European Journal of Science and Mathematics Education, 2023, **11**(1), 132-145
ISSN: 2301-251X (Online)

OPEN ACCESS

Research Article

Analyzing teachers' knowledge based on their approach to the information provided by technology

Helena Rocha [1*]
0000-0003-3865-7422

[1] CICS.NOVA, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, Lisbon, PORTUGAL
[*] Corresponding author: hcr@fct.unl.pt

Citation: Rocha, H. (2023). Analyzing teachers' knowledge based on their approach to the information provided by technology. *European Journal of Science and Mathematics Education, 11*(1), 132-145. https://doi.org/10.30935/scimath/12522

ARTICLE INFO

Received: 31 Dec 2021

Accepted: 3 Oct 2022

ABSTRACT

Teachers' knowledge plays a central role in technology integration. In this study we analyze situations, where there is some divergence between the mathematical results and the information offered by the graphing calculator (lack of mathematical fidelity), putting the focus in the teachers and in their approaches. The goal of this study is to analyze, in the light of knowledge for teaching mathematics with technology (KTMT) model, the teachers' professional knowledge, assuming the situations of lack of mathematical fidelity as having the potential to reveal some characteristics of their knowledge. Specifically, considering the teaching of functions at 10th grade (age 16), we intend to analyze: (1) What knowledge do the teachers have of technology and of its mathematical fidelity? (2) What can the teachers' options related to situations of lack of mathematical fidelity tell us about their knowledge in other KTMT domains? The study adopts a qualitative and interpretative approach based on the case studies of two teachers. Data were collected by interviews and class observation, being the analysis guided by the KTMT model. The main result points to the relevance of the mathematics and technology knowledge. However, there is evidence of some difficulties to integrate the information provided by the technology with the mathematics, and also of some interference of the teaching and learning and technology knowledge, and specifically of the knowledge related to the students. This suggests that the analysis of the teachers' actions in relation to situations of lack of mathematical fidelity, can be useful to characterize their KTMT.

Keywords: mathematical fidelity, teacher knowledge, KTMT, graphing calculator

INTRODUCTION

The potential of digital technology for teaching and learning mathematics is widely recognized (Tabach & Trgalová, 2019; Zbiek et al., 2007), as well as the central role played by teachers and their professional knowledge in its classroom integration (Chen & Lai, 2015; Clark-Wilson et al., 2020; Drijvers et al., 2016; Hoyles & Lagrange, 2010; Hoyles, 2018). One of the elements of the knowledge for teaching mathematics with technology (KTMT) model (Rocha, 2013, 2020) is the teachers' knowledge on the mathematical fidelity of the technology, and that is the focus of this article.

According to Dick (2008, p. 335), "a technological tool must stay true to the mathematics", meaning that the "characteristics and behavior of" a mathematical "object in the technological arena should reflect accurately the mathematical characteristics and behavior that the idealized object should have". And the author refers to this agreement between mathematics and the mathematics of the technology, as the mathematical fidelity of the technology.

Mathematical fidelity is obviously a desirable characteristic; however, some technological limitations can turn difficult to achieve it. Dick (2008) uses this notion to emphasize the importance of being aware that technology does not always present or represent mathematics as expected. And this may lead to situations where the mathematics experienced by the students when using technology differs from the mathematics they experience when they do not use it.

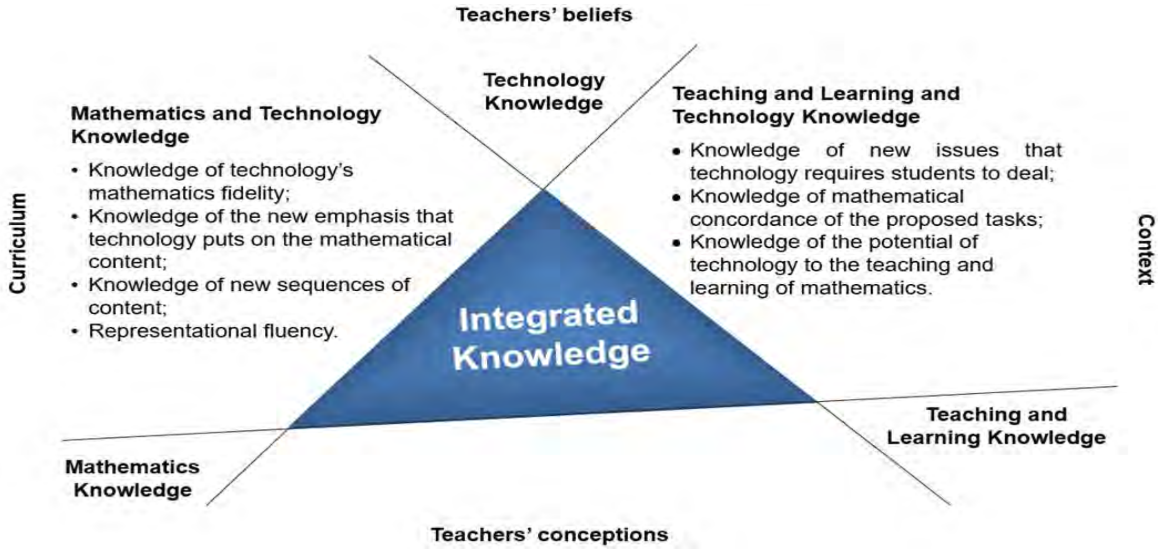

Figure 1. KTMT model (Rocha, 2014)

Several authors have devoted attention to these situations, identifying, and characterizing them (Pimm, 2014). Most of the studies developed so far focused on the students and their use of technology (Anabousy & Tabach, 2022). We intend to go forward, putting the focus on the teachers and their approaches. We assume the situations of lack of mathematical fidelity as challenging for the teacher, being a window to aspects of their professional knowledge. Thus, the goal of this study is to analyze, in the light of the KTMT model, the teachers' professional knowledge, assuming the situations of lack of mathematical fidelity as having the potential to reveal some characteristics of their knowledge. More specifically, considering the teaching of functions at the 10th grade, we intend to analyze:

1. What knowledge do the teachers have of technology and of the mathematical fidelity of technology?
2. What can the teachers' options related to situations of lack of mathematical fidelity tell us about their knowledge in other KTMT domains?

KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY

KTMT (Rocha, 2013, 2020) is a conceptualization of teachers' professional knowledge that intends to integrate in a single model the research developed on teachers' knowledge and on the integration of technology in teachers' practice (**Figure 1**). As so, it is a conceptualization that includes an emphasis on the aspects that research has pointed as essential in the integration of technology. KTMT model considers, among others, the work of Ball et al. (2008), Mishra and Koehler (2006), and Shulman (1986), and includes four base knowledge domains: mathematics, teaching and learning, technology, and curriculum. This last domain differs from the others, being conceptualized in a transversal way, and influential over all the others. Together with the curriculum knowledge it is also considered the context, where the teachers work, as well as their beliefs and conceptions. In addition to knowledge base domains, this model particularly values two sets of inter-domain knowledge developed at the confluence of more than one domain: the mathematics and technology knowledge (MTK), and the teaching and learning and technology knowledge (TLTK). MTK focuses on how technology influences mathematics, enhancing or constraining certain aspects, and includes knowledge of technology's mathematics fidelity, knowledge of new emphasis that technology puts on mathematical content, knowledge of new sequences of content, and representational fluency. TLTK focuses on how technology affects the teaching and learning process, enhancing, or constraining certain approaches, and includes knowledge of new issues that technology requires students to deal, knowledge of mathematical concordance of the proposed tasks, and knowledge of the potential of technology to the teaching and learning of mathematics. In both cases the curriculum is considered as a transversal influence (together with the context and the teachers' beliefs and conceptions). Finally, the KTMT includes integrated knowledge. This is a knowledge held by the teacher that articulates simultaneously the knowledge of each of the base domains and of the two sets of inter-domain knowledge.

KTMT AND THE MATHEMATICAL FIDELITY OF THE GRAPHING CALCULATOR

The mathematical fidelity of technology, as already mentioned, corresponds to the level of accuracy of the mathematics of the technology (Dick, 2008). Several authors present situations where the technology offers an answer that is not correct from a mathematical point of view. Here we adopt a broader understanding of this construct. Being our focus on the school mathematics and being the school syllabus often concerned about getting exact answers and not estimations of those, we assume as situations of lack of mathematical fidelity also those where the technology offers an answer that is an estimation of the intended answer, and not an exact answer.

According to Zbiek et al. (2007), the mathematical fidelity of the technology has been a matter of concern, often addressed in the literature, although many times with a focus in the information provided by the technology and not always using this designation (e.g., Pimm, 2014). Dick (2008) uses this notion to emphasize the importance of being aware that technology does not always present or represent mathematics in the way that effectively characterizes scientific knowledge, which can lead to situations in which the mathematics experienced by the students when using technology differs from what was intended. In this sense, it is particularly important that the teachers are aware of these situations.

The lack of mathematical fidelity can have different origins. It can be due to limitations inherent to the representation of continuous phenomena through discrete structures and the finite precision of numerical calculations (Dick, 2008). For example, the graphical representation of the functions *sin(2x)* and *sin(50x)* is coincident in a TI-84 Plus[1] considering $x \in [-2\pi, 2\pi]$. And yet, we know their graphs are different. This is one of the cases in which the answer provided by the technology is not faithful to mathematics.

There are many other similar situations that originate the representation of lines that do not belong to the graph (for example, the vertical lines that appear when tracing the graph of $tg(x)$ in the trigonometric standard window), or points that do not belong to the domain of the function (for example, for x=1 when plotting $\frac{x^2-1}{x-1}$ in the standard window), or points that belonging to the graph are not represented (for example, the points near the x axis in the graph of $\sqrt{81-x^2}$ drawn in the standard window), or horizontal or vertical lines in curved parts of the graph (for example, the horizontal zone at the top of the graph of $-x^2+10x-20$ plotted in the standard window).

With respect to situations related to the finite precision of the calculator, it is also possible to present several examples, such as that of the expression $10^{14}+10-10^{14}-2$, whose result is easily found to be 8, but which the calculator claims to be -2. One situation useful to address the myth that the calculator is never wrong (Olive et al., 2010).

All these limitations have been object of attention by several authors (Burrill, 1992; Cavanagh & Mitchelmore, 2003). However, the focus of their analysis is generally put more broadly, considering the limitations of the calculator together with other issues, such as those related to the viewing window used (and which prevent the main characteristics of the graph of the function from being observed or that a semicircle appears with an oval form). Nevertheless, knowledge of all these situations is valued in the context of a knowledge that is not limited to technology, but which is a mathematical knowledge of technology (Burrill, 1992).

Another area where lack of mathematical fidelity occurs is related to the discrepancy between the tool and the mathematical syntax conventions. These situations, unlike the previous ones, are not due to technology limitations, but the technology design, intending to favor an ease use of it (Dick, 2008). A situation where discrepancies can occur is in the priority of operations. In mathematics there is an established hierarchy,

[1] The examples presented relate to this model of calculator because it is the one used by the teachers involved in this study. These are situations that generally also occur for other models, namely newer models such as TI-nSpire, although for different values because the characteristics of the screen are different. Nowadays most of the graphing calculators allow updates of the operative system, what means that even for the same model it is possible to find differences (i.e., even using a calculator of the model used here, it is possible to get different results). The situations referred regarding calculation and involving values with significantly different orders of magnitude may not always occur. The quality of the graphical representation is also better in newer models, with a reduced occurrence of situations where straight parts are used in the representation of curves.

however, graphical calculators, when trying to adopt a more pleasant operation for the user, sometimes take their own rules that end up giving rise to situations that may be misleading (e.g., 3/2x should represent $\frac{3}{2}x$ but, mainly in some older models, it can represent $\frac{3}{2x}$).

Zbiek et al. (2007) reflect on the consequences of these discrepancies, pointing that some studies suggest that they have the potential to create confusion for both students and teachers. However, the results achieved so far have been inconclusive regarding whether this is just an inconvenient or whether it may have more serious consequences, with impact on students learning. According to the authors, when these situations occur, usually the implications on the students are strongly influenced by the reactions of the teachers and the interpretation that the teachers give them. The impact might be greater on the teachers than on the students. Indeed, these situations can affect the attitude of the teachers towards technology and, eventually, interfere with its future use. It seems thus that the knowledge that the teacher holds of this type of situations is important.

Most of the research related to technology discrepancies was conducted between 1990 and 2010 and has a focus on the technology and on the students. Cavanagh and Mitchelmore (2003) recognize this and highlight the need to focus on the teachers and on what is done with the technology. Tabach and Trgalová (2019) also emphasize the teachers' role on technology integration. They recognize technology has a significative impact on teachers' practices, requiring them "to develop new knowledge and skills to enable them to design relevant technology-mediated tasks, [and] monitor student work" (Tabach & Trgalová, 2019, p. 183). And Cavanagh and Mitchelmore (2003) emphasize that the teachers need to learn about the limitations and inconsistencies that may arise during calculator use, so that they can address them with their students, allowing the development of what the authors call a competent use. And Olive et al. (2010) emphasize that the main goal is a basic notion of the machine's internal representations and a generic awareness of the inherent limitations and potential problems, since an effective understanding of what the machine does internally involves mathematical knowledge that is beyond those held by most students (focusing only on internal numerical representation, the authors refer to the need to know about rational and irrational numbers, to know about numerical representation in different bases and to have a developed notion of limit).

Guin and Trouche (1999) emphasize the relevance of the tasks selected by the teachers and how they consider these situations of lack of mathematical fidelity, emphasizing that it is important to think deeply about this question. Cavanagh and Mitchelmore (2003) consider the decision regarding the moments in which it is convenient to avoid the contact of the students with this type of situations and the moments in which this contact must be deliberately promoted. The authors understand that the examples chosen by the teachers should be carefully thought out, beginning with situations in which the potential difficulties resulting from the contact of the students with this type of problem should be minimized. They point out, however, that if the teachers do not subsequently provide students with opportunities to confront more demanding examples that allow them to encounter all these situations, this will tend to prevent the students' understandings from developing as much as it could. The teachers' knowledge of these issues and the form they choose to address them seem thus to be particularly important.

Pimm (2014) designates the kind of situations related to graphs we have been discussing here as screen effects. And the author recognizes the attention given to screen effects in the past was considerable higher than it is nowadays. He assumes the reduction is due to the characteristics of the more recent models, who use better algorithms and have less problems related to pixels and screen representation, but he also points to a shift in the approach: a shift to the teachers and their role. While the focus in the past was on the impact of this misleading situations on the students learning, now a consciousness of the role and impact of the teacher has begun to emerge. This is leading to the recognition that "the teacher 'should' now know what to do with the 'opportunity' of the mistaken pixel or the rounding error or whatever else arises from attempting to look with novice eyes through the e-screen (to use Mason's interesting term) to the mathematics" (Pimm, 2014, p. vii).

Some authors have started to study the unexpected or unpredictable moments happening in the classroom. Stockero and Zoest (2013, p. 127) define the pivotal teaching moment as "an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding". This is

Table 1. Categories of analysis and their relation to KTMT model

Category of analysis	Knowledge domain
Teachers' knowledge of technical aspects related to situation of mathematical fidelity	Technology knowledge (TK)
Teachers' knowledge of situation of lack of mathematical fidelity Teachers' approach to situation from a mathematical point of view	Mathematics & technology knowledge (MTK)
Teachers' expectations about students' reaction to situation Teachers' approach to situation from a teaching & learning angle	Teaching & learning & technology knowledge (TLTK)

an interruption originated by a student that, if noticed and followed by the teacher, can generate the opportunity to create a moment for deepen the students understanding (Olawoyin et al., 2021). It requires teacher's knowledge, being related to the teacher's contingency knowledge, as conceptualized in the quartet knowledge model, by Rowland et al. (2005). Clark-Wilson and Noss (2015, p. 92) defined the notion of hiccup as "the perturbation experienced by a teacher during teaching that has been triggered by the use of mathematical technology". This means that in this case the origin of the situation lies with the teacher, while in the case of the pivotal teaching moment it lied with the students. Besides that, the notion of hiccup is directly related to the use of technology, what it was not the case of the notion of pivotal teaching moment. Clark-Wilson and Noss (2015, p. 99) consider seven hiccup types, including the teachers' perturbations arising from the "perturbations experienced by students as a result of the representational outputs of the technology". This specific type of hiccup as some connection with the idea of mathematical fidelity, although in this last case the focus is on the technology and not on the teachers. In this study we focus on this specific type of hiccup, or, in other words, we focus on the mathematical fidelity of the technology from the point of view of the teachers.

METHODOLOGY AND STUDY CONTEXT

Given the nature of the problem under study and in line with the ideas advocated by Bogdan and Biklen (1997) and Yin (2003), the study adopts a qualitative and interpretative methodological approach, undertaking two teachers case studies: the cases of Carolina and Teresa. Also participated in the study the students from a 10th grade class (16 years) of each teacher using a graphing calculator TI-84 Plus, TI-84 Plus Silver Edition, and TI-82 Stats (in the case of Carolina) and a TI-84 Plus and some TI-nSpire (in the case of Teresa). Data collection involved interviews to the teachers, observation of lessons and document gathering. Semi-structured interviews were conducted before and after each lesson, with the intention of knowing what each teacher had prepared and the reasons for these options (pre-lesson interviews) and the balance of how the lesson took place (post-lesson interviews). In these interviews the teachers were invited to share freely their plans and their reflections about the lesson. In the pre-lesson interview the goal is to find out what the teachers had planned, what use of the technology were they anticipating, and what expectations and concerns they have. In the post-lesson interview the teachers were expected to comment on the lesson, highlighting unexpected moments and reflecting on the decisions taking and on future decisions if planning a similar lesson. All the interviews focused on global aspects of the use of the graphing calculator, and not specifically on mathematical fidelity issues of the technology. This action was justified by the intension of avoiding the introduction of changes in the teachers' practice due to the objectives of the study. Both the interviews and the lessons observed were audio-recorded and later transcribed. These elements were completed by the field notes from the 14 lessons observed of each teacher and by the documents collected (such as worksheets and other materials made available by the teachers to the students). Data collection took place in Portugal, during the study of functions. The data analysis was essentially descriptive and interpretative, starting from the identification of episodes involving issues of lack of mathematical fidelity. For each episode identified, the analysis assumed as key points the teachers' knowledge of the situation of lack of mathematical fidelity and the technical aspects related, the teachers' expectations about the students' reaction to the situation, the teachers' approach to the situation from a mathematical point of view and from the teaching and learning options (namely, the teaching moment in which the episode occurred and whether or not its occurrence was intentional). These categories of analysis are related to knowledge domains of the KTMT model. The identification of one of the categories is assumed as evidence of the teachers' knowledge on the related domain (**Table 1**).

Once the case study was written, it was shared with each teacher for validation. Both teachers identified themselves with the analysis produced.

The teacher Carolina has a professional experience of over 30 years and, although clearly taking a favorable position towards the use of technology, has only an experience of three years in the use of graphing calculators with students. Teresa is a teacher with more than 30 years of professional experience and a long experience of using graphing calculators with students and a deep knowledge of the machine's operation.

RESULTS

Throughout the study of functions, the teachers enrolled in this study offer their students a wide range of tasks. The work in these tasks sometimes create situations of lack of mathematical fidelity. One common situation is the case of graphs of functions where curved parts appear as straight lines (sometimes vertical lines). However, this type of situations is not assumed as problematic or as a case of lack of mathematical fidelity by none of the teachers. Actually, they seem to be invisible for the teachers. Yet, there are other type of situations where the teachers recognize the lack of mathematical fidelity. In this section, we briefly present one episode where questions on mathematical fidelity arise. Discussing how each teacher addressed these questions, we intend to characterize aspects of their KTMT.

Carolina

Carolina has never had any training related to the graphing calculator and everything she knows about this technology was learned on her own. She assumes she does not have great knowledge regarding the operation of the machine, but beliefs she has enough knowledge to work properly with her students in the classroom. Still, she wonders if some additional knowledge might allow her the development of different approaches, going beyond what she usually does (T–teacher):

> T: Oh, I think I should know more. I think I only know the essential. I'm always discovering things I do not know, so it's because I do not know much, is not it? I mean, I think I know enough, but sometimes it does not cross my mind… I mean, if I knew more things, I might be able to think of another kind of approach, perhaps a better one (initial interview).

She expresses a particular interest in training that goes beyond the operation of the machine, focusing on ways to explore the potential of this technology and on how to deal with some of the problems and questions the use of the calculator raises. This is a very direct recognition of the relevance of technology knowledge (TK), but also of an additional type of knowledge (such as MTK and TLTK) that could help her to make a deeper exploration of the technology's potential.

The tasks Carolina proposes to her students where they are expected to use the calculator can have different characteristics. Some are explorations of families of functions, where the students are supposed to identify the impact of the variation of a parameter on the graph. Others focus on situations where the students are expected to find the values for the zeros/extremes of functions. In this case, the functions are carefully chosen by Carolina so that the values calculated by the students are integers. It is only near the end of the Function's study, when she considers that the students are already familiar with the technology, that she stops paying attention to this aspect when selecting tasks.

Carolina knows that what the machine presents in the screen is not always mathematically correct or accurate. She has already come across circumstances where the calculator does not present the exact value, but rather an approximation, for example, for the zero of the function or for the maximum or the maximizing. These are important characteristics of the internal operation of the technology that she considers necessary to approach with the students. However, she believes that such situations should be addressed in a progressive process, and not when the students are still getting used to the calculator.

This careful choice of tasks reflects the teacher's recognition of the technology's impact on the teaching and learning process. Carolina's TLTK tells her to avoid situations that could have a negative impact on the students, generating confusing situations and putting the focus on issues that differ from what is intended. As part of her TLTK, she is considering her knowledge of the students and organizing her lessons in a way that the students have time to get familiar with the basic commands of the calculator before they must face more complex situations.

After about a month and a half of experience in using the calculator, Carolina felt it was time to start addressing these questions. She then prepared a task for which she chose the function x^3-3x, intending to promote the presentation by the machine of answers that are not accurate:

> T: In this [function] it is easy for them to look for the zeros. Now, in the case they are exact values, it can happen that the calculator does not present the exact value... you know... (...) For example, here at the extremes [of this function], in this case we have a maximum for x=-1 and a minimum for x = 1 and it happens that the calculator does not present the result 1. It shows 1,0001. (...) I was thinking about addressing this kind of situations (pre-lesson 10).

Carolina knows that the viewing window chosen by the students may prevent some of them from facing this issue, however she intends to make it arise, so that all the students can examine it:

> T: I think it is important to address it because they can be seeing those five or six zeros and do not even think that it is not the right answer. (...) And maybe they are assuming... I do not know, when they are factoring the polynomial, for example, maybe they assume the zero as 1,0001 when it is 1. I think it is important for them, when such a situation arises, it is important that they check if that value is the real one or if it is 1 and it is the machine that is giving them that value. So, if it does not come up, I think I'll address it (pre-lesson 10).

In the classroom, Carolina chooses to approach the issue with the entire class, using the projector to calculate the minimum in front of all the students (S1–student 1 and S2–student 2):

> T: And now, what happened? ...
>
> S1: It gave us the minimum.
>
> T: Well, and what values have we been given?
>
> S2: One point zero zero ... it's 1.
>
> T: Well, well, that's not 1 that's there. (...) There we have 1,003. Now, with this result, what can we think? We can think the value of x for which the minimum is reached is (...) 1, although we do not see x=1, what we see is 1.003. (...) So how can we be sure that 1 is the right value?... (...) What do you think we can do to confirm that when x is equal to 1 we have a minimum? (lesson 10).

Given the results obtained, she raises the possibility of the minimizer being 1 and advances with the need to confirm this hypothesis, suggesting three different ways of doing it: to enlarge the graph, as a way to achieve a better approximation, expecting this procedure will allow us to get the exact value; to use a table properly constructed around the value 1 and with increments quite small to allow us to verify that the minimum is actually obtained when x is 1; or to calculate the image of 1. In her own words:

> T: Ask for the value. We can ask the calculator for the value of the function when x is equal to 1. (...) Or we could enlarge the graph in order to get an adequate value. Or we could make a table, set a table here around 1, with values very close to 1 (...) So what I was saying is that we could do an enlargement and see what was happening... as we get a detailed view of the graph around there, we also get a more rigorous information about what is happening. Or we could make a table. Because we saw... as the values of x change, the images also change, and we could see when the image reached a minimum value. So, the image is decreasing and then, from a certain point, it starts increasing. This means we have there the minimum value (lesson 10).

Although Carolina seems to consider important to explain to the students what they could do to confirm that the minimizer is 1, this is a confirmation that is never accomplished. Being something that the students have never done, namely the strategy that uses the table (something that the students never use), it is curious that this confirmation is never actually done.

It is equally interesting to note that this does not seem to cause any confusion to the students. When the teacher wants to address the question and asks the students for the value obtained, there is a student who begins to read the value in the calculator, but quickly interrupts the reading concluding that the answer is 1. The reason for this may lie in the previous experience of the students. They are used to get answers in the calculator with several decimal places that they quickly round off. It is therefore possible that the students consider this situation similar to the previous ones, not realizing that the values calculated by the calculator when determining zeros and extremes of a function are approximate answers.

Carolina is afraid of the confusion, from a mathematical point of view, that the situation can cause. But she does not realize that she does not know her students that well and the difficulties they have (i.e., her TLTK could be improved in relation to her knowledge of the students). This is in line with the point of view of Zbiek et al. (2007), when they suggest that situations of lack of mathematical fidelity have a greater impact on the teacher than on the students. Carolina feels the need for her students to confirm the correctness/accuracy of the value provided by the technology, but she does not seem to realize the conflict she faces. She seems to assume this confirmation as important, however, she never actually does it with the students. This suggests a certain discomfort with the use of the technology. And the origin for it can be in her TK. She is aware that she does not have a deep knowledge of the technology and she is not sure about how to explain why these situations happen. But the question here can be deeper than that. The school mathematics values procedures to get the exact answer. The technology uses estimation as the basic process to get to the answers. Finding a way to conciliate both approaches imply the development of a knowledge on the impact of technology in mathematics (MTK). Carolina values the use of technology to teach mathematics, but she is just starting to use graphing calculators. As so, she has not yet had the time to develop a deep MTK.

The strategies she proposes to find the exact answer, are not easy to implement by someone who has not much experience with the calculator. Carolina does not seem to be aware of that, showing that she does not have a thorough knowledge of the students when using technology in mathematical work. This suggests a need for her TLTK development. Besides this, the need she seems to feel for a mathematical confirmation of the value of the maximum, contradicts the fact that she never actually does it. A sign of some difficulty in reconciling technology and mathematical knowledge, indicating once again a need to develop her MTK.

Teresa

The tasks proposed by Teresa to her students where the use of the graphing calculator is expected are quite diversified. They include explorations on families of functions, where the students are required to identify the effect of the parameter variation on the graphs. They also include diversified explorations where an analytical approach (without a calculator) and a comparison with the graphical approach is not uncommon. Sometimes an exploration should end with the students formulating a conjecture and proving it afterwards. In some other cases the tasks include situations, with a context of reality or a strictly mathematical context, where the students are expected to find the values for the zeros or extremes of functions. Teresa also proposes modeling tasks, that is, situations with a context of reality where the students should start by collecting data and then find the mathematical model that best fits the situation. This diversity of tasks suggests knowledge about work proposals that can take the most of the technology's potential (i.e., teacher's TLTK).

Teresa never expresses the intention to confront students with situations of lack of mathematical fidelity, yet they arise in the classroom. Many were the situations in which graphs appeared with straight parts in curved zones. Nevertheless, these situations were never the object of attention by the teacher, seeming to have not caused any difficulty to the students. There were also situations in which the answer given by the calculator did not correspond to the exact answer. In these cases, it was possible to observe some confusion in the students, particularly when the values or expressions presented by the machine involve scientific notation. This was what happened in a task where the students had to find out the rectangle of maximum area that could be delimited with 20 cm of string. The students should start by collecting data taking some measures, and then look for the function that best fits their data.

The students used the calculator and obtained the expression $-x^2+10x+2E-12$. Some students were perplexed and questioned why they did not get the exact answer, which at that time they already knew to be $-x^2+10x$. Teresa explains the calculator uses a method of approximate calculation, which involves mathematical knowledge that is beyond the knowledge of the students. She adds that the calculator could never work on the problem as they did because it does not even know the problem. And she briefly addresses another field of mathematics, different from what the students have studied so far, and focused on getting estimations instead of exact answers. She further explains that this is the mathematics used by the calculator, being that the reason for the differences they can find in the results. Teresa is relying on her mathematics knowledge (MK) and using it based on her knowledge of the students (part of her TLTK) to give the students an explanation for the difference in the results offered by the technology and the results they could expect. The teacher then focuses in helping the students understand that the answer given by the calculator is rather close to the exact one. With this intention, she explains that the part of the expression added by the calculator to the exact answer represents an extremely small value:

> S: In this case, if there is an exact expression why does it not give us that expression? The calculator still puts there $-2E-12$.
>
> T: Exactly. Let's see then. What was the expression of the function? It was $10x-x^2$. And what did the calculator say? $-x^2+10x-2E-12$. What does $2E-12$ mean? We have seen this before. 2×10^{-12}, that is $2:10^{12}$. Well $2:10^{12}$ is… a large or a small number?
>
> S: Small.
>
> T: Small. Zero point and how many decimal places?
>
> S: 12.
>
> T: Then, do it there. $2:10^{12}$ … Is it a number close to 0 or not? It is. But it is not 0. It's always an approximate value but seeing this I'm realizing that this is a number very close to 0. It has to do to how the calculator does the calculations to get to that function. Notice, it came to that function not reasoning about the problem, because the calculator does not know the problem… What it knows is this set of points that we put on the calculator and from this set of points it said ok, that function goes quite well through these points, it fits very well as we can see here on the screen. Is the calculator making an error? Well, yes, but it's a small error (lesson 12).

At the time this task was carried out, the study of functions was almost at the end and, therefore, most of the students had already been confronted with situations in which the calculator presented answers in scientific notation. As so, the student's question is more about understanding the underlying reason for this answer than asking for help in making sense of the answer itself. In these circumstances, Teresa is using her MK but mainly her knowledge about the technology and the mathematics (MTK). But it was not always like that. At an initial phase, such an answer tended to be interpreted as a calculator error. This was what happened during an early task. The students marked two points on a parabola, draw the straight line that goes through them, and used the calculator to find its slope, getting $-4E-3$. They assume it is an error and called the teacher. She tries to get them to realize that, considering that the line they draw is horizontal, the slope will have to be 0.

When the teacher mentions the value presented by the calculator is an approximate answer, the students react, evidencing that they have not identified the value represented. Teresa then tries to get them to realize what value is represented by that notation. She also draws their attention to the number of decimal places they are working with, pointing that it limits the quality of the approximation of the answers made available by the machine:

> T: In this case we have a horizontal line. What is the slope of a horizontal line? (…)
>
> S: 0. (…)

T: It is 0. This is an approximate value.

S: It's very approximate indeed!

T: Is not it? -4×10^{-3}, what does it mean?... What does it mean? $1/10^3$. That is 0.001. And you are getting this because you're working with three decimal places. Maybe if you define the calculator to work with five, you get a better answer. You already know that in these values the calculator sometimes is not very exact. We must be critical (lesson 2).

In this case the teacher is using her knowledge about the notation used by the technology (TK), together with her MK and, of course, a knowledge that includes both (MTK).

Throughout the functions' study, there were often situations in which the value provided by the calculator was an approximation. It happened, for example, in the 7th lesson when the students were supposed to determine the height of a box constructed in a certain way, so that the volume was as large as possible. In this case the students have to use the calculator to determine the maximum of the function, since they did not know a way to carry out the calculation analytically because it was a third degree polynomial function. The value obtained for the height of the box showed some differences with respect to the last decimal places (x=.15695067 or x=.15695034), without this having been felt by the students as a problem. It also emerged in another situation, in the 9th lesson, where the students worked on a quadratic function that modeled the flight of an aeromodelling plain. In this case the maximum height reached was 5.05504 or 5.05505 (depending on the conditions under which the calculation was carried out). As an interesting point, this situation had the fact that the maximum was not obtained for the value 5.

The way how the students easily round off any value obtained is a noteworthy aspect. Indeed, in a task where the students were collecting data by means of measurements, the teacher had to intervene on several groups of students who were recording the measurements rounded to centimeters, apparently without worrying about the implications that option will have latter on the function that would model the phenomenon in question. And some students have found it difficult to realize that rounding all the data collected will turn difficult to obtain a function that actually is a good model of the situation and that is identical to the function obtained by an analytical approach to the situation.

DISCUSSION

Teachers' Knowledge About the Technology and Its Mathematical Fidelity

The teachers' knowledge about the graphing calculator is different. Teresa has a deep knowledge of its operation and has been using it with students for several years. Carolina has knowledge of the main commands, but she is aware that her knowledge could be deeper. Her experience in using this specific technology with students is also limited, although she has experience in the use of computers.

Both teachers offer their students opportunities to come across graphs of functions where curved parts appear as straight parts (a situation common in graphs of quadratic functions) and situations where the calculator is used to determine zeros or extremes and the answer provided by the calculator differs from the exact one. In the literature (e.g., Olive et al., 2010), and in terms of the content worked in the 10th grade, it is possible to find also reference to situations of lack of mathematical fidelity due to the considerable discrepancy of the values involved in calculations. However, this type of situation was not approached by none of the teachers.

Although the situations of lack of mathematical fidelity proposed to the students are of the same type for the two teachers involved, the way they arise in class is different (see **Figure 2**). Carolina considers important to ensure that situations like these do not arise too soon when the students are not yet familiar with the use of the calculator. As so, she carefully selects the functions for the tasks she proposes to her students, showing preference for questions that lead to integer values as answers. However, she believes it is important that students are aware of how the calculator may not always provide the exact/correct answer. Therefore, she deliberately prepares a task involving values that are not integers and through which she expects to confront students with a situation of lack of mathematical fidelity. Still, the option to work around integer numbers is

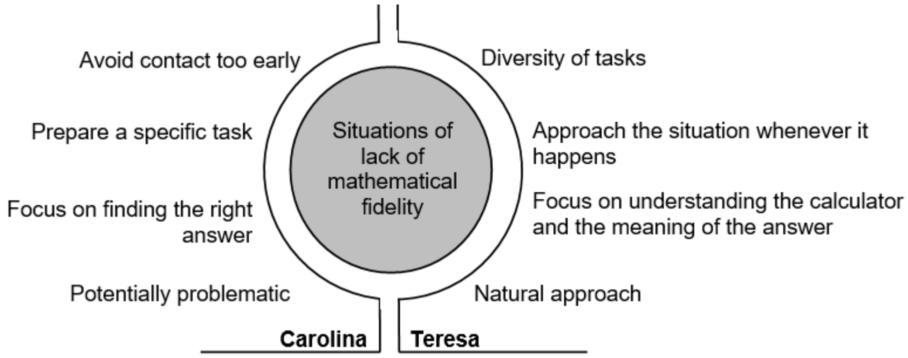

Figure 2. Mathematical fidelity–Synthesis of teachers' options

not necessarily a guarantee that a situation of lack of mathematical fidelity does not arise. These situations are, however, rare and restricted to one or two students, so Carolina chooses to individually help the student involved to overcome the situation. The options of Carolina are aligned with what is proposed by Cavanagh and Mitchelmore (2003), in what refers to avoid addressing complex situations too earlier, however the authors emphasize that afterwards it is important to address them.

In turn, Teresa deals with the calculator and the specificities of its operation in a very natural way. That is, she does not develop any specific task to deal with situations of lack of mathematical fidelity, nor takes any measure to prevent them from arising. The diversity of tasks proposed to the students, involving situations of reality, situations in which students work with data collected by themselves and also strictly mathematical situations, turns possible that situations of lack of mathematical fidelity arise and that this happens in different circumstances and in different ways. This option gives the students the opportunity to find these situations early and to become familiar with them progressively and naturally.

As so, both teachers are aware of some situations of technology's lack of mathematical fidelity, but they consider them in different ways, approaching them differently in the classroom.

Teachers' Knowledge

The way how the teachers support their students when they are facing a situation of lack of mathematical fidelity is also different for the two teachers. For Carolina it is important the students are aware of these situations. She encourages them to suspect that the value they are seeing does not correspond to the exact value. And she points out strategies they can use to confirm their suspicion. This suggests knowledge about the limitations of technology and even points to an intention of using that to promote students' reflection, in line with Thurm and Barzel (2021). However, she never implements any of these confirmation strategies. This action seems to suggest some discomfort on the part of Carolina regarding these less exact answers given by the calculator. On one hand, she feels the need to present ways of confirming the exact answer, but on the other hand, she does not seem to feel the need to actually confirm it. In what relates to situations of a graphic nature (such as those in which straight parts are represented in curved parts of the graph), they occur often in Carolina's classroom without receiving any attention from her. Conciliate the technology and the mathematics is not always easy to Carolina, suggesting the need for development of her MTK.

Teresa, when confronted in class with a situation of lack of mathematical fidelity, explains to the students that this apparent lack of accuracy is due to the fact that technology uses approximate methods of calculation, using mathematical knowledge that is beyond the one of the students. This option seems to be aligned with Hoyles (2018, p. 213) point of view, when she states it is important that "the black box of the digital tool" should be "opened 'just a little'". It is therefore a matter of making students understand the notation used by the calculator (in particular when it uses scientific notation), so that they understand the values presented. It is thus clear that the discrepancies between the values presented by the machine for a given calculation are not considered by the teacher as problematic, being the focus on students' understanding of the meaning of the values in the light of their mathematical knowledge. As so, the technology seems to be easy to articulate with the MK for this teacher, suggesting a well-developed MTK.

The difficulties that the teachers anticipate in the students, when confronted with situations of lack of mathematical fidelity, also seem to have some differences. Carolina is concerned that the students may not be aware of the situation and consequently may not be able to deal with it adequately. However, the truth is that the students are so used to rounding out the answers involving decimal values that they do it immediately, showing no concern in knowing what the exact value is. The fact that an integer value is almost always the exact answer can also be an important aspect. Teresa's students are confronted with a greater diversity of tasks, several of them involving real data where an integer value is not always the exact answer and where it is necessary to pay attention to the precision of the measurements made. It is true that they also show a propensity to round up the values, but in the end of the task there is a class discussion of these issues. Perhaps that is why, while in Carolina's classes there is a focus on the value of the answer, in Teresa's classes the focus is more on the process and on the understanding of the answer offered by the calculator, assuming the technology offers estimations of the answer and not necessarily the exact answer. Globally, Carolina seems to have some difficulty in anticipate her students' difficulties and ways of dealing with the technology, suggesting a need to development of her TLTK. Teresa also needs to address the way how students round decimal numbers, however there is no evidence about moments in which her expectations about her students diverge from reality.

Final Comments

One of the teachers enrolled in this study show some hesitation about how to approach situations of lack of mathematical fidelity (which translates into the suggestion to the students of strategies that are not implemented), and some mismatch between the teacher's expectations of the students' difficulties and their real difficulties. This corroborates the ideas of Zbiek et al. (2007) when they suggest that situations of lack of mathematical fidelity have a greater impact on the teacher than on the students. Nevertheless, avoiding the contact of students with this kind of situations at an initial stage, as suggested by Cavanagh and Mitchelmore (2003), does not seem to be very important, according to the results of this study. The characteristics of the tasks proposed to the students, as emphasized by Guin and Trouche (1999), seem to be more important than the moment of contact. The case of Teresa, where the situations are approached as they occur, is a very clear example of that. But here, the diversity of tasks proposed to the students is certainly an important factor, as stated by Cavanagh and Mitchelmore (2003), who point to the diversity of tasks as an important element to allow the students to develop a deeper understanding.

CONCLUSION

One of the aims of this study was to characterize the teachers' knowledge about the technology and about its mathematical fidelity. And the data analysis suggests the two teachers' knowledge about the technology is different. One of the teachers has a deep knowledge about how to operate it and the other, having some knowledge, is not so familiar with all the commands. This difference does not seem to translate into a different knowledge about the mathematical fidelity of the technology. In fact, both teachers seem to have the same level of knowledge about situations of lack of mathematical fidelity. The differences identified are related to the knowledge used by the teachers to deal with this kind of situations.

The second aim of the study was to understand what the teachers' options related to situations of lack of mathematical fidelity can tell us about their knowledge in other KTMT domains. And, in this case, the data point to the relevance of MTK. The different approaches assumed by the teachers provide evidence of the difficulties faced by one of the teachers to integrate the information provided by the technology with the mathematics. This suggests that the analysis of the teachers' actions in relation to situations of lack of mathematical fidelity, can be useful to characterize their KTMT. The analysis of situations of lack of mathematical fidelity also provide evidence about the relevance of the teachers' knowledge of students and their use of technology, being the diversity of tasks offered to the students also emphasized. These elements are an important part of the TLTK domain and, according to this study, this approach also allows us to get information about them.

This study followed the practice of the two teachers for a relative long period (all the lessons addressing functions), however it focuses on only two teachers, being that these are teachers with a very positive attitude about the use of technology to promote mathematical learning. It would be interesting to study some more teachers and to understand if (and how) differences on what Tabach and Trgalová (2019) call the affective orientation of the teachers towards technology, impact the results.

Finally, the situations of lack of mathematical fidelity addressed raise questions about the use of the decimal numbers by the students and about how they round them, that seem important for further reflection and investigation.

Funding: This study was supported by national funds through FCT–Foundation for Science and Technology, I.P., within the scope of the project 2022.03892.PTDC.

Declaration of interest: The author declares no competing interest.

Data availability: Data generated or analyzed during this study are available from the author on request.

REFERENCES

Anabousy, A., Tabach, M. (2022). In-service mathematics teachers' pedagogical technology knowledge development in a community of inquiry context. *Mathematics, 10*, 1-18. https://doi.org/10.3390/math10193465

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177/0022487108324554

Bogdan, R., & Biklen, S. (1997). *Investigação qualitativa em educação–Uma introdução à teoria e aos métodos* [*Qualitative research in education–An introduction to theory and methods*]. Porto Editora.

Burrill, G. (1992). The graphing calculator: A tool for change. In J. Fey, & C. Hirsch (Eds.), *Calculators in mathematics education* (pp. 14-22). NCTM.

Cavanagh, M., & Mitchelmore, M. (2003). Graphics calculators in the learning of mathematics: Teacher understandings and classroom practices. *Mathematics Teacher Education and Development, 5*, 3-18.

Chen, J.-C., & Lai, Y.-L. (2015). A brief review of researches on the use of graphing calculator in mathematics classrooms. *International Journal of Learning, Teaching and Educational Research, 14*(2), 163-172.

Clark-Wilson, A., & Noss, R. (2015) Hiccups within technology mediated lessons: A catalyst for mathematics teachers' epistemological development. *Research in Mathematics Education, 17*(2), 92-109. https://doi.org/10.1080/14794802.2015.1046476

Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM, 52*(7), 1223-1242. https://doi.org/10.1007/s11858-020-01196-0

Dick, T. (2008). Keeping the faith: Fidelity in technological tools for mathematics education. In G. Blume, & M. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Cases and perspectives* (pp. 333-340). NCTM.

Drijvers, P., Ball, L., Barzel, B., Heid, M., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Springer. https://doi.org/10.1007/978-3-319-33666-4_1

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*, 195-227. https://doi.org/10.1023/A:1009892720043

Hoyles, C. (2018) Transforming the mathematical practices of learners and teachers through digital technology. *Research in Mathematics Education, 20*(3), 209-228. https://doi.org/10.1080/14794802.2018.1484799

Hoyles, C., & Lagrange, J. (Eds.) (2010). *Mathematics education and technology–Rethinking the terrain*. Springer. https://doi.org/10.1007/978-1-4419-0146-0

Mishra, P., & Koehler, M. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Olawoyin, O., Kribs, C., & Joswick, C. (2021). Embracing pivotal teaching moments: Elementary teachers' role in advancing high cognitive levels of mathematics discourse. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-021-00374-x

Olive, J., Makar, K., Hoyos, V., Kor, L., Kosheleva, O., & Strässer (2010). Mathematical knowledge and practices resulting from access to digital technologies. In C. Hoyles, & J. Lagrange (Eds.), *Mathematics education and technology–Rethinking the terrain* (pp.133-178). Springer. https://doi.org/10.1007/978-1-4419-0146-0_8

Pimm, D. (2014). Integrated circuits. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. v-xii). Springer.

Rocha, H. (2013). Knowledge for teaching mathematics with technology–A new framework of teacher knowledge. In A. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (pp. 105-112). PME.

Rocha, H. (2014). Different representations in mathematics teaching with technology. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (pp. 384). PME.

Rocha, H. (2020). Using tasks to develop pre-service teachers' knowledge for teaching mathematics with digital technology. *ZDM, 52*(7), 1381-1396. https://doi.org/10.1007/s11858-020-01195-1

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255-281. https://doi.org/10.1007/s10857-005-0853-5

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14. https://doi.org/10.3102/0013189X015002004

Stockero, S., & Zoest, L. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16*, 125-147. https://doi.org/10.1007/s10857-012-9222-3

Tabach, M., & Trgalová, J. (2019). The knowledge and skills that mathematics teachers need for ICT integration: The issue of standards. In G. Aldon, & J. Trgalová (Eds.), *Technology in mathematics teaching* (pp. 183-203). Springer. https://doi.org/10.1007/978-3-030-19741-4_8

Thurm, D., & Barzel, B. (2021). Teaching mathematics with technology: A multidimensional analysis of teacher beliefs. *Educational Studies in Mathematics, 109*, 41-63. https://doi.org/10.1007/s10649-021-10072-x

Yin, R. (2003). *Case study research–Design and methods*. SAGE.

Zbiek, R., Heid, M., Blume, G., & Dick, T. (2007). Research on technology in mathematics education. In F. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169-1207). NCTM.