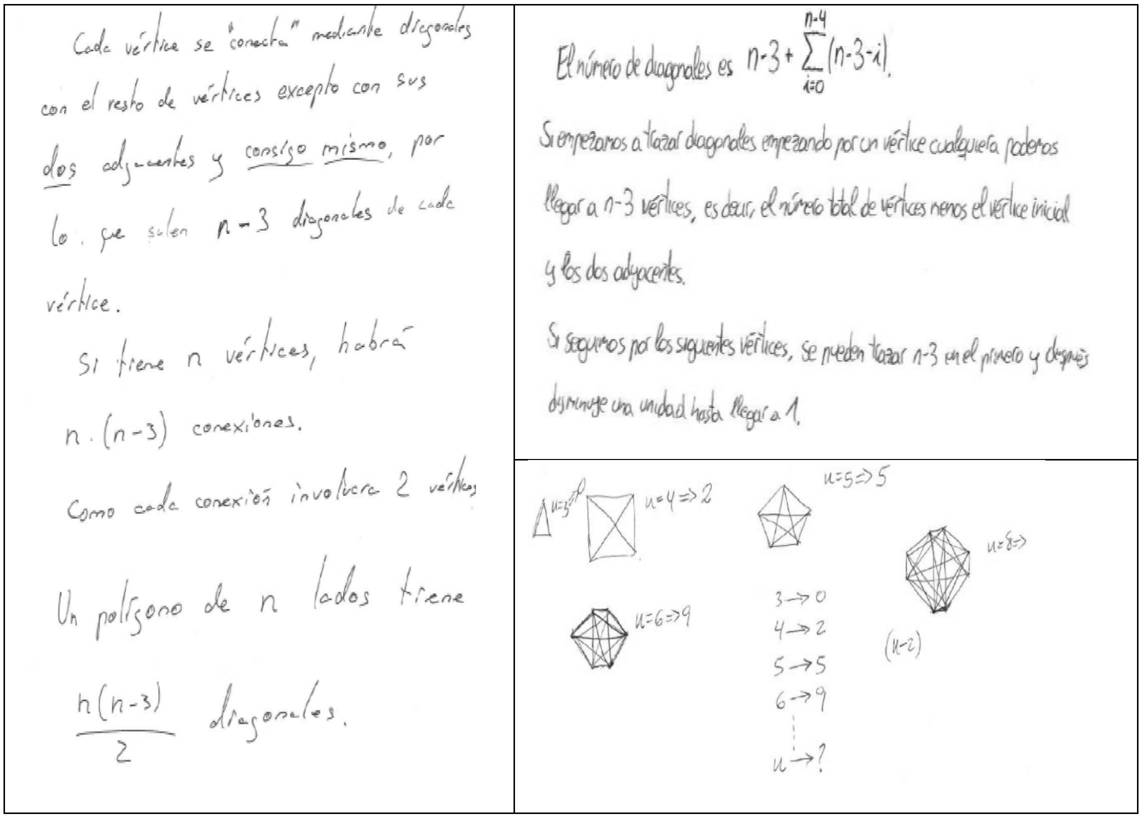


Figure 1. Examples of students' answers to sub-item 5.1 of C1 (left), C2 (above-right) and C3 (below- right)

Concerning the specific characteristics of the proof, in this item there was no room to refer to previous mathematical results such as theorems or propositions. We have not founded any examples of accumulative discourse, no matter the different types of grounding they use on its correctness. Attending to the grounding of the arguments used, these are mostly of a geometrical nature in C1 (See Figure 1-left: "Each vertex is 'connected' by diagonals with the rest of the vertices but the adjacent ones and it self, thus we have n-3 diagonals at each vertex."). Arguments combining the geometrical and the numerical ones showed up in C2 (See Figure 1 above-right, instead of multiplying by n the student states at the end of his/her explanation that "(…) If we follow by the other vertices, the number decreases to one."). In C3, most of the arguments are of a numeric nature (See Figure 1 below-right: the student looks for numerical regularities observed counting the total number of diagonals in particular cases). Some of the students in C2 that had given mixed arguments in 5.1, gave only geometric arguments in 5.2. In C3, some of the students moved from numerical arguments in 5.1 to geometric arguments in 5.2. In C1, 100% of the students carried out an analytical proof, while in C2, this percentage decreased to 36,4%. In C3, every student showed an empirical proof scheme.

Item 6

Item 6 consists of three sub-items (6.1, 6.2 and 6.3). 6.1 asks to (unsupported) prove that the sum of angles of any acute-angled triangle is 180º. In 6.2 (proof with hints) it is recalled that a pair of parallel lines crossed by a secant form several angles with the same size. Finally, in 6.3 (proof with hints) it is described a complete proof of the statement in 6.1 and it is required to perform similar proofs for straight triangles first and obtuse triangles later. The functions of the proof present in this item are verification/conviction. Sub-item 6.1 enables to classify students at levels 2, 3 and 4 while 6.2 and 6.3 allows distinguishing between levels 2 and 3.

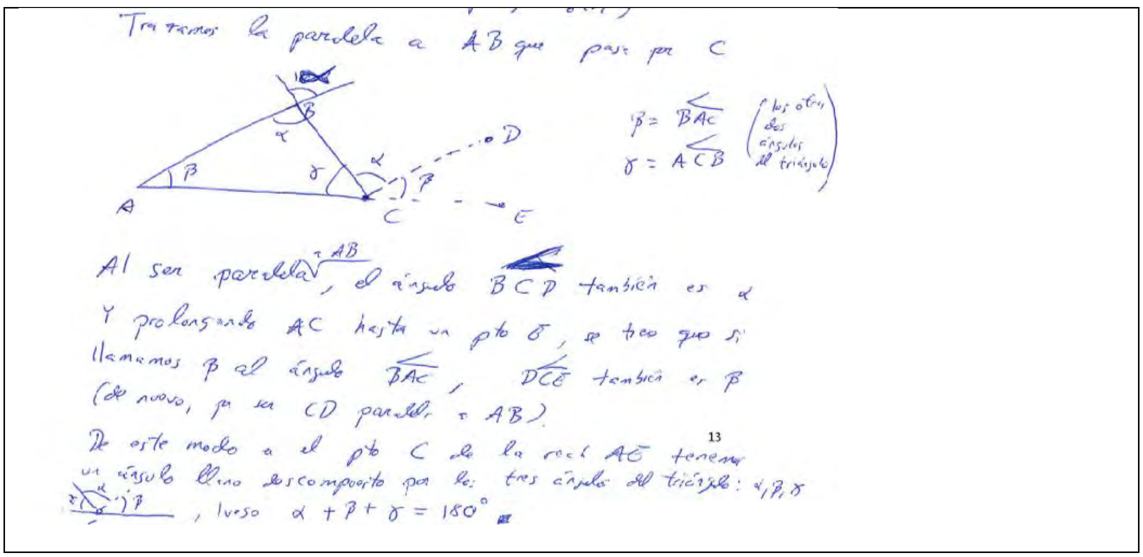
It can be observed that the answers of students in C1 are longer than those given by C2, which at the same time are longer than the ones written by C3. The completion rate observed is very high in C1 and C2 (over 90%), while less than 50% of the students in C3 completed this item.

Characteristics	Variable	C 1	C 2	C 3
General characteristics	Length of the answer (item 6)	157,2	78,2	42
	Completion rate	94,4%	90,9%	45,8%
Graphic characteristics	Number of pictures (6.1)	3	2,2	2
	Uses of the pictures (6.1)	E:16,7% I:50,0% P:33,3% WP:0%	E:27,2% I:36,4% P:36,4% WP:0%	E:62,5% I:0% P:25,0% WP:12,5%
Proof characteristics	Reference to previous results	63,1%	49,4%	25,0%
	Use of properties that are consequence of the result	13,3%	22,7%	66,7%

Proof Scheme (6.1)	E:0% E/A:0% A:66,6% NE:33,4%	E:9,1% E/A:18,2 % A:54,5% NE:18,2%	E:25% E/A:0% A:12,5% NE:62,5%

Table 7. Characteristics observed in item 6.

With respect to the number of pictures used by the students, there are no significant differences in the number of pictures presented by students of the different clusters. However, there exist differences in the use of them since in C1 and C2 its uses are informative or perceptive, 83,3% and 72,8% in the aggregate respectively, while in C3 the main use is as exploratory examples (62,5%).



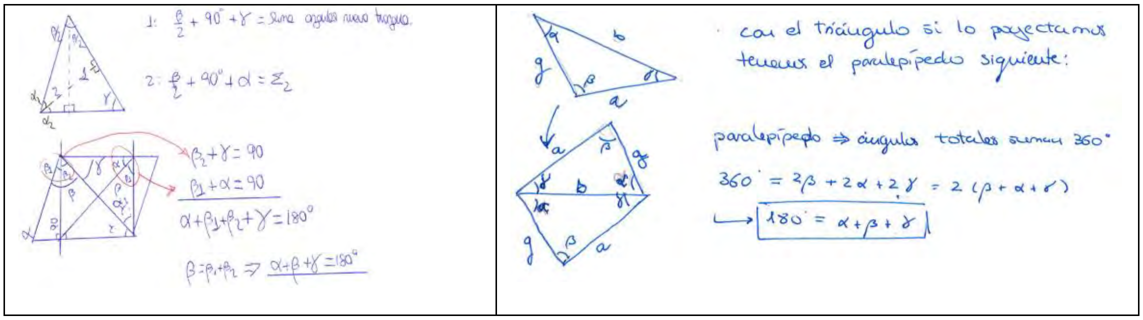


Figure 2. Examples of students' answers to sub-item 6.1 of C1 (above), C2 (below-left) and C3 (below-right).

In 6.1, most of the students from C1 made clear references to the results they were using (see Figure 2 above) whilst references to previous mathematical results are almost non-existent in C2 and C3. In 6.2 and 6.3, it is appreciated that C2 students are more explicit describing the previous mathematical results being used. The explanations given by C3 students have no references to previous results and tend to be limited to a series of computational steps expressed algebraically with very few textual descriptions. Moreover, it has been observed that C1 and C2 students present a better comprehension of the proof techniques since, in general, they do not use properties which are consequence of the result that wants to be proved, but some exceptions can be found (see Figure 2 below-left). This mistake is very common in C3 (66,7%); for instance, in Figure 2 below-right the student uses the sum of the internal angles of a quadrilateral to proof that the sum of the internal angles of a triangle is 180º. In C1 and C2, most proofs follow an analytical scheme, while in C3 in the majority of cases, the answers are blank or with very little content.

Item 7

Item 7 includes two sub-items (7A and 7B). 7A asks to prove that two diagonals of any rectangle are congruent while 7B asks to prove that any point in the perpendicular bisector of a segment is equidistant from its endpoints. Both sub-items focus in the verification and discovering functions through unsupported proof tasks. This item allows distinguishing the students at levels 2, 3 and 4.

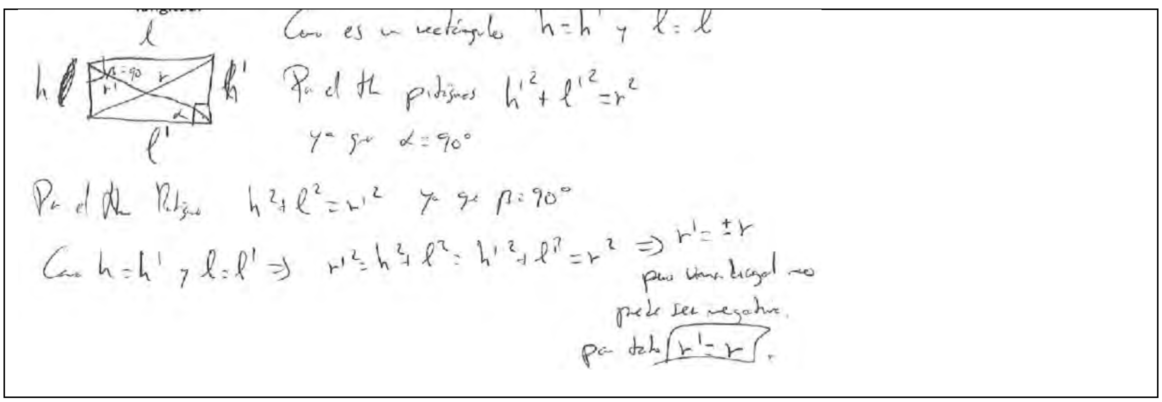
Characteristics	Variable	C 1	C 2	C 3
General characteristics	Length of the answer (item 7)	66	58	24
	Completion rate (7A)	100%	100%	100%
Graphic characteristics	Number of pictures (item 7)	2.8	2.3	3.3
	Uses of the pictures (7A)	E:16,7% I:33,3% P:50% WP:0%	E:0% I:18,2% P:81,8% WP:0%	E:0% I:12,5% P:87,5% WP:0%
Specific characteristics	Reference to previous results	100%	50%	33,3%
	Justification of the use of a previous result	50%	30%	0%
	Proof Scheme	E:16,7% E/A:0% A:83,3% NE:0%	E:9,1% E/A:9,1% A:81,8% NE:0%	E:25% E/A:25% A:50% NE:0%

Table 8. Characteristics observed in item 7.

According to the data shown in Table 8, the answers to item 7 are longer in the case of C1 and C2 students (66 and 58 on average respectively) than in those of C3 (24 words on average). In this item, all the students completed the task meaning they reached a conclusion, even with errors or inaccuracies in some of the intermediate argumentations.

Concerning the graphic characteristics, we can see that every student drew at least one picture and the number of them is higher in C3 than in the other clusters in 7A. However, the use of these pictures is different in every cluster; for instance, some students of C1 used their pictures to face the problem before starting their verbal argumentation, this use has not been identified in other clusters. When changing from C1 to C2 or from C2 to C3 we appreciate that the use of the pictures as information decreases and the use as perception increases. Most of the students of C3 (87,5%) made pictures depicting perceptions against only a 50% of the students in C1. Giving more details about these perception-related uses, we can say that in C1, nobody used the same letter to name both diagonals versus 11% and 38% of the students of C2 and C3 who did. In the case of the sides, 60% of the students of C1 named with the same letters versus 86% and 75% of C2 and C3, respectively (see Figure 3 below-right).

We paid attention to the references of previous mathematical results (mainly the Pythagorean Theorem) and if they justified the appropriateness of its use. 100% of the students of C1 using the Pythagorean Theorem in their proof referenced it, this percentage decreased up to 50% and 33,3% in C2 and C3 respectively. Whether they referenced it or not, the fact is that almost every student used it in their item 7 proofs but only 50% of the students in C1 justified the appropriateness of its use (see Figure 3 above, student in C1 claims "Using Pythagorean Theorem $h^2+l^2=r^2$ given $\alpha=90^{o}$"). Only 30% of the students in C2 (see Figure 3 below-left, students in C2 only mentions the Theorem) and none of the C3 students justified the appropriateness of its use. Most of the proofs were classified as analytical regarding its scheme. However, a significant percentage of students in C3 performed an empirical proof.



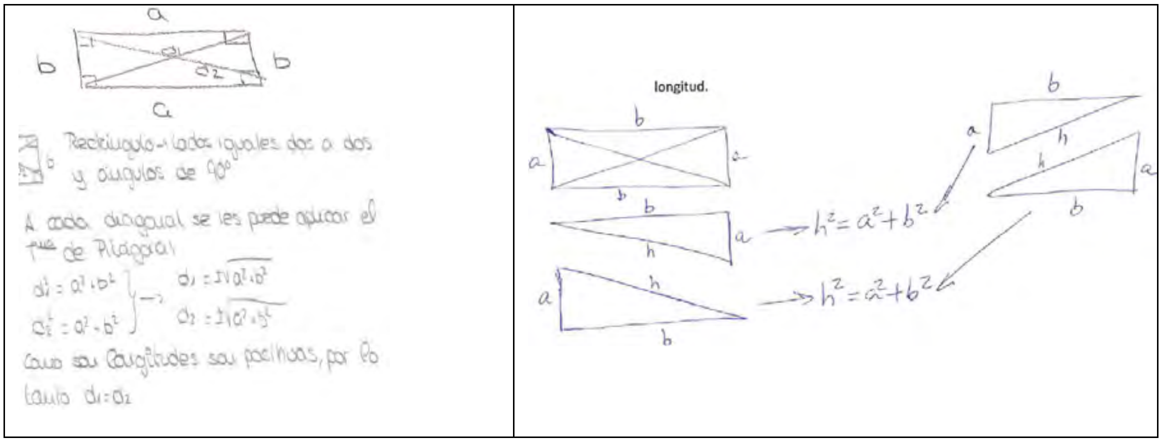


Figure 3. Examples of students' answers to item 7 of C1 (above), C2 (below-left) and C3 (below right)

Item 8

Item 8 asks to (unsupported) prove the equivalence between the definitions of parallelogram as "quadrilateral having two pairs of parallel sides" (to what they call usual definition) and "quadrilateral in which the sum of any two consecutive angles is 180º". In case of a negative answer, it is required to draw a counterexample. The function of the proof shown in this item is systematization. This item allows distinguishing students at levels 3 and 4.

Characteristics	Variable	C 1	C 2	C 3
General characteristics	Length of the answer	78	39.2	20.5
	Completion rate	100%	100%	75%
Graphic characteristics	Number of pictures	2,2	1,7	1,6
	Use of the pictures	E:0% I:50% P:50% WP:0%	E:9,1% I:36,4% P:54,5% WP:0%	E:37,5% I:0% P:62,5% WP:0%

Specific characteristics	Reference to previous results	50%	36,4%	0%
	Non-substitutive features	16,7%	36,4%	NA
	Sensibility to double implication	33,3%	18,2%	0%
	Proof Scheme	E:0% E/A:0% A:100% NE:0%	E:0% E/A:18,2% A:81,8% NE:0%	E:50% E/A:50% A:0% NE:0%

Table 9. Characteristics observed in item 8.

Responses from students in C1 are clearly longer than in C2, which at the same time are longer than in C3. The completion rate observed is very high in the three clusters being complete on C1 and C2.

Concerning the pictures drawn by the students, it is observed that the number of these in C1 is slightly higher than in the other two clusters. However, the greatest difference appears in their use: in C1 and C2 there is a balance between information and perception uses whilst in C3 the main use is perception and the rest are exploratory examples.

Half of the students of C1 refer to the previous results that are being used while this references decrease in C2 (see Figure 5 below-left, where the student states at the beginning "if we have 2 parallel sides, we can represent it in the following form" without further justification) and are non-existent in C3. Most students in C1 and C2 proof using substitutive discourse; however, in C2 we observe some accumulative features such as unnecessary repetitions of arguments or the proof of the same implication twice. Discourse of C3 cannot be studied due to the scarcity of arguments produced. Concerning the sensibility to double implication, one out of three students in C1 were conscious of the relevance in their use in order to prove the equivalence (see Figure 4-above). Nevertheless, this fact is minority in C2 (18,2%) and does not appear in C3.

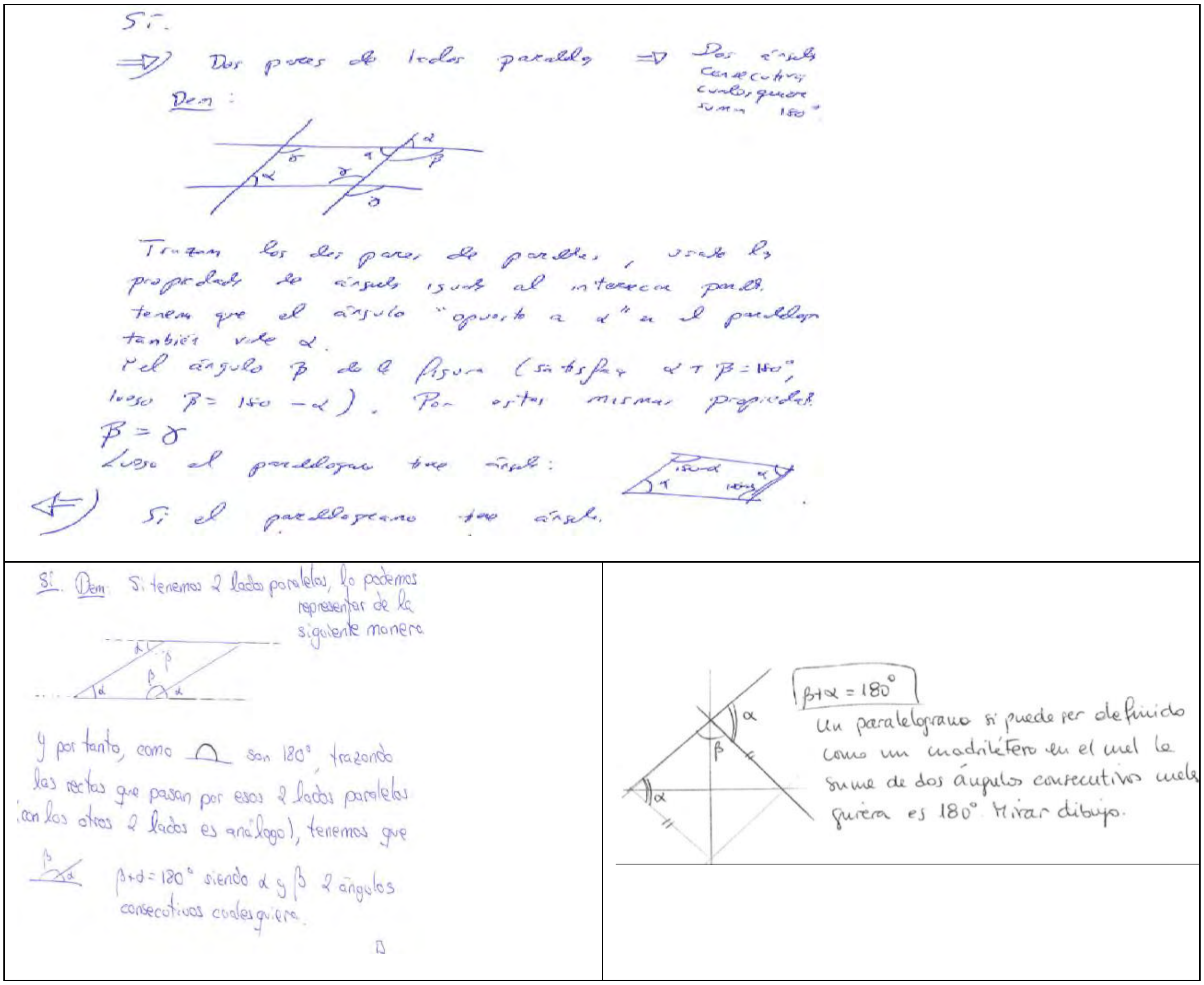


Figure 4. Examples of students' answers to item 8 of C1 (above), C2 (below-left) and C3 (below-right).

In C1, and C2, most of the students carry out an analytical proof, whereas in C3 we find a majority of empirical schemes. For instance, the student in Figure 5 below-right bases the proof on the example drawn: he/she only confirms the validity of the statement by copying it and ends the explanation with "Look at the picture".

Characterization of the clusters

Based on the results obtained for the four items being studied, a characterization of the three clusters of pre-service secondary teachers involved in the study is achieved. In what follows we present a summary of the more relevant characteristics regarding the categories and subcategories analyzed.

Cluster 1

Cluster 1 consists of 6 individuals (24%) of the sample. These individuals show a high level of acquisition of the fourth van Hiele level. They always complete formal proofs without extra help following an analytical scheme. The most frequent use of pictures supporting the responses is informational. Their discourse is always substitutive and their reasoning is mathematically grounded in previous results that are explicitly stated, or at least referred to. These students can operate with the idea of equivalent definitions understanding that a two-way-proof has to be done.

Cluster 2

Cluster 2 consists of 11 individuals (44%) of the sample. These individuals show a medium level of acquisition of the fourth van Hiele level and they have completely acquired all the previous levels. They usually complete formal proofs without extra help following most of the times an analytical scheme. Their discourse is usually substitutive, but showing some accumulative practices. Nevertheless, the most frequent use of pictures is perceptual. Their reasoning is mathematically grounded; however, they do not refer to the previous results in which the reasoning is based.

Cluster 3

Cluster 3 is formed by 8 individuals (32%) of the sample. These individuals show a low level of acquisition of the third second van Hiele level and a high level in the third level. They frequently need hints to start the activities following an empirical scheme in their proofs. The most frequent use of pictures supporting the responses is perceptual. The reasoning shown in their answers is frequently based in numerical patterns or in wrongly deduced properties from the pictures. In most cases, their discourse has not enough content to be described as accumulative nor substitutive. They do not refer to the previous results in which the reasoning is based and, frequently, they use properties that are consequence of the result that is to be proven.

DISCUSSION AND CONCLUSION

With respect to the first objective, "To identify different profiles of pre-service secondary-school mathematics teachers according to the geometric reasoning shown in their proof practices" the use of the questionnaire (Gutiérrez & Jaime, 1995) has been proved adequate. It gave correct consistency values and showed coherence between the answers to each individual item and the levels that such item was supposed to identify. In particular, we considered the items containing proof tasks to analyze their geometrical reasoning. The use of this questionnaire led us to the construction of three statistically different clusters containing 25 PSMTs graduated in Mathematics, Physics, Engineering and Statistics. As Wang and Kinzel (2014) stated, the Van Hiele model gave not enough information to differentiate the specific characteristics of the clusters, what made necessary to delve more deeply into their qualitative characteristics.

With respect to the second objective, "To describe pre-service secondary mathematics teachers proof practices in Geometry", the analysis of the students' answers has been based on the study of some variables to explain their main aspects: general characteristics, graphic characteristics and proof characteristics.

In the analysis of the graphic characteristics of the answers, we found an extensive use of pictures with different uses (examples, information or perception). The only exception to this was that some students in C1 did not draw anything to solve item 5 since they could base all the work in mathematical properties. The analysis of the pictures showed differences between clusters: while in C1 the most common use was informative, most of our PSMTs in C2 and C3 used their pictures with perception purposes. Students in C2 and C3 included not only the information written in the given statement but also some other facts that were directly inferred by them without writing down the reasons to do so; these practices prevent geometrical reasoning from developing (Mesquita, 1998; Sandoval, 2009). This could be related with practices observed in pre-service primary school teachers assuming that the illustration accompanying a geometrical problem had an object value to infer conclusions from (Arnal-Bailera & Oller-Marcén, 2020).

Concerning the proof characteristics of the answers, we worked with Duval's ideas (accumulation and substitution). All the proofs in C1 and C2 were classified as substitution proofs; however, some of the students showed some non-substitutive features such as the unnecessary repetition of an argumentation. This can be explained since most of our students are Mathematics or Physics graduates and have an important mathematical background which prevented them from having some of the problems shown in similar studies with other PSMTs enrolled in degrees with less mathematical contents (Demiray & Işiksal, 2017; Uğurel et al., 2015). As we delved into the analysis of the answers, we needed to detail the different levels of substitution that our students achieved: most of C1 students made reference to previous results, the percentage decreased as we moved to C2 and C3; however, when the result to mention was a well-known Theorem (Pythagoras) only half of C1 students justified its use and none of the C3 students did so. In this regard, C1 contains the students with a deeper understanding of the proving processes, including ideas as the need of proving the double implication to state the equivalence between definitions, and showing a more formal use of previous results. The students in C2 support their proofs in a more informal use of mathematical properties, showing conceptual mistakes as the use of properties in the process of proof that are actually a consequence of what they were proving. In this respect, Stylianides et al. (2007) showed that most PSMTs struggle with the logic rules involved in the equivalency of two statements and Uğurel et al. (2015) found that, frequently, PSMTs fail to define logical structure of the statements in the theorems. Finally, C3 is formed by students having difficulties to carry out the given proofs and, in many occasions, get to progress on them backing their reasoning in the perceptions of their pictures and in numerical regularities over the mathematical results or the properties of the mathematical objects involved in the proof. The mere existence of this cluster is a source of concern for us, given their background degrees, and make us agree with Karunakaran et al. (2014) that propose a medium- to long-term work to improve PSMTs abilities when these cannot even construct a generic example proof.

Regarding the different types of proof (Harel & Sowder, 1998), our data showed that in C1 most of the proofs were analytical while in C3 most of the proofs were empirical. In C2 we found similarities with both: the proofs of item 5 were empirical while the others were mainly analytical. The extensive use of empirical proofs by students in C3 suggest that they share some characteristics with the undergraduate pre-service middle school mathematics teachers studied by Demiray and Işiksal (2017) or Makovski (2020) that preferred examples over mathematical proofs.

Official documents pay attention to the teaching of proof from a formal point of view, especially in the last years of secondary school. The NCTM (2000) establishes that high school students "should develop a repertoire of increasingly sophisticated methods of reasoning and proof" (p.342). The Spanish's curriculum official contents (LOMCE, Jefatura de Estado, 2013) correspond with Van Hiele level 4 since this document compels to teach formal proof methods. These curriculums are really challenging for both teachers and students and require teachers with a strong understanding of the mathematical argument (Makovski, 2020). In our opinion, official curricula should explain more precisely the desired level of the different mathematical processes that students should acquire. This concreteness will make it possible to determine the desired level for the PSMTs what, in the end, could give us ideas to improve the education of these future teachers. According to our data, students in C1 showed the appropriate level to develop the official curriculum while students in C2 could find some difficulties and students in C3 have a level lower than the expected in their own students and the required to develop the aforementioned contents in order to make them progress from level 2 to the following levels. These results are consistent with Pandiscio and Knight (2010) who showed that most of the PSMTs were statistically under level 4. It is obvious that, if the teacher cannot proof, it would be difficult for him to teach the different proof methods to his students. Moreover, according to Demiray and Işiksal (2017) and Sears (2019), this weakness in the abilities of the (future) teachers could lead to the avoiding of proofs and the discussions about concepts and relations between them or to the relying solely on the textbook as the expert for how to write the proof.

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