# A visualization approach to multiplicative reasoning and geometric measurement for primary-school students: a pilot study

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*Abstract: Understanding the concept of area requires an understanding of the relationship between geometry and multiplication. The multiplicative reasoning required to find the areas of regular figures is used in many courses in elementary mathematical education. This paper explores various methods in which multiplicative reasoning is incorporated into the measurement of area. The main goal is to provide tasks that encourage the application of multiplicative reasoning when students are asked to measure the areas of geometric figures. Student performance is analyzed in two pilot studies of the relationship between geometric measurement and multiplicative reasoning.*

Keywords: Area measurement, geometric measurement, matrix array structure, mathematics education, multiplicative reasoning.

## INTRODUCTION

In mathematics education, multiplicative reasoning (MR) is a well-studied topic. According to Jacob and Willis (2003), "MR results in a multiplicative reaction to a circumstance by identifying or constructing a multiplicand, a multiplier and their simultaneous coordination in that context". It involves paying attention to the multiplicative relationship between magnitudes and quantities, as well as the ability to deal with such situations numerically. MR can be used to understand the inverse relationship between multiplication and division, the part-whole relationship, fractions and proportions, among other things. Recent studies (Al Farra et al., 2022; Khairunnisak et al., 2021; Lyublinskaya, 2009; Migon & Krygowska 2007; Putrawangsa et al., 2021) demonstrated that various in-classroom activities, by implicitly incorporating MR, help students' learning on various mathematical concepts in elementary and middle schools. By contrast, the measurement of area is less studied, with even fewer studies discussing directly the link between measuring and MR (Cavanagh, 2007; Chen & Herbst, 2013; Moyer, 2001; Nohda, 2000; Schoenfeld, 2016; Susac, et al., 2014; Yeo 2008).

In geometry, a measurement involves calculating a new quantity---`the number of units'---by deriving it from known quantities, such as the unit's magnitude and the magnitude of the space to be measured. The target magnitude is the product of the unit and the space to be measured. The concept of geometric measurement is usually based on MR rather than direct counting, and several authors have argued that elementary students are being taught how to measure rather than how to develop the mathematical concept behind measuring certain objects (Carpenter et al., 2003; Ellis, 2007; Fernández et al., 2014; Fujii & Stephens, 2001; Fujii & Stephens, 2008; Greens & Rubenstein, 2008; Hunter 2007; Kieran, 2004; Lins & Kaput 2004; Medov et al., 2020; Molina & Ambrose, 2008; Mulligan et al., 2009; Naik et al., 2005; Tan Sisman & Aksu, 2016; Widjaja & Vale, 2021). Smith III et al. (Smith III et al., 2013) stated that only some students understood that the unit of measurement might be broken down further into smaller subunits to improve precision. In the present study, we look at how MR is used in geometric area measurement, with a particular focus on developing tasks that link the use of MR to finding the area of geometric figures.

In the literature on MR, most of the situations and contexts examine proportionality and involve a linear relation between two one-dimensional measures (Abramovich & Pieper 1995; Ayalon et al., 2016; Cheng et al., 2017; Carpenter et al., 2003), such as those between weight and cost, wage and time, distance and speed, etc. Each of these one-dimensional measures is analogous to length, and many of the issues examined in the context of proportionality have their geometric equivalents in the case of length measurement. For example, in proportional reasoning, unitisation (the process of cognitively chunking discrete units into a bigger, more convenient unit or dividing a unit into smaller units) is crucial (Lamon 2007), and flexible unitisation is used in operations that demand the creation of a `unit of units' (Reynolds & Wheatley 1996). In proportional reasoning, the number line---which is a direct representation of length---can be useful. Curry, Mitchelmore, and Outhred (Curry et al., 2006) outlined the following five measuring principles: (i) the need for congruent units, (ii) using an appropriate unit, (iii) using the same unit for comparing objects, (iv) the relation between the unit and the measure, and (v) the structuring of unit iteration. Each of the five principles listed above requires an understanding of the multiplicative relationships between various quantities in the context of geometric measurement.

As we move from linear measurements to other types of geometric measurement, we discover that MR is used in new ways. In the case of area measurement, MR is first required in the same manner as in length measurement: (i) using sub-units and chunked units (unit of units) in estimating area and (ii) the inverse relationship between size of the unit and the measure. On the other hand, MR also appears in ways in which length measurement does not, such as the array structure of units in the case of rectangles or other two-dimensional figures, leading to the area formula as the product of length and width. There is also a multiplicative relationship between the rectangle's area and the unit, as well as among the area, the length and the width. Similarly, the area and the magnitude of the area unit---which depends on the length and width of the unit---have an inverse relationship. In addition, the transition to non-rectangular figures involves triangulation, and the base case for triangulation, i.e. finding the area of a right-angled triangle by taking half of a rectangle, naturally requires a multiplicative relationship. Consequently, we see that MR is used explicitly and implicitly in various ways in various geometric measurements.

### Geometric Measurement of Area: Two Pilot Studies

The present study is part of a larger one aimed at enhancing teachers' knowledge of students' thinking by designing tasks that support reflective teaching. The phase of the study reported herein explores students' algebraic reasoning when exposed to early algebraic ideas through contexts such as number sentences, pattern generalization, proof and justification, etc. Based on students' reasoning, we plan to prepare student cases for discussion among mathematics teachers and teacher educators. By analyzing students' strategies for area measurement, the present study aims to examine the importance of MR in understanding the relation between spatial attributes (e.g. area) and the numerical values assigned to them. The study aims to develop tasks that have the potential to stimulate different modes of MR in area measurement.

As an example, in the study by Reynolds and Wheatley (Reynolds & Wheatley 1996), students were asked how many cards of a given size (3 cm x 5 cm) are required to cover a rectangle of dimensions 15 cm x 30 cm. The researchers were initially unaware of the fact that the students had understood that the two dimensions of the small card have to completely divide those of the large rectangle to ensure that the number obtained by dividing the bigger area by the smaller area of a card is a whole number. This gives an instance in which the unit is related to the target measure of area of the space in terms of not only the multiplicative relationship between the magnitude of the unit and that of the target measure but also that between either the length or breadth of the unit and the target measure.

Herein, we report two pilot projects labeled 1 and 2 that were done by means of task-based interviews of students. The aim of the two pilot projects was to create tasks that explore MR in activities related to area measurement, and gather preliminary data in the form of student replies and their performances in the given tasks. We picked two different groups of students for these two pilot studies.

## METHODOLOGY

### Pilot study 1. Data Collection

The first task in the study involved tiling exercises in which students had to determine the number of cards of given size required to cover the area of a rectangular piece of paper. The interviews of the students were recorded with their permission and used for analysis. The concept behind pilot study 1 was that it involved not only the numerical relationship between the unit and the measure but also the spatial structuring of the units.

The sample for pilot study 1 comprised ten grade-5 children in two classes from the same school, which was BSC Public Secondary School Kheroda in Udaipur, Rajasthan, India. Five children were picked randomly from each class, and of the ten children, five were labeled as above average, three were labeled as average and two were labeled as below average, with the labeling based on evaluation by their class teachers.

**Tasks involved in Pilot Study 1**

We constructed three tiling tasks (see Table 1) that are comparable to those used by Reynolds and Wheatley (Reynolds & Wheatley 1996). In the first two tasks (T1 and T2), the chosen tile area was a factor of that of the given rectangle to be covered, but a dimension of a tile was not necessarily a factor of the corresponding dimension of the given rectangle; for example, each of the tiles used in the second task (see case S4 of task T2) had dimensions of 3 cm x 2 cm, and the tile width (2 cm) was not a factor of the length (19 cm) of the given rectangle (see Figure 1).

| Task | Material provided | Dimension of rectangle | Shape and dimensions of tile(s) |
|---|---|---|---|
| **T1** | Rectangular paper sheet and three different paper tiles | 21 cm x 12 cm | Rectangle<br>2 cm x 2 cm (S1),<br>3 cm x 4 cm (S2),<br>6 cm x 2 cm (S3) |
| **T2** | Dimensions of rectangle and tiles | 19 cm x 6 cm | Rectangle<br>3 cm x 2 cm (S4) |
| **T3** | Dimensions of rectangle and tiles | 15 cm x 8 cm | Right-angled triangle<br>height: 5 cm, base: 2 cm (S5) |

Table 1: Description of tasks in pilot study 1

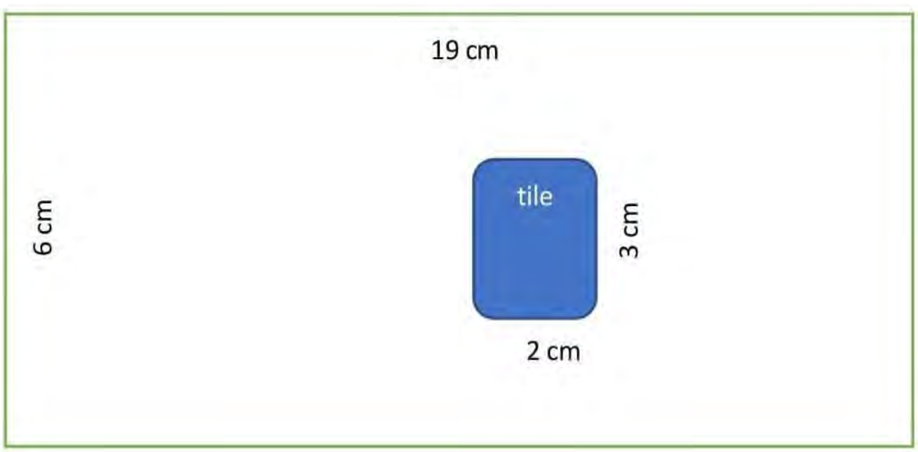


Figure 1: The tile’s width (2 cm) is not a factor of the rectangular length (19 cm)

In each of the tiling tasks, students were asked whether the given tiles, glued over and over, might cover the rectangle and about the number of tiles required to do that. In task T1, students were given a rectangular sheet of the given size and three different types of paper tiles (see S1, S2 and S3). In tasks T2 and T3, students were simply informed verbally about the sizes of the rectangle and tiles, with no tangible objects with which to work; in task T3, students were told to use right-angled triangular tiles to cover the rectangle.

In task T1, students were given physical objects with which to work. In case S2 of task T1, both the length and width of the given rectangle are clearly divisible by the corresponding length and width of the given tiles, so they can tile the rectangle. In cases S1 and S3 of task T1, although the area of the given rectangle is divisible by that of the given tiles, they cannot tile it because the given tiles have even-numbered dimensions and can cover only rectangles of even-numbered dimensions. Specifically, in cases S1 and S3 of task T1, the dimensions of the tiles are 2 cm x 2 cm and 6 cm x 2 cm respectively. Hence, they can tile rectangles of even-numbered length and even-numbered width only. But, in the task T1, the given rectangle has an odd-numbered length (21 cm). As a result, the rectangle given in task T1 cannot be covered by tiles in cases S1 and S3.

The most challenging task is T2. Although the length of the given rectangle is divisible by neither the length nor the width of the given tiles, they can still tile it. Doing so requires a special arrangement of the tiles in the rectangle, and Figure 2 shows a solution for task T2.

In case S5 of task T3, it is obvious that the given tiles can tile the given rectangle. Two of the given right-angled triangles can be glued together to form a 5 cm x 2 cm rectangle, and clearly these rectangles can tile the given 15 cm x 8 cm rectangle.

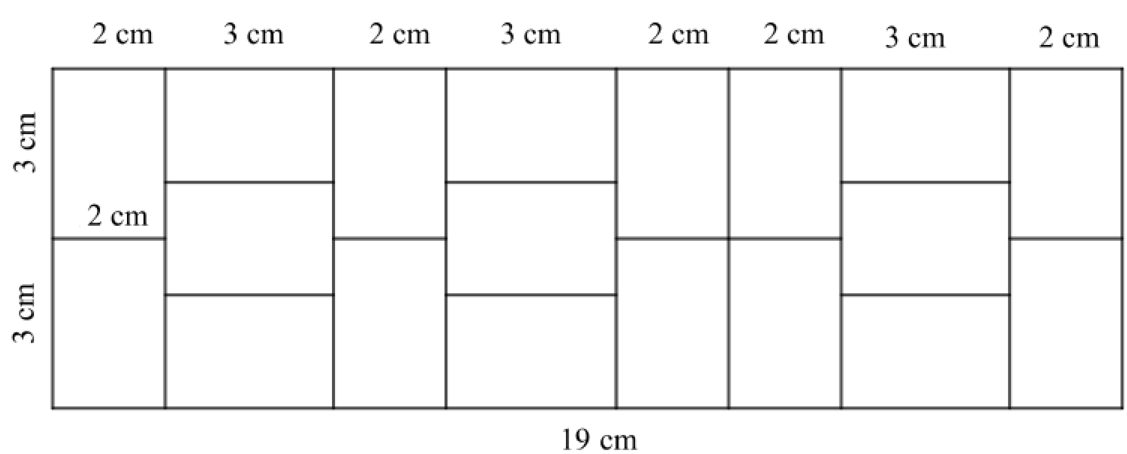


Figure 2: A solution for task T2.

**Pilot Study 2. Data Collection**

In pilot study 2, the student interviews were analyzed with the agreement of the children and their parents, and the data from the interviews were collected and analyzed. The sample comprised eight grade-5 students from the same school as in pilot study 1, and all the students were given identical tasks on which to work.

**Tasks involved in Pilot Study 2**

There were four tasks in total, which are described below.

Task U1: The students were asked to compare two sets of rectangular sheets that differed slightly in either length or width but not both. The goal of this activity was to observe the tactics used by the students in the given task, which were either (i) direct comparison of the sizes of the two sheets by physically trying to cover one with the other or (ii) comparison of one-dimensional quantities such as the lengths or widths of the two sheets. For example, in Figure 3, students were given two sets of rectangular sheets. The first one is a 5 cm x 6 cm rectangle; while the second one is a 5 cm x 7 cm rectangle.

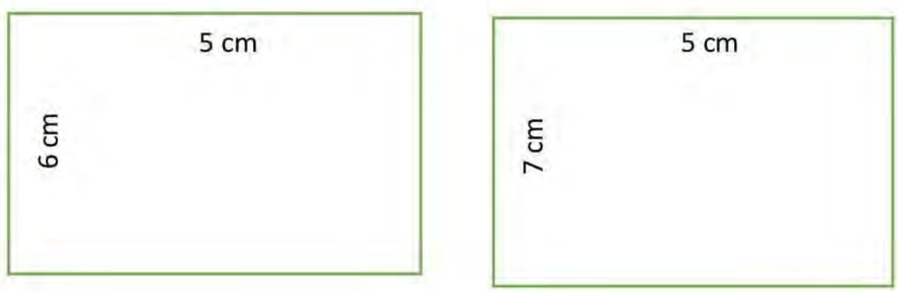


Figure 3: Task U1: Comparison of two rectangle sheets that differ in length but not in width.

Task U2: This task consisted of three closely related but different exercises.

In the first exercise, each student drew randomly from a set of composite numbers and then was asked to collect that number of 1 cm x 1 cm cards. Next, they were asked to construct a rectangle by using all the cards.

The second exercise was essentially the same as the first one, but the students had to complete the task without being given any physical objects: each student drew randomly from a set of composite numbers and then was asked to say whether a rectangle could be constructed by using exactly that number of 1 cm x 1 cm cards.

In the third exercise, each student drew randomly from a set of both prime and composite numbers and then was asked to say whether a rectangle could be constructed from that number of 1 cm x 1 cm cards; if their answer was yes, then they were asked to describe the dimensions of such a rectangle.

The first exercise gave students the opportunity to connect the number of cards with the size of a rectangular array, while the second and third exercises gave students the opportunity to strengthen their mental arithmetic skills. Implicit in this challenge is the multiplicative relationship between the given number and its factors as the length and width of a rectangle. Because the first exercise required physical action by the students, we could determine whether they were implicitly aware of the stated multiplicative relationship, even if they could not articulate it explicitly.

Task U3: Students were asked to compare the sizes of a 7 cm x 7 cm sheet and an 8 cm x 6 cm sheet. They were also given a box full of 1 cm x 1 cm cards to use if necessary (see Figure 4). This task allowed us to investigate the various tactics used by the students to compare the sizes of the sheets (e.g. array structuring, complete covering, etc.). Physically trying to cover one sheet by the other is not helpful in reaching the correct answer, and we investigated whether the students had understood the mathematical concepts from the two preceding tasks (U1 and U2), such as whether they used multiplicative relationship or repetitive addition to measure the two different regions.

Task U4: Students were given an A4 page for this task and were free to use any of the items from the preceding task. They were asked to use either the rectangular sheets or the 1 cm x 1 cm cards from the preceding task to calculate the size of the A4 sheet, and then were asked to use it to calculate the size of a table. This activity was designed to assess whether the students could apply their knowledge of area measurement to larger regions. The use of tangible materials allowed the students to work with the items in whatever way they wished, and the use of repeated multiplicative operations was required in this task to reduce the number of steps needed to reach the final answer.

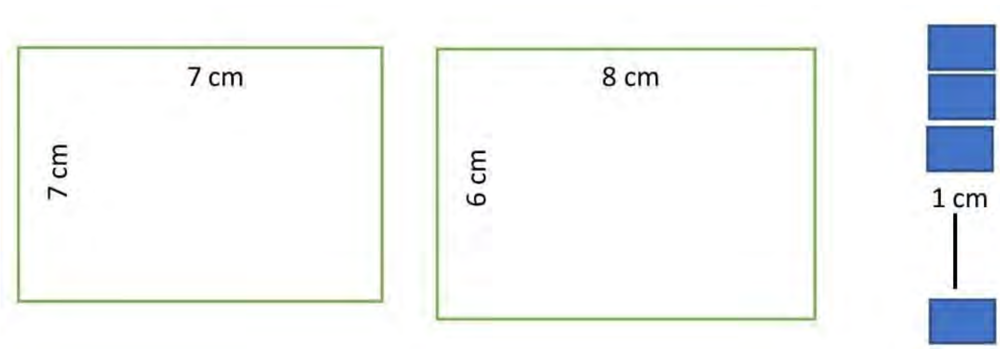


Figure 4: Task U3: Comparison of two rectangular sheets. Students can choose to complete the task by using 1 cm x 1 cm cards.

## RESULTS AND DISCUSSION

### Results and Analysis of Pilot Study 1

After all students had completed the tiling exercises, we noticed that most had completed tasks T1 and T2 using two different tactics, which are explained as follows.

(1) Dividing the area of the given rectangle (determined by numerically multiplying its length by its width) by that of a tile to find out whether the tiles can be used to cover the rectangle fully.

Four students completed all the tasks by using this method. Therefore, in the cases in which a tile dimension is not a factor of the corresponding rectangle dimension (see cases S1 and S3 of task T1), those students provided incorrect answers to the exercises. The tricky point in these situations is that the given tiles (of even-numbered dimensions) cannot cover the given rectangle of length 21 cm, even though the area of the given titles (4 square centimeters in case S1 and 12 square centimeters in case S3 respectively) divides the area of the given rectangle (252 square centimeters).

In particular, in task T1, they did not take advantage of being given physical objects with which to work (a piece of rectangular paper and the tiles of given size). Their performance showed that they did not fully understand the importance of the dimensions of the objects in such tiling tasks, nor did they understand the close relationship between the spatial structures of the tiles and that of the rectangular paper.

(2) Checking the dimensions of the tiles along the dimensions of the rectangular sheet.

Four students used this strategy in all the tiling tasks, giving all the answers correctly in task T1 but not in task T2. After pilot study 1, several students indicated that they would have solved task T2 correctly had they been given physical objects with which to work. Clearly, those students understood the importance of measuring the dimensions of the objects in such tiling tasks, and they showed a deeper understanding of the geometrical structures of the objects. Also, a few of them may even have understood that the rectangular sheet can be covered fully by arranging the tiles in special ways, as mentioned in the `Remarks' section.

Interestingly, two students started task T1 by using the second strategy but then switched to the first strategy for tasks T2 and T3. This shows that some students may be inclined to look at the dimensions of the objects when they are given physical objects with which to work (task T1), rather than merely the numerical values of the areas of the figures. They solved task T1 correctly, which may have been simply by trial and error with the physical objects in their hands; it is unclear whether they really understood the importance of the dimensions of the objects in these tiling exercises. Further investigation and studies may be required to see whether these students understood implicitly the multiplicative relationship between the dimensions and areas of the figures.

None of the students solved task T2 completely correctly, which is perhaps understandable given that they were given this task only verbally; it may be necessary to give students the physical objects with which to play. Tasks in this format (with no physical objects given to the students) may be too challenging even for students with a good understanding of the dimensions of objects in tiling exercises.

In task T3, every student stated initially that the triangular tiles could not be used to cover the rectangle entirely, but two students changed their minds later after realizing that two such triangles could be combined to form a rectangle. This showed that a few of them were at least implicitly aware of the multiplicative relationship based on the geometric division of a figure into two equal parts.

These examples show that it is not enough to understand the multiplicative relationship between the area of a tile and that of the rectangular paper: it is also necessary to understand the multiplicative relationship between the dimensions of the tiles and those of the rectangular paper. In particular, as shown in cases S1 and S3 in task T1, it is clear that the area of the given rectangle (252 square centimeters) is divisible by the area of the tiles (4 square centimeters in case S1 and 12 square centimeters in case S3 respectively). But the given tiles (of even-numbered dimensions) cannot be used to cover a rectangle which has an odd-numbered length (21 cm).

In addition, in task T2, the students' performance could have been much better had they been given physical objects. Further investigation may be required to understand how physical exercises such as this can improve students' learning in mathematics education in a classroom setting.

**Discussion about Pilot Study 1**

Before pilot study 1, eight of the ten students had learned the formula for the area of a rectangle (as the product of its length and width), and so they calculated the area by using the product formula during the tasks. Furthermore, they divided the area of the rectangular sheet by that of a tile of given size so that they could relate the tiling problem to the areas of the objects. However, they did not give correct answers to all the tiling tasks (in particular, cases S1 and S3 of task T1). This was mainly because they failed to check whether the tile dimensions divided the rectangle ones. This shows that several students may have learned how to compute the area of a figure simply by memorizing the formula correctly, without a deeper conceptual understanding of the multiplicative relationship between the dimensions and area of a figure.

As a side note, it is worth comparing the students' academic performance in school with their results in these tasks. All the students labeled above average or average (eight in total) had learned the formula for the area of a rectangle, but two used the first strategy to tackle all the tasks. In particular, they did so by ignoring the dimensions of the objects in task T1, for which the physical objects were provided. This shows that several students who achieved good academic results in mathematics may not have fully understood the concepts behind the formulas learned in class. To foster students' conceptual understanding in mathematics education, it may be important to provide physical tasks (such as those in pilot study 1) more frequently to students; such hands-on experience may help students to understand the multiplicative relationship between the dimensions and area of a figure.

It is unclear whether the students who used the formula for the area of the rectangle were aware of the array of unit squares that covered the rectangle. The data from pilot study 1 show that more research is needed into students' understanding of the relationship between MR and area measurement. In particular, it is necessary to create challenges that could elicit such thinking.

In pilot study 2 discussed next, we devised a series of activities focused on the array structure of square units in rectangles of various dimensions.

**Results and Analysis of Pilot Study 2**

In the following subsections, we report the results pertaining to the students' performances in pilot study 2. We make detailed observations on their behavior throughout the tasks, and we analyze their performances as follows.

Task U1: The difference in length or width (but not both) was not visually clear, and the students could not easily tell the difference between the paper sheets by simply glancing at them. When the rectangular pieces of paper were laid flat on the table next to each other, all but one of the eight students tended to compare them by length or width. Later, they compared the sheets by overlapping them. To compare the sizes of the rectangles, the natural tendency of most students was to compare their sides, which reflects an intuitive awareness of the relationship between the dimensions and area of a rectangle.

Task U2: After a few failed attempts, four of the students gradually understood the relationship between the factors of the given composite number and the rectangle that could be constructed from it. However, the remaining four students were unable to spot this relationship during the exercises. It seems that several of the students used the multiplicative relationship between the number of cards and their arrangements in a two-dimensional array in some of these exercises. On numerous occasions, we found that students used a factor of the given composite number to make the first row of cards, but they were unable to explain why they chose that particular number to start with. The conceptual understanding of the multiplicative relationship between numbers and their factors was not solid for these students; this was clear from the fact that they did not use the same technique consistently to solve the exercises.

In one instance, a student constructed a 4 cm x 3 cm rectangle and a 6 cm x 2 cm rectangle by using 12 square cards for each of them; he also said that a 3 cm x 5 cm rectangle could be created by using 15 cards. However, when asked afterwards about the dimensions of rectangles that could be created with 10 and 13 cards, he responded incorrectly that they would be 3 cm x 7 cm and 3 cm x 10 cm, respectively. Another student, who correctly formed a 7 cm x 2 cm rectangle by using 14 cards, said that an 8 cm x 6 cm rectangle could be formed by using 14 cards. Apparently, he wrongly thought of the relationship between the area and dimensions of the rectangle as being 6+8=14; this mistake arose even though he answered correctly in the preceding exercises. Interestingly, several students attempted to form rectangles with fractions as their widths and lengths. For example, two of the eight students cut a few of the cards in half in an attempt to create rectangles: one student was able to make a 7.5 cm x 2 cm rectangle out of 15 cards, while another student created a 5.5 cm x 4 cm rectangle out of 22 cards. It seems that some students attempted the exercises using this strategy when the given composite number (i.e. 15 and 22 in these cases) had fewer non trivial factors.

After completing the tasks, one of the eight students stated explicitly that he was looking at the factors of the number in order to create the rectangle. We also observed that three students used multiplication tables explicitly and consistently. The remaining four students switched between additive and multiplicative relationships throughout the exercises, without sticking to the same strategy consistently. This shows that the latter four students may not have had a solid understanding of MR. More specifically, we note some interesting facts about the latter four students. Two of them began with a factor of the provided number. One of them was unable to explain why he chose that particular number of cards. Another student claimed that the number occurred to her without her having to think about it. On another occasion, one of the students said that a square can be created using a product of the same number, such as 6 x 6, 1 x 1, 10 x 10 and 3 x3. This shows that a few of them understood the relationship between a number and a square, even though they may not have had an excellent understanding of MR in geometric measurement. It appears that those students (i.e. the first four) who had a better understanding of the concepts of multiplication were able to make the measurement--multiplication connection faster than those who were unaware of it. To certain extent, the remaining four students may have understood multiplication partially, but we cannot confirm that their understanding was solid and complete.

The exercises in task U2 are intriguing because the students had the option to break down a given number by using either additive or multiplicative relationships, and the two ways of thinking---at least in the context of the given exercises---are related somehow to a certain degree. The students may have viewed rectangles in two different ways: either as a two-dimensional array or as the border of an empty rectangle. The latter way of viewing a rectangle was observed in two students, who started to put the cards on the sides of the given rectangle while leaving its interior empty. The impact of these two distinct ways of visualizing a geometric figure while learning about area is intriguing, and it would be interesting to investigate this subject further.

Based on pilot study 2, it is true to say that the students' performances at the tasks improved significantly when they were allowed to work on the exercises physically with the given items. A few of the students may have obtained the correct answers simply by trial and error, without a full understanding of the MR required for the exercises, but the fact that several students took quite a bit of time to get the correct answers when they were given the physical objects to work with suggests that a few of them may have gradually understood the required MR when they worked through the exercises.

Task U3: One student compared the additional space left on both the square and the rectangle after the two sheets were placed one above the other; he discovered that the width of the extra space left on the square was one unit wide, while the length of the extra space left on the rectangle was also one unit long. Consequently, he stated incorrectly that both sheets had the same size. However, he failed to take into account the fact that the extra unit of length left in the square could be filled by using seven 1 cm x 1 cm cards, while the extra unit of width left in the rectangle could be filled by using only six 1 cm x 1 cm cards. The remaining students completed this task correctly but by using different approaches.

The first approach was to cover one sheet by the other and then use the 1 cm x 1 cm cards to measure the size of the extra space left over on the square and the rectangle. Four students took this approach.

The second approach was to find the total number of 1 cm x 1 cm cards required to fill up each sheet. Three students took this approach. To calculate this number, the students first filled the outermost sides of the two sheets with the square cards, then most of them repeated this action until the whole sheets were filled with the cards. This essentially means that these students obtained the area of the sheets by the process of repetitive addition. Only one student calculated the total number of cards required by multiplying the numbers of cards used on the length and width. This is essentially the process of getting the area of a rectangle by the product of its length and width.

Interestingly, most of the students preferred to measure the size of a rectangle by using the given square cards to fill its empty space repeatedly. This is essentially an additive process rather than the more advanced multiplicative process. Even among the students who completed task U2 by using MR, they preferred the additive approach to task U3. The reason for their preference is unclear because they did not mention any specific reason in the interviews after the tasks; it is probably because students of this age group prefer to complete such tasks physically in the most elementary way (if the objects are all given), rather than thinking about the hands-on tasks mathematically.

Task U4: Six students completed this task, while the remaining two students did not. Three of the students who completed the task used the multiplicative relationship, while the remaining three used repetitive addition; the students who did not complete the task simply did not understand the MR required for it, and they were also the students who performed poorly in task U2. For example, the students who used the multiplicative relationship to complete the task computed the total number of cards required to fill the whole table as the product of 10 x 100 once they realized that an A4 sheet could be filled by 100 cards and a table could be filled by ten A4 sheets.

Note that the students were required to use a nonstandard chunked unit (i.e. the A4 paper) rather than a standard square unit to measure the area of the table, and we consider task U4 to be more challenging than task U3. In task U4, the understanding of geometric division of the measure in terms of this new multiplicative unit is required. Put another way, the students had to understand the connection between the numerical and geometrical aspects of the given figures to complete the task. A student was considered to have excellent understanding of MR if they completed this task by using the multiplicative relationship between the chunked unit and the standard square unit.

**Discussion about Pilot Study 2**

In this pilot study, the students were given opportunities to connect area measurement to multiplication directly. As well as the results stated above, we list the following three key findings.

(1) Several of the students tended to focus on the sides of a rectangle only, rather than on the area that it covered. Consequently, some students tended to compare the sides in task U1, and in task U2, a few of them occasionally failed to fill the inside of the rectangle.

(2) In task U2, when the students were asked to construct rectangles with a given number of cards, a few of them incorrectly used the additive relationship between the numbers rather than the multiplicative one. We consider this error to be a conceptual one, and several of the students clearly lacked the understanding of the MR required in area measurement.

(3) It seems that several of the students understood the multiplicative relationship between numerical quantities and the area of a geometric figure but were unable to explain it explicitly. Consequently, they were unable to articulate the multiplicative concepts involved in the physical tasks of area measurement, even though they got the answer correctly.

From a methodological perspective, the tasks used in pilot study 2 allow students to examine the array structures of geometric figures and their relationships to the multiplication of numbers. Students are also able to explore the concepts of MR based on physical objects in hands, even when they are unable to articulate it.

We recorded and analyzed the students' performances in each of the four tasks separately in this pilot study. However, we did not study how exposure to one task, such as task U2, may have influenced the students' performance in other tasks, and further study may be required in that regard.

## CONCLUSIONS

The use of MR has been investigated in various fields in mathematics education. The present pilot studies showed that by connecting geometric measurement to MR, students may improve their understanding of geometric measurement. The two pilot studies allowed students to explore various geometric quantities and structures and their connections to the addition and multiplication of numbers. Providing tasks similar to those in our pilot studies may be useful in fostering students' learning in elementary mathematics education. We note that the sample sizes taken in our pilot studies were small. Inspired by the results shown in these studies, our next goal is to conduct similar studies on a larger sample size of students in classroom settings. On the other hand, it would be equally important for education researchers to conduct similar studies on students chosen from other countries. In the future, we would be interested in collaboration with school teachers in the region. Our main focus is to understand how students, regardless of their nationality and social upbringing, connect MR to different tasks of geometric measurement.

## Acknowledgments

The authors are grateful to the financial support given by the Ministry of Education in the United Arab Emirates. The authors are also grateful to the support given by the United Arab Emirates University.

### Declarations

**Author contribution:** Sonal Jain: Conceptualization, Writing - Original Draft & Editing, Formal analysis, and Methodology; Ho-Hon Leung: Writing - Original Draft, Review & Editing, Formal analysis, and Supervision; Firuz Kamalov: Writing - Review & Editing.

**Funding Statement:** This research was funded by the Collaborative Research Program Grant 2019 (CRPG 2019), which is supported by the Ministry of Education in the United Arab Emirates.

**Conflict of Interest:** The authors declare no conflict of interest

## REFERENCES

[1] Al Farra, N. K., Al Owais, N. S., & Belhase, S. (2022). Computational, Logical, Argumentative, and Representational Thinking in the United Arab Emirates Schools: Fifth Grade Students' Skills in Mathematical Problem Solving. *Mathematics Teaching-Research Journal*, 14(1), 215-252. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/04/v14n1-Computational-Logical-Argumentative.pdf

[2] Abramovich, S., & Pieper, A. (1995). Fostering recursive thinking in combinatorics through the use of manipulatives and computing technology. *The Mathematics Educator,* 6(2). Retrieved from https://openjournals.libs.uga.edu/tme/article/view/1789/1697

[3] Ayalon, M., Watson, A., & Lerman, S. (2016). Progression towards functions: Students' performance on three tasks about variables from grades 7 to 12. *International Journal of Science and Mathematics Education*, 14(6), 1153-1173. https://doi.org/10.1007/s10763-014-9611-4

[4] Battista, M. T. (2007). *The development of geometric and spatial thinking.* In F. K. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (pp. 843-908). Charlotte, NC: Information Age.

[5] Cheng, S.-C., She, H.-C., & Huang, L.-Y. (2017). The impact of problem-solving instruction on middle school students' physical science learning: Interplays of knowledge, reasoning, and problem solving. *Eurasia Journal of Mathematics, Science*

*and Technology Education,* 14(3), 731-743. https://doi.org/10.12973/ejmste/80902

[6] Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School.* Portsmouth, NH: Heinemann.

[7] Carraher, D.W. & Schliemann, A. D. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 669-705). Charlotte, NC: Information Age.

[8] Cavanagh, M. (2007). Year 7 students' understanding of area measurement. In K. Milton, H. Reeves, & T. Spencer (Eds.), *Proceedings of the 21st biennial conference of the Australian Association of Mathematics Teachers Inc.* (pp. 136-143). Adelaide: Australian Association of Mathematics Teachers.

[9] Chen, C.-L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics*, 83(2), 285–307. http://doi.org/10.1007/s10649-015-9652-9

[10] Curry, M., Mitchelmore, M., & Outhred, L. (2006). Development of children's understanding of length, area and volume measurement principles. In J. Novotna, H. Moraova, M. Kratka, N. Stehlikova (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 377-384), Prague, Czech Republic. Retrieved from https://research-management.mq.edu.au/ws/portalfiles/portal/17070807/mq-10586

[11] Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics education*, 38(3), 194-229. https://www.jstor.org/stable/30034866

[12] Fernández, C. H., De Bock, D., Verschaffel, L., & Van Dooren, W. (2014). Do students confuse dimensionality and "directionality"? *Journal of Mathematical Behavior*, 36, 166-176. http://doi.org/10.1016/j.jmathb.2014.07.001

[13] Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: The role of quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12 th ICMI Study Conference* (Vol. 1, pp. 258–264). Melbourne, Australia.

[14] Fujii, T., & Stephens, M. (2008). *Using number sentences to introduce the idea of variable.* In *Algebra and algebraic thinking in school mathematics: Seventieth* Yearbook (pp. 127-140). National Council of Teachers of Mathematics.

[15] Greens, C. E., & Rubenstein, R. N. (2008). *Algebra and Algebraic Thinking in School Mathematics; Seventieth Yearbook.* National Council of Teachers of Mathematics.

[16] Hunter, J. (2007). Relational or calculational thinking: Students solving open number

equivalence problems. In J. Watson & K. Beswick (Eds), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 421-429), Adelaide: MERGA.

[17] Jacob, L. & Willis, S. (2003) The development of multiplicative thinking in young children. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia* (Vol.1, 460–467), Geelong, Vic: MERGA.

[18] Khairunnisak, C., Johar, R., Yuhasriati, Zubainur, C. M., Suhartati, & Sasalia, P. (2021) Learning Trajectory of Algebraic Expression: Supporting Students' Mathematical Representation Ability. *Mathematics Teaching-Research Journal*, 13(4), 27-41. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/01/v13n4-Learning-Trajectory-of-Algebraic-Expression.pdf

[19] Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8(1), 139-151. Retrieved from https://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=7ED762CE131CB4011BCE097E747328C6?doi=10.1.1.513.5908&rep=rep1&type=pdf

[20] Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: The current state of the field. In In K. Stacey, H. Chick, & M. Kendal (Eds.). *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study* (Vol. 8, pp. 45-70), Springer, Dordrecht: Kluwer Academic Publishers. https://doi.org/10.1007/1-4020-8131-6_4

[21] Lyublinskaya, I. (2009). Developing Number Sense with Technology-Based Science Experiments: Reflections on Classroom Practice in Primary Grades and Pre-service Education. *Mathematics Teaching-Research Journal*, 3(2). Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2018/12/v3n2-Developing-Number-Sense-with-Technology.pdf

[22] Medová, J., Bulková, K. O., & Čeretková, S. (2020). Relations between Generalization, Reasoning and Combinatorial Thinking in Solving Mathematical Open-Ended Problems within Mathematical Contest. *Mathematics*, 8(12), 2257. https://doi.org/10.3390/math8122257

[23] Migon, J., & Krygowska, P. (2007). Diagnostic procedure in algebra course of the first grade of the middle school. *Mathematics Teaching-Research Journal,* 1(2). Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2018/12/v1n2-Diagnostic-procedure-in-algebra.pdf

[24] Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics*, 30(1), 61-80. Retrieved from https://core.ac.uk/download/pdf/12341174.pdf

[25] Moyer, S. P. (2001). Using representations to explore perimeter and area. *Teaching Children Mathematics*, 8(1), 52-59.

[26] Mulligan, J., Vale, C., & Stephens, M. (2009). Understanding and developing structure—Its importance for mathematics learning. *Mathematics Education Research Journal*, 21(2), 1-4. https://doi.org/10.1007/BF03217541

[27] Nohda, N. (2000). Teaching by Open-Approach Method in Japanese Mathematics Classroom. In T. Nakahara, & M. Koyama (Eds.) *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 39–53), Hiroshima, Japan. Retrieved from https://files.eric.ed.gov/fulltext/ED466736

[28] Naik, S., Banerjee, R., & Subramaniam, K. (2005). Understanding Student's Reasoning while Comparing Expressions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.) *Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 569–576), Sydney: MERGA.

[29] Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 1, pp. 629-668). Charlotte, NC: Information Age Publishing.

[30] Putrawangsa, S., Febrian, & Hasanah, U. (2021). Developing Students' Understanding of Percentage: The Role of Spatial Representation. *Mathematics Teaching Research Journal*, 13(4), 4-26. Retrieved from https://commons.hostos.cuny.edu/mtrj/wp-content/uploads/sites/30/2022/01/v13n4-Developing-Students-Understanding.pdf

[31] Reynolds, A., & Wheatley, G. H. (1996). Elementary students' construction a coordination of units in an area setting. *Journal for Research in Mathematics Education*, 27(5), 564-581. https://doi.org/10.2307/749848

[32] Smith III, J. P., Males, L. M., Dietiker, L. C., Lee, K., & Mosier, A. (2013). Curricular treatments of length measurement in the United States: Do they address known learning challenges? *Cognition and Instruction*, 31(4), 388-433. https://www.jstor.org/stable/23525183

[33] Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. *Journal of Education*, 196(2), 1-38. https://doi.org/10.1177/002205741619600202

[34] Susac, A., Bubic, A., Vrbanc, A., & Planinic, M. (2014). Development of abstract mathematical reasoning: the case of algebra. *Frontiers in Human Neuroscience*, 8, 679. https://doi.org/10.3389/fnhum.2014.00679

[35] Tan Sisman, G., & Aksu, M. (2016). A study on sixth grade students' misconceptions and errors in spatial measurement: length, area, and volume. *International Journal of Science*

*and Mathematics Education*, 14(7), 1293-1319. https://doi.org/10.1007/s10763-015-9642-5

[36] Widjaja, W., & Vale, C. (2021). Counterexamples: Challenges Faced by Elementary Students When Testing A Conjecture about the Relationship between Perimeter and Area. *Journal on Mathematics Education,* 12(3), 487-506. http://doi.org/10.22342/jme.12.3.14526.487-506

[37] Yeo, J. K. (2008). Teaching area and perimeter: Mathematics-Pedagogical-Content-Knowledge-in-Action. In M. Goos, R. Brown, & K. Makar (Eds.) *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 621-627). Brisbane: MERGA.