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Mathematical Knowledge of Pre-Service Early Childhood and Primary Education Teachers: An Approach Based on the Design of Tasks Involving Patterns

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Abstract This study analyzes the mathematical knowledge of 40 pre-service Chilean Early Childhood and Primary Education teachers when designing mathematical tasks on patterns, in the context of teaching early algebra. Based on the domains of the Mathematical Knowledge for Teaching (MKT) model, we have adopted a descriptive qualitative methodological approach that relies on the content analysis technique. The results show that pre-service teachers exhibit little mathematical knowledge in the description of the mathematical tasks they pose, addressing partial aspects of the subdomains of specialized content knowledge and knowledge of content and teaching. We conclude that training experiences should be given that allow students to further their acquisition of mathematical knowledge in order to achieve effective learning of the contents that early algebra promotes, such as patterns, in Early Childhood and Primary Education classrooms.

Keywords: Mathematical knowledge for teaching, design of tasks, early algebra, patterns, pre-service teachers, Early Childhood Education, Primary Education.

Introduction

In recent decades, early algebra has emerged as a proposal for curricular improvement, the goal being to promote the development of algebraic thinking from the initial years of schooling (Cai & Knuth, 2011; Carpenter et al., 2003; Kaput, 2000). This approach seeks to "algebrize" the curriculum in order to integrate algebraic thinking through all phases of schooling and facilitate a better understanding of mathematics (Kaput, 2000).

Early algebra has led to a growing transformation in terms of curricular adjustments, both for Early Childhood (ages 3 to 6) and Primary (ages 6 to 12) education. As a result, contemporary curricula (e.g., Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015; Ministry of Education [MINEDUC], 2012; 2018; Ministry of Education, Republic of Singapore, 2012; 2013; National Council of Teachers of Mathematics [NCTM], 2000) have explicitly incorporated algebraic knowledge into their study plans from an early age (Pincheira & Alsina, 2021a).

This knowledge includes a study of patterns as a driver of early algebraic thinking, since it contributes to the development of mathematical representation and abstraction (Papic, 2015). Likewise, an early development of patterns and an understanding of their structure facilitate mathematical performance and provide an essential basis for promoting the process of generalization (Mulligan & Mitchelmore, 2009).

This scenario of curricular changes and integration poses a real challenge for teachers in the early educational stages, and prompts them to build a solid base for understanding and dealing with experiences involving early algebra (NCTM, 2000). Against this backdrop, the design of mathematical tasks is part of the development of teaching practice to organize classes, since these tasks play an important role in classroom experiences for students and teachers alike (Wake, 2018). However, the design of tasks is influenced by the mathematical knowledge that teachers have for teaching (Sullivan et al., 2015). Therefore, when designing a task, teachers leave evidence of their mathematical knowledge and perception about the learning and teaching of mathematics (Thanheiser, 2015).

Based on the guidelines posed by early algebra for introducing algebraic knowledge into early levels of schooling and the importance of incorporating tasks that elicit the effective teaching of the knowledge derived from this content block, such as patterns, it is necessary to pay more attention to the mathematical knowledge demonstrated by teachers when designing these tasks, since teachers are the key to the opportunity to learn mathematics (Even & Ball, 2009).

From this perspective, this study focuses on the mathematical knowledge possessed by pre-service Early Childhood and Primary Education teachers, given the impact on the performance of their educational practices, which, by introducing the teaching of early algebra from the first years of schooling, are transformed into key agents for implementing the knowledge promoted by this field. From this point of view, we ask ourselves: What mathematical knowledge do pre-service Early Childhood and Primary Education teachers possess when designing mathematical tasks involving patterns?

Research on teachers' knowledge to teach mathematics has given rise to a variety of analysis models, including: the Knowledge Quartet (KQ) proposed by Rowland, Huckstep and Thwaites (2005); Mathematical Knowledge for Teaching (MKT) raised by Ball, Thames and Phelps (2008); Didactic-Mathematical Knowledge and Competences (CCDM) model by Godino et al. (2017); and Mathematics Teacher's Specialized Knowledge (MTSK) model proposed by Carrillo et al. (2018).

To answer the research question, we turn to the Mathematical Knowledge for Teaching (MKT) model, since it considers a set of knowledge and skills that teachers need to manage the recurring tasks and problems involved in teaching mathematics (Ball et al., 2008). Based on this framework, the goal of our study is to analyze the mathematical knowledge that pre-service Early Childhood and Primary Education teachers evoke when designing a mathematical task to promote the study of patterns.

Theoretical Foundation

In the sections that follow, we discuss patterns as a mathematical object of the study, followed by the design of tasks as part of the development of teaching practice, and finally, the MKT model, which provides the theoretical/analysis tool used to delve into the nature of the mathematical knowledge involved in designing tasks. These three main areas will help us understand the aspects to consider when designing mathematical tasks involving patterns.

Mathematical Patterns

Early algebraic thinking is defined as "the reasoning engaged in by 5- to 12-year-olds as they build meaning for the objects and ways of thinking to be encountered within the later study of secondary school algebra" (Kieran, 2022, p.1131). Accordingly, a key element in general mathematical activity, and in algebraic thinking in particular, is the process of generalization (Kaput, 2008; Papic, 2015).

Generalization is a mental process that is regarded as a prerequisite for achieving mathematical abstraction, since "to generalize is to derive or induce from particulars, to identify commonalities, to expand domains of validity" (Dreyfus, 2002, p.35).

The recognition and analysis of mathematical patterns, defined as "any predictable regularity, usually involving numerical, spatial or logical relationships" (Mulligan & Mitchelmore, 2009, p.34), offers children the opportunity to observe and verbalize generalizations, and to register them symbolically (Threlfall, 1999). Various authors (e.g., Clements & Sarama, 2009; Mason et al., 2009) believe that the exploration of patterns lays the foundations to promote generalization and encourage algebraic thinking, since it allows children to coordinate their perceptive and symbolic inference skills such that they are able to build a plausible structure that is algebraically useful (Rivera, 2010). The ability to observe regularities is developed by children intuitively from the first years of schooling (Carpenter et al., 2003) through actions, behaviors, visual representations, musical melodies, and in other ways (Liljedahl, 2004). Accordingly, "patterns provide a way for children to recognize, order and organize their world" (NCTM, 2000, p.95).

Mulligan and Mitchelmore (2009) state that in mathematical patterns, it is necessary to differentiate between a pattern as an ordered sequence or serialization (e.g., ABABAB), and as a structure, that is, the rule or core underlying the pattern (e.g., AB). Thus, patterns exhibit a cognitive component linked to the recognition of their structure, and a meta-cognitive component associated with the ability to find and analyze patterns. The complexity of the repeating patterns is related to adding more elements and to the variability of the core pattern, that is, its color, orientation or shape (Liljedahl, 2004).

There is a series of tasks that can be used to operationalize work with patterns and develop skills. Lüken and Sauzet (2020) define these skills as "children's competencies regarding repeating patterns" (p. 29).

The most frequent tasks in the literature are: duplicating the same pattern; finding missing elements in a sequence; expanding the sequence; building the same pattern with different materials; identifying the unit of repetition; and inventing a pattern. The skills to make patterns that mobilize these tasks are: copy, interpolate, extend, abstract, or translate, recognize the unit of repetition, and create, respectively (Clements & Sarama, 2009; Lüken & Sauzet, 2020; Rittle-Johnson et al., 2013; Wijns et al., 2019). The level of difficulty between the tasks is progressive (Lüken & Sauzet, 2020).

Various studies have reported that children progress in their pattern-making skills in Early Childhood Education and early Primary Education when working with mathematical patterns (e.g., Clements & Sarama, 2009; Lüken and Sauzet, 2020). At about the age of three or four, children are able to perform tasks that require copying skills, since they have a basic level of difficulty (Clements & Sarama, 2009; Rittle-Johnson et al., 2015). They are then successfully initiated in the interpolation and extension skills from the age of four (Clements & Sarama, 2009). Finally, at around the age of five or six, children develop the ability to identify the unit of repetition and transfer that knowledge to translate and create a certain pattern (Lüken & Sauzet, 2020; Rittle-Johnson et al., 2013).

According to Threlfall (2005), understanding the unit of repetition of a sequence is a crucial step in the mathematical development of children. However, identifying the unit of repetition is one of the most difficult tasks, even for nine-year-olds (Warren & Cooper, 2007).

Design of Mathematical Tasks

Mathematical tasks play a fundamental role in teaching, since student learning is determined by the type of task that is presented to them (Sullivan et al., 2010). According to Smith and Stein (1998), mathematical tasks should promote high levels of thinking in students. Thus, to achieve effective mathematics learning, teachers need to involve students in tasks that are challenging (NCTM, 2015).

By mathematical task we mean the information that drives the work with the students, including representations, context, questions and instructions (Sullivan et al., 2012). Designing tasks is part of the professional endeavor that teachers undertake as part of teaching mathematics (Llinares, 2011). Therefore, by working with mathematical tasks, teachers improve their mathematical knowledge and their capacity for mathematical-didactic design (Pepin, 2015).

Liljedahl et al. (2007) concluded that the design of a mathematical task constitutes a recursive process that involves both the creation of completely new tasks and the adaptation or refinement of existing tasks. According to Liljedahl et al. (2007), to develop this process, teachers must consider four phases: 1) *predictive analysis* is related to the personal experience that teachers have with solving the task, as well as to their experience using similar tasks. This experience helps with the design of the task and its implementation; 2) *testing* refers to the period of trying out or implementing the task in a classroom context. This phase allows analyzing a variety of interpretations and solutions in the task; 3) *reflective analysis* to evaluate the pedagogical aspects and decision-making put into practice in the previous phase; and 4) *the adjustment* is related to re-designing the task and/or its implementation. The interaction of the teaching staff in this recursive process allows them to broaden their knowledge and acquire a better understanding of the possibilities that the task provides for student learning.

Another important aspect to consider in the design of tasks is the proposal by Jones and Pepin (2016), who suggest that the design of mathematical tasks requires paying attention in terms of: what to design, that is, individual tasks, groups of tasks or task sequences; what tools are needed or beneficial to design the task and analyzing the possibilities and limitations of the tools or resources for the thematic area involved in the task; finally, under what conditions the task is designed, that is, the purpose of the task and who the agents involved in its design will be.

A mathematics task can be practiced through various teaching contexts. Alsina (2019; 2020) refers to the term teaching itineraries as an intentional sequence that begins in informal contexts and can be used to visualize mathematical ideas concretely (everyday life situations, manipulative materials and games); it continues in intermediate contexts that, through exploration and reflection, lead to the progressive schematization and generalization of mathematical knowledge (literary and technological resources); and ends in formal contexts, in which the representation and formalization of mathematical knowledge is practiced through conventional procedures and notations, thus completing the learning from the concrete to the symbolic (graphic resources and textbooks).

Thus, the design of mathematical tasks is something that teachers do on a daily basis, and that requires a broad view of what to consider for their development. From our point of view, designing a mathematical task requires teachers to know the learning objective that is intended to be addressed, and the teaching context on which the design of the task will be based. They must also have broad knowledge of the mathematical content to be taught and its didactics so as to consider sequences of tasks that enhance learning. Finally, they must be clear as to the depth of the content they want to present, based on the intended educational level of the task.

In the case of designing pattern tasks, it is important to consider the tasks for making patterns mentioned in the previous section: duplicate the same pattern; find missing elements of a sequence; expand a sequence; build the same pattern with different elements; identify the unit of repetition; invent a pattern (Clements & Sarama, 2009; Lüken & Sauzet, 2020; Rittle-Johnson et al., 2013; Wijns et al., 2019).

Mathematical Knowledge for Teaching (MKT)

The MKT model proposed by Ball et al. (2008), emerges from the advances put forward by Shulman (1986,1987) in the framework of the teacher professional knowledge model. This model has been developed as an analytical tool of teacher knowledge and is defined as “the mathematical knowledge that teachers uses in the classrooms to produce instruction and student growth” (Hill et al., 2008, p.374).

MKT considers two major domains of mathematical knowledge for teaching: subject matter knowledge and pedagogical content knowledge.

Subject matter knowledge includes three sub-domains: common content knowledge (CCK), which refers to “mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p.399), that is, it corresponds to the management that can be achieved throughout the educational levels and that any person who faces a mathematical task possesses; specialized content knowledge (SCK), which refers to “mathematical knowledge and skill unique to teaching” (Ball et al., 2008, p.400), knowledge that is specific to the teacher and is used to develop teaching tasks related to: “how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems” (Hill et al., 2008, p.377-378); and the knowledge at the mathematical horizon, which is described as “awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p.403), this knowledge allows the teacher to establish the way in which the mathematical contents are related to others in the mathematics curriculum throughout the various educational stages. In other words, this knowledge lets teachers know how the mathematics they teach relates to the mathematics that students will learn in later years so as to lay the foundations for what will come later.

Pedagogical content knowledge is also composed of three subdomains: knowledge of content and students (KCS), which is defined as the "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill et al., 2008, p.375), the knowledge that the teacher manages regarding students' knowledge, allowing them to predict situations and anticipate the concerns, attitudes or difficulties of the students; knowledge of content and teaching (KCT), which is defined as knowledge that "combines knowledge about teaching and knowing about mathematics" (Ball et al., 2008, p.401), this knowledge integrates specific mathematical knowledge, and pedagogical and didactic aspects of the teaching processes involved in student learning; and, finally, knowledge of curriculum, is "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, and the variety of instructional materials available in relation to those programs" (Ball et al., 2008, p.391), it is related to the orientations and approaches corresponding to the programs designed for each educational level in the area of mathematics and the materials available in relation to them.

The tools provided by the mathematical knowledge model for teaching (Hill et al., 2008) are very rich in the field of Mathematics Education, since they allow us to categorize the knowledge that a teacher must manifest in the development of their practice to teach mathematics.

For the purposes of our study, we will specifically draw from the subdomains of specialized content knowledge and content knowledge and teaching, as these are more critically linked to the design of mathematical tasks (Sullivan et al., 2015).

Some studies have analyzed the mathematical knowledge that teachers have of early algebra and teaching patterns. Bair and Rich (2011), for example, analyze specialized knowledge for the development of algebraic reasoning in pre-service Early Childhood and Primary Education teachers. The results reveal a lack of ability to exemplify the nature of the relationships between quantities, and difficulties establishing connections between different representations of a number sequence. McAuliffe and Lubben (2013) analyze a teacher's performance when planning and presenting an early algebra lesson on patterns. These authors note the difficulty of helping students move from focusing solely on the number pattern to simultaneously focusing on the function, a central transition in early algebra teaching.

Elsewhere, Wilkie (2014) analyzes the mathematical knowledge that 105 in-service teachers have of functions, relationships and variations. The results show that two-thirds of teachers exhibit knowledge of the content in pattern generalization tasks. However, less than half demonstrated a reasonable pedagogical knowledge of the content, especially when providing suitable examples for the development of functional thinking. In a subsequent study, Wilkie (2016) delves into mathematical knowledge for teaching functional thinking by generalizing patterns with 10 in-service Primary Education teachers. After a year-long intervention, the results reveal an increase in certain aspects of their mathematical knowledge: a greater capacity to generalize; and an improvement in the choice of representations and examples used in the lessons.

Likewise, Zapatera and Callejo (2017) analyze the mathematical knowledge of 40 pre-service teachers in the context of pattern generalization, obtaining as a result a low level of specialized knowledge, since they exhibit difficulties identifying the mathematical elements used by students, and in abstracting observed regularities to interpret the characteristics of understanding generalization.

Methodology

In keeping with the purpose of our research, which, as noted, consists of analyzing the mathematical knowledge that pre-service Early Childhood and Primary Education teachers possess when designing a mathematical task to promote the study of patterns, we designed a qualitative descriptive study (Creswell, 2009) that relies on the content analysis technique. This technique lays out a "strict and systematic set of procedures for the rigorous analysis, examination and verification of the contents of written data" (Cohen et al., 2011, p. 475). It is used to study the nature of discourse in detail, and is able to reveal the internal structure of texts by studying their semantic content (Rico & Fernández-Cano, 2013); in our case, the written productions corresponding to the tasks designed by the pre-service teachers.

To analyze the content, the following stages were considered:

1. Individual reading of each of the mathematical tasks to explore and organize the information present in each of them.
2. Determine analysis indicators from the review of the literature. In our case, we considered a series of components to observe in the design of tasks for teaching patterns based on the SCK and KCT subdomains of the MKT model (Ball et al., 2008).
3. Encode math tasks based on established components.
4. Systematize the information through statistical tables to facilitate the descriptive analysis.
5. Support the descriptive analysis by selecting examples of tasks that consider the components analyzed.

Participants and context

Forty pre-service teachers participated in the study. They were deliberately taken from the 2020 academic year, and 18 of them were pre-service Early Childhood Education teachers who were in their fifth semester of training (out of eight), and 22 were pre-service Primary Education teachers who were in their fifth semester of training (out of ten) at a university in southern Chile.

Specifically, the pre-service Early Childhood Education teachers were taking the Mathematics Didactics subject, which is a didactic-disciplinary course in which they receive didactic training on algebra and other areas of content. It should be noted that these participants had internship experience, and only took one previous related course on understanding mathematical logical thinking. As for the pre-service Primary Education teachers, they were taking the Learning and Teaching Algebra course, where they received specific didactic training on algebra to complement their disciplinary training. These participants previously took courses on teaching arithmetic and geometry, and had also taken part in internships.

Task design is part of the teacher training process, since both in their previous subjects, as well as during their internship, they had to plan class sessions, which required them to design mathematical tasks.

It should be noted that pre-service teachers of both Early Childhood Education and Primary Education take a mathematics test beforehand in order to be admitted into the program. This test ensures a baseline knowledge of mathematics to start studies in the Education Degree.

Design and procedure

In order to assess the mathematical knowledge of pre-service teachers, teaching situations have been formulated using "vignettes" (Tab. 1), which consist of placing the participants in a fictitious situation and from there posing semi-structured follow-up questions about the topic under investigation (Schoenberg & Ravdal, 2000), in our case the mathematical patterns.

The vignettes are based on situations that pre-service teachers of Early Childhood and Primary Education must face to propose a mathematical activity according to the objectives proposed by the Chilean school curriculum linked to early algebra in the Third Level (4-5 years of age) of Early Childhood Education (MINEDUC, 2018) and the First Year (6-7 years old) of Primary Education (MINEDUC, 2012). Objectives that share common teaching knowledge have been selected, although they differ in the degree of depth that each educational level deserves, such as working with patterns, with the intention of analyzing the transition and evolution that this content receives in accordance with the school curriculum.

Teaching situation for pre-service teachers of Early Childhood Education
A teacher of Early Childhood Education should address the following learning objective: *"Create sound, visual, gestural, body or other patterns of two or three elements".* She has 30 minutes to carry out an activity with her students. As a pre-service teacher, if you had to recommend to Camila an activity to address this objective with her students, what would you propose? Describe and justify the activity that you would propose considering the concepts and procedures that will be implemented during the development of the activity, the possible orientations that must be addressed and anticipated to generate interaction, discussion and feedback with the students.
Teaching situation for pre-service teachers of Primary Education
The teacher of the first year of Primary Education should address the following learning objective: *"Recognize, describe, create and continue repetitive patterns (sounds, figures, rhythms ...) and numerical patterns up to 20, increasing and decreasing, using concrete, pictorial and symbolic material, manually and / or through educational software".* You have 45 minutes to carry out an activity with your students. As a pre-service teacher, if you had to recommend to Carlos an activity to address this objective with his students, what would you propose? Describe and justify the activity that you would propose considering the concepts and procedures that will be implemented during the development of the activity, the possible orientations that must be addressed and anticipated to generate interaction, discussion and feedback with the students.

Table 1. Vignettes used in pattern teaching situations

The teaching situation presented in each vignette aims to place pre-service teachers in a teaching context, in order to show and describe the components of mathematical knowledge that they reveal through the design of a mathematical task.

The vignettes were presented to the participants, in the context of a regular class of the participants' training process (90-minute session), in the respective courses they were taking, with the authorization and collaboration of the academic trainers in charge. The development of the study also considered the informed consent of the participants.

For the design of the tasks, the pre-service teachers had access to the internet and mathematics textbooks, since the didactic training they receive promotes the design of tasks in different contexts, such as informal, intermediate and formal contexts (Alsina, 2019; 2020). The Internet and textbooks are used to search for activities in different contexts that let them design mathematical tasks based on the training received.

Data analysis

To analyze the responses of pre-service teachers, deductive analysis categories were established from the literature that consider the adaptation and evolution of the theoretical tools provided by the MKT model (Ball et al., 2008) in the context of the design of tasks for teaching patterns.

Thus, considering the aspects to observe in the design of a task involving patterns, we identified components of the subdomains of specialized content knowledge (SCK) and knowledge of content and teaching (KCT), since they are directly linked to this professional activity (Sullivan et al., 2015).

These components were first drafted and submitted for review by 4 experts in the field of Mathematics Didactics in Early Childhood and Primary Education from Chile and Spain, and then their comments and observations were considered for the elaboration of the final components. The expert judgment allowed us to adjust the designed components, shown in Table 2, in terms of clarity, to evaluate their formulation and wording, and in terms of coherence, to assess whether they are logically related to the subdomains of the MKT model that we intend to investigate in our study.

Specialized content knowledge
C1. Specify and expand the formal language of an algebraic nature associated with patterns based on the educational level.
C2. Manage the complexity and depth of a task involving patterns while understanding its potential.
C3. Identify theoretical concepts and/or mathematical properties of a task on patterns.
C4. Select and use various representations to present a task involving patterns, such as natural language, algebraic language, symbolic, graphical, or tabular representation.
C5. Present a variety of situations to which the study of patterns can be applied.
C6. Demonstrate rules for forming pattern sequences, numerical, geometric, or graphical regularities.
C7. Use definitions and/or properties to justify the validity of the results and procedures when solving a task involving patterns.
Knowledge of content and teaching
C8. Select sequences of tasks that can be used to acquire or reinforce algebraic knowledge through teaching itineraries, such as description of patterns, regularities and number relationships.
C9. Propose tasks that allow the regularities of a sequence and/or the rule for creating it to be described in natural language.
C10. Propose teaching strategies based on mathematical games to motivate the use of algebraic language and promote an understanding of patterns.

Table 2. Components of mathematical knowledge for teaching patterns when designing mathematical tasks

In order to analyze the data, the authors coded the written productions of the pre-service teachers based on the distribution of the presence or absence of the components.

Scores were assigned to identify the presence (1 point) or absence (0 points) of the components.

To guarantee the reliability of the coding process, the authors carried out a calibration process, through joint coding and debate sessions to standardize the criteria in order to then carry out the individual coding. The tasks designed were analyzed through cyclical and deductive reviews, considering the formulation of the task on a general level, the questions that were asked in the task and how the early algebraic content was steered to obtain the answers. This made it possible to evaluate the levels of inter-rater reliability, yielding a reliability coefficient greater than 85%, above the minimum acceptable (Tinsley & Brown, 2000).

For the coding process, the authors used a log sheet for the components described in Table 2 and evidence of the segments of the tasks, and then assigned the corresponding scores.

Finally, with regard to the examples selected, a criterion was established that involved showing excerpts from tasks that exhibited a greater presence of the components proposed in Table 2.

Results

According to our study objective, the results are based on the analysis of the mathematical tasks proposed by 18 pre-service teachers of Early Childhood Education and 22 of Primary Education, to promote the study of the patterns.

Mathematical knowledge that teachers possess when designing tasks involving patterns

The description of the mathematical tasks designed by the pre-service Early Childhood and Primary Education teachers made it possible to analyze the presence/absence of the components of MKT for patterns organized according to the SCK and KCT subdomains. Table 3 shows an overview of the distribution of both subdomains based on their presence in the tasks proposed for working with patterns in Early Childhood and Primary Education classrooms.

Subdomain of mathematical knowledge	Early Childhood Education Tasks	Primary Education Tasks
Specialized content knowledge (SCK)	83.3%	90.9%
Knowledge of content and teaching (KCT)	61.1%	45.4%

Table 3. Presence of MKT subdomains for teaching patterns in Early Childhood and Primary Education

Note that the SCK subdomain is more prevalent than KCT for both Early Childhood and Primary Education tasks (above 75%). In the case of KCT, its presence is higher in Early Childhood Education tasks (above 50%), while in Primary Education, its presence is low, being found in fewer than 50% of the tasks.

Analysis of the Components of SCK and KCT

To show the trends of each subdomain of the MKT model observed in the written productions of pre-service Early Childhood and Primary Education teachers when designing a mathematical task, our analysis focuses on the presence of MKT components for teaching patterns. The distribution of the presence of the components defined and validated in this study is presented below.

Components of SCK

Table 4 shows the presence of the components involved in the development of SCK. Note that in the case of the tasks proposed by pre-service Early Childhood Education teachers, three components (C5, C6 and C7) are below 25%, and in Primary Education only two (C5 and C6). This shows the participants' indifference to incorporating in the mathematical tasks a variety of situations in which to apply the study of early algebra, in this case: the use of patterns to determine a generality; presenting rules for creating sequences of patterns and numerical regularities; using definitions and/or properties to justify the validity of the results and procedures when solving an algebraic task.

Components	Early Childhood Education Tasks	Primary Education Tasks
C1. Specify and expand the formal language of an algebraic nature associated with patterns based on the educational level.	77.7%	90.9%
C2. Manage the complexity and depth of a task involving patterns while understanding its potential.	44.4%	54.5%
C3. Identify theoretical concepts and/or mathematical properties of a task on patterns.	44.4%	90.9%
C4. Select and use various representations to present a task involving patterns, such as natural language, algebraic language, symbolic, graphical, or tabular representation.	33.3%	54.5%
C5. Present a variety of situations to which the study of patterns can be applied.	11.1%	18.1%
C6. Demonstrate rules for forming pattern sequences, numerical, geometric, or graphical regularities.	5.5%	13.6%
C7. Use definitions and/or properties to justify the validity of the results and procedures when solving a task involving patterns.	16.6%	31.8%

Table 4. Presence of SCK components for teaching patterns in Early Childhood and Primary Education

Component 1 stands out with a significantly higher presence in relation to the other six components observed in the proposed Early Childhood Education tasks (77.7%), demonstrating that the participants were able to specify and expand the formal language of an algebraic nature for this level of education. Similarly, this component is prominent in the Primary Education tasks (90.9%), as is component 3 (90.9%). The presence of the latter shows that pre-service Primary Education teachers identified theoretical concepts and/or mathematical properties in the algebraic tasks they proposed, such as repetition patterns, number sequence, increasing and decreasing sequences, and others.

Components of KCT

In relation to the components that comprise this subdomain, Table 5 shows a greater presence in the tasks proposed by the pre-service Early Childhood Education teachers than in those of the Primary Education teachers. Component 8 is present in over 50% of the tasks proposed by the pre-service Early Childhood Education teachers, while the scope of the components is low in the Primary Education tasks, ranging between 18.1% and 45.4 %.

Component 1 shows that the participants mostly considered the selection of sequences of tasks that allow students to acquire algebraic knowledge over those that allow them to describe, using natural language, the regularities of a sequence and/or the rules for creating one (C9). Finally, we see that component 10, which is related to the implementation of strategies based on mathematical games to motivate the use of algebraic language and promote an understanding of early algebra, is the component that is present the least in this area of knowledge.

Component	Early Childhood Education Tasks	Primary Education Tasks
C8. Select sequences of tasks that can be used to acquire or reinforce algebraic knowledge through teaching itineraries, such as description of patterns, regularities and number relationships.	61.1%	45.4%
C9. Propose tasks that allow the regularities of a sequence and/or the rule for creating it to be described in natural language.	33.3%	22.7%
C10. Propose teaching strategies based on mathematical games to motivate the use of algebraic language and promote an understanding of patterns.	22.2%	18.1%

Table 5. Presence of KCT components for teaching patterns in Early Childhood and Primary Education

Below, Figure 1 provides an example of a task proposed by a Primary Education pre-service teacher, showing the presence of the components described above.

> Through the game the following situation is modeled: Clap- stamp- say "A"- Clap- stamp- say "A"- Clap
>
> Questions: How do we continue the sequence? What's next? What is the pattern? How many elements does the repeating pattern have? How do these elements repeat? The teacher should explain to his students that a pattern is a set of elements ordered according to a rule, which when repeated several times forms a sequence.
> Can we form another pattern with these elements? What would the sequence look like?
> Children represent the new sequence with math link cubes.

Figure 1: Extract task PF-10, Primary Education

With regard to SCK, the task put forth shows that the pre-service teacher proposes a rhythmic repetition pattern with three elements. Both the questions presented - What is the pattern? How many elements does the repeating pattern have? - and the way in which the definition of a pattern is formalized show that the task requires the formal language of an algebraic nature (C1) and identifies theoretical concepts related to the development of the early algebraic task (C3).

The task focuses on expanding the sequence, identifying the unit of repetition, and, finally, inventing a new pattern by rearranging the elements in the sequence, mobilizing the skills to make patterns gradually, extend, recognize the unit of repetition, and create a pattern, respectively. This makes it evident that the pre-service teacher can handle the complexity and depth of the task to teach patterns, understanding their potential (C2).

Another aspect observed in the task is that the pre-service teacher proposes various situations where the study of patterns (C5) can be applied, by suggesting the expansion of a sequence using a rhythmic pattern and then representing a pattern created with the multilink cubes. This last indication provides evidence for the presence of the C4 component, since the pre-service teacher considers the graphical representation of an ABC repetition pattern to carry out the task.

When designing the task, the teacher considers the use of the definition of pattern to justify the validity of the results, revealing the presence of the C7 component. However, the absence of the C6 component in the task designed is of note.

Regarding the KCT demonstrated in the task, there is an absence of component 8, since the task designed does not present a sequence of activities through teaching itineraries, it is only developed in an informal context, which allows visualizing the mathematical ideas around the patterns in a concrete manner through the game and manipulatives. However, the task can be used to describe in natural language the regularity of the sequence (C9) by prompting the identification of the unit of repetition.

Finally, the task promotes an understanding of patterns through game-based teaching strategies (C10), since the task begins by reproducing a sequence with a mathematical game that models a situation with a rhythmic pattern sequence.

As an example, Figure 2 shows the analysis of an extract from a task proposed by an Early Childhood Education pre-service teacher.

> To introduce the content, children can be shown a video (https://www.youtube.com/watch?v=-SxF9TZG5_w) that proposes sequences where they can complete the series with the missing term and go observing the pattern. The teacher of Early Childhood can guide the presentation of the video through the questions: What do you observe? What elements are repeated? What elements must we add to continue the sequence?
> Then the children can work on a worksheet by painting the geometric figures that form a pattern: blue triangle- green triangle, blue triangle- green triangle, blue triangle- green triangle. You can work with two different figures such as: yellow square-red circle, yellow square-red circle, yellow square - red circle. In both cases we are working with visual patterns of two elements (AB).
> Finally, the children are asked to get together in pairs to create gestural patterns of three elements (ABC), for example: surprise-sadness-anger. Each pair can present their composition. To reflect on the experience, the teachers may ask: Can you create a different pattern with the same gestures? How do we achieve this?

Figure 2: Extract task PF-04, Early Childhood Education

Regarding the SCK, the pre-service teacher proposes the task with repetition patterns of two and three elements. The design of the task and the questions formulated by the teacher exhibit the presence of components C1 and C3, since they use a formal language of an algebraic nature and identify theoretical concepts associated with the patterns based on the use of terms such as sequence, series, visual patterns, gesture patterns.

The first part of the task focuses on expanding a sequence, then duplicating a pattern, and, finally, inventing a pattern, mobilizing the skills of extending, copying, and creating, respectively. The presence of component C2 is evident, since the pre-service teacher handles the complexity and depth of the task to teach patterns, understanding their potential.

The design of the task also shows that the pre-service teacher handles a variety of situations to work in more detail with patterns (C5), such as, for example, the observation of patterns with two and three elements to extend a series, the use of visual patterns through geometric figures and the use of gesture patterns. However, there is no evidence of the use of various representations when planning the task (C4) or definitions to justify the validity of the results when solving the task (C7).

With regard to KCT, the description of the task reveals a sequence of activities proposed by the pre-service teacher to enhance the study of patterns through teaching itineraries (C8). The task begins at an intermediate context through the use of a technological resource (video), and progresses towards a formal context where the content is formalized with a worksheet. However, the task as presented does not consider informal contexts.

The task proposes continuing sequences, identifying the core of a pattern and creating patterns with two and three elements. In this sense, the task allows students to describe, using natural language, the regularities observed in each sequence (C9) through the questions that are posed, for example: What do you observe? What elements are repeated? What elements must we add to continue the sequence?

The design of the task lacks component C10, since the task does not consider game-based teaching strategies to motivate the use of algebraic language and promote an understanding of patterns.

Final Considerations

This research has analyzed the mathematical knowledge possessed by 40 pre-service Chilean Early Childhood and Primary Education teachers when designing mathematical tasks to teach patterns. It considers the fact that task design can be used to link the knowledge needed to perform various professional tasks, such as organizing mathematical content, interpreting learning and managing teaching (Llinares, 2011).

We focused on the design of mathematical tasks that can be used to work with patterns in the third level of Early Childhood Education (4-5 years of age) and the first year of Primary Education (6-7 years of age), in accordance with the learning objectives contained in the Chilean school curriculum for both learning stages (MINEDUC, 2012; 2018). In order to analyze these tasks, we relied on the MKT model (Ball et al., 2008), and more specifically on the SCK and KCT subdomains, since they are directly involved in the design of mathematical tasks (Sullivan et al., 2015).

The pre-service teachers have designed a variety of mathematical tasks on patterns based on specific teaching levels. These tasks encourage the observation of regularities involving different actions and representations (Liljedahl, 2004).

The analysis of the tasks designed to promote the study of patterns made it possible to accentuate aspects involving the mathematical knowledge of pre-service teachers, showing that the SCK subdomain is more developed than the KCT subdomain, attaining average presences of 87.1% and 53.2%, respectively. Despite the fact that both subdomains are above 50%, the description of the mathematical tasks consider partial aspects of this knowledge. In the area of SCK, for example, the component that involves demonstrating rules for creating sequences of patterns, or numerical, geometric or graphic regularities has the lowest presence in the mathematical tasks of Early Childhood (5.5%) and Primary (13.6%) Education. Another component that exhibits gaps in this subdomain is presenting a variety of situations in which the study of patterns can be applied, and using definitions or properties to justify the validity of the results and procedures in solving a pattern task, which is present in fewer than 50% of the tasks proposed for both school levels.

Regarding KCT, which is related to presenting tasks that can be used to describe, using natural language, the regularities of a sequence or the rule for creating it, its presence is below 50%, as is proposing teaching strategies based on mathematical games to motivate the use of algebraic language and promote an understanding of patterns.

Consequently, the low level of detail of mathematical knowledge displayed by the study participants has shown that the tasks designed to teach patterns do not guarantee a high cognitive potential, as Smith and Stein (1998) suggests. However, instructing the pre-service teachers to craft mathematical tasks has underscored their ability for mathematical-didactic design (Pepin, 2015).

In general terms, considering the results of various studies analyzed by Authors (Pincheira & Alsina, 2021b) as part of a systematic review of the mathematical knowledge of Early Childhood and Primary Education teachers to teach early algebra, our findings match those of similar research that analyzed SCK and KCT, which revealed that teachers are able to progress in the analysis of specific aspects of a pattern task (Bair & Rich, 2011) and have a good handle on the representations available to teach this content, such as manipulatives, function tables, and diagrams (McAuliffe & Lubben, 2013); however, the explanations they provide about the tasks selected to describe the relationships that lead to generalization are not always considered successful (Zapatera & Callejo, 2017).

Moreover, Wilkie (2014; 2016) and Wilkie and Clarke (2015) show that teachers identify pattern generalization strategies, but they exhibit problems in representing the generalizations symbolically.

These results provide evidence on the mathematical knowledge that pre-service teachers bring to bear to promote the teaching of patterns in Early Childhood and Primary education. More specifically, the analysis carried out has allowed us to identify the mathematical knowledge that should be addressed most urgently during the process of initial teacher training if we are to promote an ideal learning of patterns from the earliest levels of education.

However, the previous training received by teachers can influence the result of the tasks they propose, since preschool and primary education teachers do not exhibit uniform knowledge when designing mathematical tasks. In this regard, two limitations are considered: not having delved into the CCK of the teachers, as it would have allowed us to establish links with the SCK; and not having conducted follow-up interviews to identify and confirm the knowledge promulgated by the pre-service teachers in the design of the tasks.

Another limitation of our study is that it does not consider formative experiences in the design of mathematical tasks that promote progress and improvement of the initial proposal, as noted by Liljedahl et al. (2007). As a result, future studies will need to investigate educational resources and strategies that can be used to enhance the mathematical knowledge of pre-service teachers to ensure that patterns are taught effectively from the first years of school, since the quality of mathematics teaching depends on the knowledge of teachers (Ball et al., 2008), meaning that their professional development positively impacts the quality of teaching and student performance (Cohen & Hill, 2001). From this perspective, the development of skills to design challenging mathematical tasks that promote in-depth learning of this content block, accompanied by training experiences that provoke reflection around the analysis, implementation and redesign of these tasks, will allow pre-service teachers to progress towards the acquisition of the mathematical knowledge they need in order to teach patterns effectively.

References

ACARA (2015). The Australian Curriculum: Mathematics. https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

Alsina, Á. (2019). Itinerarios de enseñanza de las matemáticas en educación primaria. *Aula de innovación educativa, 286*, 12-17.

Alsina, Á. (2020). El Enfoque de los Itinerarios de Enseñanza de las Matemáticas: ¿por qué?, ¿para qué? y ¿cómo aplicarlo en el aula? *TANGRAM – Revista de Educação Matemática, 3*(2), 127-159. http://10.30612/tangram.v3i2.12018

Bair, S. L., & Rich, B. S. (2011). Characterizing the development of specialized mathematical content knowledge for teaching in algebraic reasoning and number theory. *Mathematical Thinking and Learning*, *13*(4), 292-321. https://doi.org/10.1080/10986065.2011.608345

Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes it Special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Cai, J., & Knuth, E. (2011). *Early algebraization. A Global dialogue from multiple perspectives*. Springer. https://doi.org/10.1007/978-3-642-17735-4

Carpenter, T. P., Franke, M. L. y Levi, L. (2003). *Thinking mathematically: Integrating arithmetic y algebra in elementary school*. Heinemann.

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L.C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, A., Ribeiro, M., & Muñoz-Catalán, Mª.C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. Research in Mathematics Education, 20, 236–253. https://doi.org/10.1080/14794802.2018.1479981

Clements D., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge. https://doi.org/10.4324/9780203883389

Cohen, D. K., & Hill, H. C. (2001). *Learning policy*: *When State Education Reform Works***.** Yale University Press.

Cohen, L., Manion, L. & Morrison, K. (2011). *Research methods in education*. Routledge. https://doi.org/10.12987/yale/9780300089479.001.0001

Creswell, J. W. (2009). *Research Design: qualitative, quantitative, and mixed methods approaches* (3ª edition). Sage
Dreyfus, T. (2002). Advanced mathematical thinking processes. En D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47203-1_2
Even, R., & Ball, D. L. (2009). *The professional education and development of teachers of mathematics the 15th ICMI Study* (Eds.), Springer. https://doi.org/10.1007/978-0-387-09601-8
Godino, J. D., Giacomone, B., Batanero, C., & Font, V. (2017). Enfoque ontosemiótico de los conocimientos y competencias del profesor de matemáticas. *Bolema: Boletim de Educação Matemática, 31*(57), 90-113. https://doi.org/10.1590/1980-4415v31n57a05
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-specific Knowledge of Students. *Journal for Research in Mathematics Education, 39*(4), 372-400. https://doi.org/10.5951/jresematheduc.39.4.0372
Jones, K., & Pepin, B. (2016). Research on mathematics teachers as partners in task design. *Journal of Mathematics Teacher Education*, *19*, 105-121. https://doi.org/10.1007/s10857-016-9345-z
Kaput, J. (2000). *Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum*. National Center for Improving Student Learning and Achievement in Mathematics and Science.
Kaput, J. (2008). What is algebra? What is algebraic reasoning? En J. J. Kaput, D. W. Carraher, M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Lawrence Erlbaum. https://doi.org/10.4324/9781315097435-2
Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM Mathematics Education*, 1-20. https://doi.org/10.1007/s11858-022-01435-6
Liljedahl, P. (2004). Repeating pattern or number pattern: The distinction is blurred. *Focus on learning problems in mathematics, 26*(3), 24-42.
Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in task design: A tale of one task. *Journal of Mathematics Teacher Education*, *10*(4-6), 239-249. https://doi.org/10.1007/s10857-007-9047-7
Llinares, S. (2011). Tareas matemáticas en la formación de maestros. Caracterizando perspectivas. *Números, Revista de Didáctica de las Matemáticas, 78*, 5-16.
Lüken, M., & Sauzet, O. (2020). Patterning strategies in early childhood: a mixed methods study examining 3- to 5-year-old children's patterning competencies. *Mathematical Thinking and Learning*, *23*(1), 28-48. https://doi.org/10.1080/10986065.2020.1719452
Mason, J., Stephens, M., & Watson, A. (2009). Appreciating structure for all. *Mathematics Education Research Journal, 2*(2), 10–32. https://doi.org/10.1007/BF03217543
McAuliffe, S., & Lubben, F. (2013). Perspectives on pre-service teacher knowledge for teaching early algebra. *Perspectives in Education*, *31*(3), 155-169.
MINEDUC (2012). *Bases Curriculares 2012: Educación Básica Matemática*. Unidad de Curriculum y Evaluación.
MINEDUC (2018). *Bases Curriculares 2018: Educación Parvularia*. Unidad de Curriculum y Evaluación.

Ministry of Education Singapore (2012). *Mathematics Syllabus: Primary on to six*. Ministry of Education Singapore: Curriculum Planning and Development Division.
Ministry of Education Singapore (2013). *Nurturing Early Learners: A Curriculum for Kindergartens in Singapore: Numeracy: Volume 6.* Ministry of Education.
Mulligan, J. T., & Mitchelmore, M.C. (2009). Awareness of Pattern and Structure in Early Mathematical Development. *Mathematics Education Research Journal, 21*(2), 33-49. https://doi.org/10.1007/BF03217544
National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics.* NCTM
National Council of Teachers of Mathematics (2015). *De los principios a la acción: Para garantizar el éxito matemático para todos*. NCTM
Papic, M.M. (2015). An Early Mathematical Patterning Assessment: identifying young Australian Indigenous children's patterning skills. *Mathematics Education Research Journal, 27*(4), 519-534. https://doi.org/10.1007/s13394-015-0149-8
Pepin, B. (2015). *Enhancing mathematics/STEM education: A 'resourceful' approach*. Inaugural lecture, 27 November 2015, Technische Universiteit Eindhoven.
Pincheira, N., & Alsina, Á. (2021a). Hacia una caracterización del álgebra temprana a partir del análisis de los currículos contemporáneos de Educación Infantil y Primaria. *Revista Educación Matemática 33*(1), 153-180. https://doi.org/10.24844/EM3301.06
Pincheira, N. & Alsina, Á. (2021b). Teachers' mathematics knowledge for teaching early algebra: a systematic review from the MKT perspective. *Mathematics, 9 , 2590.* https://doi.org/10.3390/math9202590
Rico, L., & Fernández-Cano, A. (2013). Análisis didáctico y metodología de investigación. En L. Rico, J.L. Lupiáñez y M. Molina (Eds.), *Análisis didáctico en educación matemática* (pp.1-22). Comares.
Rittle-Johnson, B., Fyfe, E. R., McLean, L. E., & McEldoon, K. L. (2013). Emerging understanding of patterning in 4-year-olds. *Journal of Cognition and Development, 14*(3), 376–396. https://doi.org/10.1080/15248372.2012.689897
Rittle-Johnson, B., Fyfe, E. R., Loehr, A. M., & Miller, M. R. (2015). Beyond numeracy in preschool: Adding patterns to the equation. *Early Childhood Research Quarterly, 31,* 101-112. https://doi.org/10.1016/j.ecresq.2015.01.005
Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics, 73*(3), 297-328. https://doi.org/10.1007/s10649-009-9222-0
Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255-281. https://doi.org/10.1007/s10857-005-0853-5
Schoenberg, N. E., & Ravdal, H. (2000). Using vignettes in awareness and attitudinal research. *International Journal of Social Research Methodology*, *3*(1), 63-74. https://doi.org/10.1080/136455700294932
Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School,* 3, 344-350. https://doi.org/10.5951/MTMS.3.5.0344
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22. https://doi.org/10.17763/haer.57.1.j463w79r56455411
Sullivan, P., Clarke, D., & Clarke, B. (2012). *Teaching with tasks for effective mathematics learning*. Springer Science & Business Media. https://doi.org/10.1007/978-1-4614-4681-1
Sullivan, P., Clarke, D., Clarke, B. & O'Shea, H. (2010). Exploring the relationship between task, teacher actions, and student learning. *PNA, 4*(4), 133-142.
Sullivan, P., Knott, L., & Yang, Y. (2015). The relationships between task design, anticipated pedagogies, and student learning. In A. Watson & M. Obtain (Eds.), *Task design in mathematics education, an ICMI study 22* (pp. 83-114). Springer. https://doi.org/10.1007/978-3-319-09629-2_3
Thanheiser, E. (2015). Developing prospective teachers' conceptions with well-designed tasks: Explaining successes and analyzing conceptual difficulties. *Journal of Mathematics Teacher Education, 18*(2), 141-172. https://doi.org/10.1007/s10857-014-9272-9
Threlfall, J. (1999). Repeating patterns in the primary years. En A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 18–30). Cassell.
Threlfall, J. (2005). Repeating patterns in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–30). Continuum.
Tinsley, H. E. A. & Brown, S. D. (2000). *Handbook of applied multivariate statistics and mathematical modeling*. Academic Press. https://doi.org/10.1016/B978-012691360-6/50002-1
Wake, G. C. (2018). A case study of theory-informed task design: what might we, as designers, learn? En L.J. Rodríguez-Muñiz, L. Muñiz-Rodríguez, A. Aguilar-González, P. Alonso, F. J. García, A. Bruno (Eds.), *Investigación en Educación Matemática XXII* (pp. 94-109). SEIEM.
Warren, E., & Cooper, T. (2007). Repeating patterns and multiplicative thinking: Analysis of classroom interactions with 9-year-old students that support the transition from the known to the novel. *The Journal of Classroom Interaction Research Library, 4142*(7), 7–17.
Wijns, N., Torbeyns, J., De Smedt, B., & Verschaffel, L. (2019). Young children's patterning competencies and mathematical development: A review. En K. Robinson, H. Osana, y D. Kotsopoulos (Eds.), *Mathematical Learning and Cognition in Early Childhood* (pp. 139–161). Springer International Publishing. https://doi.org/10.1007/978-3-030-12895-1_9
Wilkie, K. J. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education, 17*(5), 397-428. https://doi.org/10.1007/s10857-013-9251-6
Wilkie, K. J. (2016). Learning to teach upper primary school algebra: changes to teachers' mathematical knowledge for teaching functional thinking. *Mathematics Education Research Journal, 28*(2), 245-275. https://doi.org/10.1007/s13394-015-0151-1
Wilkie, K. J., & Clarke, D. (2015). Pathways to Professional Growth: Investigating Upper Primary School Teachers' Perspectives on Learning to Teach Algebra. *Australian Journal of Teacher Education, 40*(4), 87-118. https://doi.org/10.14221/ajte.2015v40n4.6

Zapatera, A., & Callejo, M. L. (2017). Mathematical knowledge and professional noticing of prospective teachers in the context of pattern generalization. Characterization of Profiles. *Revista Complutense de Educación, 24* (1), 35-38. https://doi.org/10.5209/RCED.55070

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