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Incoherencies in elementary pre-service teachers' understanding of calculations in proportional tasks

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ABSTRACT

In this study we investigated pre-service teachers' (PSTs) proportional reasoning and how they interpret their calculations in proportional tasks. We administered a written questionnaire to 199 PSTs and used an inductive content analysis approach for data analysis. We found that one item, in which PSTs were asked to interpret the meaning of the results of their calculations, had unusually low coherency, and applying open coding to the responses revealed several common errors. We argue these common errors cannot be dismissed as simple unit or rounding mistakes but rather reflect problems in how respondents think about quantities, story problems, and the nature of mathematics itself. We end with suggestions on how to address these problems.

Keywords: pre-service teacher education, teacher knowledge, proportional reasoning, problem solving, quantitative reasoning

INTRODUCTION

Proportional reasoning, cited in the common core as one of the eleven mathematical domains that span mathematics from elementary to high school (National Governors Association Center for Best Practices, 2010), has also been widely investigated as a key type of reasoning with which both students and teachers in K-12 mathematics classrooms struggle (Beckmann & Izsak, 2015; Byerley & Thompson, 2017; Weiland et al., 2021). According to Lamon (2007), proportional reasoning is defined as

> "detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships between two co-varying quantities" (p. 647).

It is applicable to many situations in daily life and plays an important role in developing students' mathematical reasoning for elementary school arithmetic and higher mathematics (NCTM, 2000). In prior studies, the two types of proportional relationship problems that have most often been used to investigate students or pre-service teachers' (PSTs) proportional reasoning are missing-value and comparison problems (Lamon, 2007; Pelen & Artut, 2019). In missing-value problems, students are usually asked to determine the missing-value when three of the four values are given. In comparison problems, students are asked to compare two ratios to determine whether they are equal (Artut & Pelen, 2015). In this study, we focused on a comparison problem (see the Car Gas Problem in **Figure 1**) to investigate PSTs' ability to interpret their own calculations in a proportional word problem.

Dr. Lee drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline? Solve this problem and justify your reasoning.

Figure 1. The Car Gas Problem

In a larger study, we surveyed 199 preservice elementary school teachers using 10 tasks that involve both missing-value and comparison problems in order to investigate their proportional reasoning. One finding of this larger study was that the PSTs struggled to interpret the meanings of their own calculations, especially in a comparison problem that required PSTs to explain the process by which they calculated their answers even when they were correct. That is, while in many prior studies PSTs' major difficulties with proportional reasoning were attributed to their judging non-proportional relationships as proportional (Arican, 2019; Izsak & Jacobson, 2017) without further investigation as to why, this study suggested another type of difficulty in solving word proportional reasoning problems, one which focused on the nature of the reasoning processes underlying the mathematical procedures. These findings are in line with Inoue's (2005) research about PSTs' strong tendency to solve arithmetic word problems by mechanically calculating numbers even when the answers of their calculations produce seem unrealistic.

In this paper, we share the major themes we found in the work and explanations of PSTs who had difficulty solving the Car Gas Problem, which required a realistic interpretation of an everyday issue. Our study includes two research questions, as follows:

1. How do PSTs solve the Car Gas Problem including proportional reasoning?
2. What kinds of challenges in reasoning could cause PSTs to struggle with solving this proportional reasoning problem?

THEORETICAL FRAMEWORK & BACKGROUND

PSTs' Quantitative Reasoning

We approached our research questions and data analysis within a theoretical framework of quantitative reasoning (Lee, 2017b; Thompson, 2011). A quantity is a measurable attribute (such as length, elapsed time, volume, etc.) of an object (such as a car, a person, the Earth, etc.), and in the words of Thompson (2011), a person thinks about a quantity when they have

> "conceptualiz[ed] an object and an attribute of it so that the attribute has a unit of measure" (p. 37).

This way of thinking in words, as in story problems, stands in marked contrast to more procedural ways of reasoning such as key-term approaches (replacing "and" with "+", "less than" with "-", etc.). Quantities as defined by Thompson (2011) occur only in the mind of a thinker, who conceptualizes them by making sense of a quantitative situation. A person is reasoning quantitatively when he or she is reasoning about actual quantities of something instead of numbers, undefined variables, or memorized procedures. Our study focuses on two different aspects of PSTs' quantitative reasoning: the extent to which they engaged in proportional reasoning with quantities, and the extent to which they identified the quantitative meaning of the results of their calculations.

PSTs' Proportional Reasoning

The ability to reason proportionally is not the same as the ability to set up proportions and cross-multiply to find missing values. As Hines and McMahon (2005) observe, to develop proportional reasoning, learners need to understand

> "the multiplicative co-variation within proportions that involves interpretation of both the meaning of two individual ratios and the comparison of ratios" (p. 88).

According to Vergnaud (1983), the development of proportional reasoning is determined by both students' ability to recognize the correct relationship between quantities and the strategies they use to solve problems. Several studies (Artut & Pelen, 2015; Baxter & Junker, 2001; Son, 2013) identified three correct strategies in their study to investigate developmental levels of proportional reasoning associated with correct and incorrect strategies for proportion tasks, as follows:

1. *the within ratio strategy*, in which a ratio is found in one context and applied to another context;
2. *the between ratio (functional) strategy*, in which a ratio is found between contexts and applied to another context; and
3. *the scale factor strategy*, in which unit rate is found and applied to find a missing value in other contexts.

Several studies have demonstrated that PSTs have problems with understanding proportional reasoning (Arican, 2019; Ben-Chaim et al., 2007; Glassmeyer et al., 2021; Hines & McMahon, 2005; Izsak & Jacobson, 2017; Livy & Herbert, 2013; Weiland et al., 2021). For example, Izsak and Jacobson (2017) reported that PSTs have difficulty with distinguishing proportional relationships from other relationships between two co-varying quantities, and not surprisingly, elementary students have been shown to have similar struggles (Ucar & Bozkus, 2018; Vershaffel et al., 1997). Arican (2019) suggested that PSTs tend to solve missing-value problems using cross-multiplication without considering whether the quantities have a proportional relationship. Similarly, Arican (2019) found that PSTs struggled with representing and interpreting proportional and non-proportional relationships and tended to rely on cross-multiplication and across multiplication strategies. Arican (2019) also noted that PSTs' excessive attention to simultaneous increases or decreases and constancy of the rate of change limited their understanding of what constitutes proportional and non-proportional relationships. Furthermore, Livy and Herbert (2013) pointed out that PSTs often have low levels of the mathematical content knowledge required to teach proportional reasoning. Hines and McMahon (2005) found that PSTs struggled with interpreting middle school students' various solution strategies in solving proportional reasoning problems.

To address PSTs' known weaknesses in the content and pedagogical knowledge needed to teach proportional reasoning, Ben-Chaim et al. (2007) created and implemented authentic investigative proportional reasoning tasks and found that completing the tasks was effective in improving PSTs' content and pedagogical knowledge as well as their attitudes. The researchers argued that using hands-on experience with a variety of authentic proportional reasoning tasks helped PSTs gain insight into proportional relationships because of their interest in the real-world contexts of the problems and collaboration with classmates in discussing a variety of strategies drawn from their daily experience.

Most of the studies mentioned above (e.g., Arican, 2019; Ben-Chaim et al., 2007; Izsak & Jacobson, 2017; Livy & Herbert, 2013) either used missing-value problems to investigate PSTs' proportional reasoning or suggested designing and implementing authentic proportional reasoning tasks to support students' understanding of proportional reasoning. In this study, we report difficulties related to proportional reasoning in comparison problems that may arise in solving a realistic word problem.

PSTs' Interpretations of Quantitative Meanings of Calculations

There are several cognitive actions must be taken in order to determine the situational appropriateness of a solution to a word problem (Thevenot, 2017). More work has been done on students' than on teachers' interpretations of calculations, though more recently researchers have gravitated toward analyzing teachers' conceptualizations in this area. Verschaffel et al. (1997) conducted a study on how Belgian elementary students responded to standard textbook word problems, in which the solution strategy was straightforward and involved one or more basic operations compared to parallel problems that required interpreting the real-world constraints and quantitative meanings of the values in the situation, and found that very few answers to parallel problems were based on realistic considerations. Similar studies have been conducted with United States college students (Inoue, 2005) and compared with Belgian and Japanese students (Yoshida et al., 1997). Despite the differences in mathematical performance regularly expected from these countries, they found that students in all three exhibited similar levels of difficulty dealing with non-standard problems.

A common theme in the research on students' assessment of situational appropriateness is the assumption that over time students build an understanding of what a math problem in a classroom setting entails (Greer et al., 2002), which may be insufficient for authentic problem-solving. Therefore, it is clear that the teacher plays a crucial role in helping students to draw connections between their real-world knowledge and classroom mathematics. Rosales et al. (2012) analyzed the actions of in-service teachers (ISTs) in the classroom to determine whether they used a narrative approach with their students that took into account the situational realities of a task, or a paradigmatic approach, which emphasized comparatively context-free aspects of problem-solving, and found that most teachers emphasized paradigmatic approaches and placed a greater emphasis on correct algorithmic procedures than on assessing the quantitative appropriateness of the resultant calculation.

Verschaffel et al. (1997) studied Belgian PSTs and found that they both excluded real-world considerations from their own answers and did not value them in students' answers. Also, Chen et al. (2011) found that, although Chinese teachers were better at checking the situational appropriateness of answers than had been demonstrated in prior studies, they were also far more lenient with situational inappropriateness in students' responses than with incorrect calculations. Indeed, across studies involving students, PSTs and ISTs, there are strong themes of devaluing the importance of evaluating the quantitative meaning of a calculation and assessing its appropriateness in the given context.

METHODOLOGY

The proportional reasoning questionnaire was given to 199 elementary preservice teachers over three semesters at a large Southwestern university in the USA. All participants were juniors enrolled in an elementary mathematics content course covering patterns, functions, and modeling. They were on average 1.5 years away from becoming fully certified elementary school teachers in grades K through 8.

In the middle of the spring 2015, fall 2015, and spring 2016 semesters, all PSTs in six sections of a mathematics content course were assigned to complete a 10-problem questionnaire as homework right after learning proportional reasoning. In the questionnaire, PSTs were explicitly asked to solve all problems first using non-standard approaches (such as drawing pictures) rather than a standard algorithm, and to provide clear justification along with their work. This paper presents the results of our analysis of their responses to one item, the Car Gas Problem (**Figure 1**).

This prompt deliberately did not ask for a numerical answer; instead, it asked a yes/no question that required respondents to engage in three steps, as follows:

1. decide what calculations would be relevant and useful to answering the question,
2. carry out those calculations accurately, and
3. interpret the meaning of their calculated results to answer the prompt.

Our analysis focuses mainly, though not exclusively, on the PSTs' written work that illustrates the third part of this process.

To analyze the data, we used both qualitative and quantitative analyses based on an inductive content analysis approach (Grbich, 2007), including organizing raw data, identifying correctness of the responses, creating coding schemes, coding all data, and interpreting the data quantitatively and qualitatively. Accordingly, we first organized PSTs' responses to the 10 proportional reasoning tasks on a spreadsheet and read each PST's responses to identify their correctness. Then we came up with a first-draft coding strategy for PSTs' correct or incorrect strategies based on the format of the questions. We then coded over half the data to assess how well the coding strategy worked, suggested modifications, and agreed on the next draft of the coding rubric. All the data were then coded during which several smaller coding strategy changes were also implemented and those sections recoded.

Another mathematics educator then coded a random subset of responses to verify that the strategy was being implemented uniformly, resulting in 100% agreement on the coding of 95% of the examples. Once the initial coding was completed, we focused on a specific phenomenon: the cases in which PSTs either incorrectly interpreted the results of their own calculations or did not interpret their results at all. We then found that PSTs showed unusually low coherency in interpreting the meanings of the results of their calculations in the Car Gas Problem. Thus, we chose the problem for deep data analysis by applying open coding to the responses, which revealed several common errors.

Table 1. Summary of all responses

	Correct answer: "He can't make it"	Incorrect answer: "He can make it"	No final answer	Total
Total responses	152	36	11	199
Responses with interpretation errors	9	32	5	46

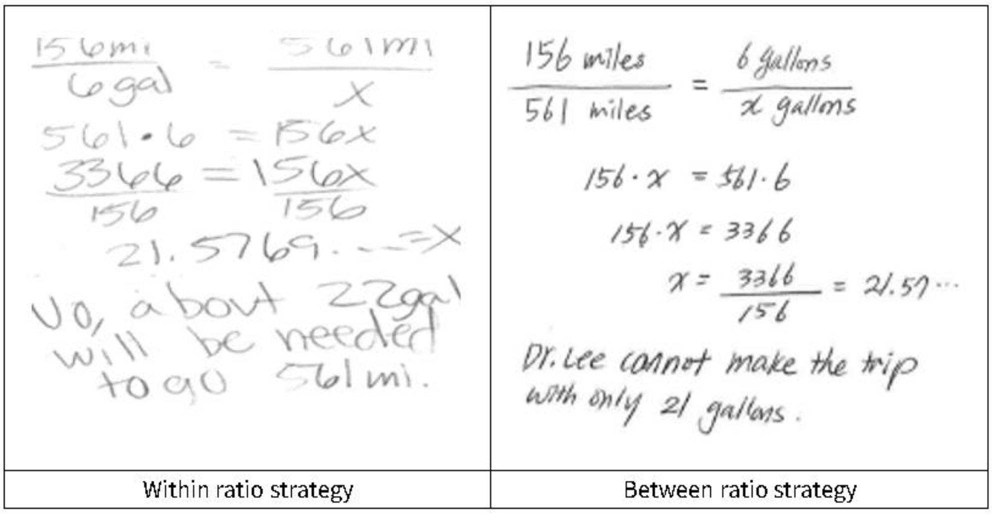

Figure 2. Examples of the strategy "finding and interpreting gas use"

FINDINGS AND DISCUSSION

Since the prompt asked whether Dr. Lee could drive a certain distance on a certain amount of gas, not for a numerical answer, the correct answer was "no" or some variation of it, which was given by 152 PSTs who wrote such responses as "He cannot go 561 miles," "He can go only 546 miles," "He has miles left over," or "He would need more than 21 gallons." Despite their correct answers, nine of these PSTs still had problems interpreting their own results. Of the 36 PSTs who answered "yes" or "He can go 561 miles," only four gave incorrect answers due solely to arithmetic errors, while the other 32 had problems interpreting their own results. 11 PSTs gave no final answer, five of whom completed all of the necessary calculations yet did not interpret them to answer the question. Overall, 46 out of 199 PSTs, or 23.11%, demonstrated some problem with interpreting the results of their own calculations. **Table 1** shows the summary of all responses.

Out of 199 PSTs, 143 made relevant and accurate computations and correctly interpreted these results to answer the prompt, using one of three effective solution techniques. However, 56 out of 199 PSTs provided either reasoning or answers that were incorrect or missing. Among these, 73% used computations similar to those used by the 143 successful PSTs but drew the wrong conclusion by using a faulty combination of the realistic constraints of the problem and their own real-world knowledge along with the limitations of their ability to reason proportionally. In the next section, we first present the three strategies PSTs used to solve the task, and in the following section, we address the challenges PSTs encountered in solving the problem.

PSTs' Three Strategies Used to Solve the Task

Strategy 1: Finding & interpreting gas use

Ninety-three PSTs chose to focus on mileage by

a) calculating the number of gallons needed to complete the desired trip,
b) comparing that number of gallons to the available gallons, and
c) concluding that the car could not make the trip on a full tank.

The crucial real-world information is that in the realistic context given, the drive is possible only if the needed gas is less than or equal to a full tank.

Figure 2 (left) shows an example of a response in which the PST set up a proportion by using *the within ratio strategy*, with which the PST found the ratio in one context (i.e., 156 miles/6 gallons) and applied it to another context (i.e., 561 miles/x gallons). The PST then cross-multiplied to find a missing value for the gas needed for a trip of 561 miles and concluded the Dr. Lee would not be able to travel 561 miles with 21 gallons because more gas was needed.

As seen in the example on the right side of **Figure 2**, some PSTs set up a proportion by using the Between Ratio strategy, with which the PSTs found the ratio between contexts (i.e., 156 miles/561 miles) and applied it to find a missing value (i.e., 6 gallons/x gallons). After finding the gas needed for a trip of 561 miles through cross-multiplication, the PSTs concluded that Dr. Lee could not complete the trip with 21 gallons.

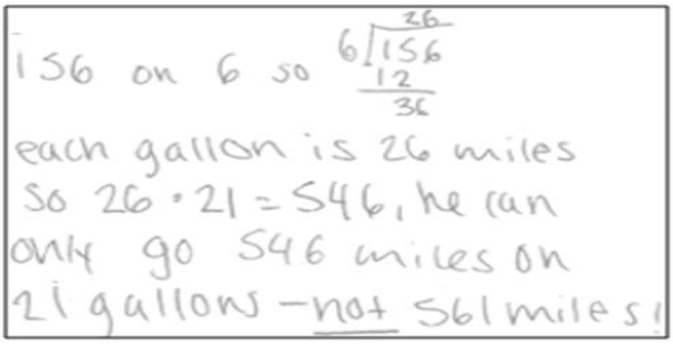

Figure 3. An example of the strategy "finding and interpreting mileage"

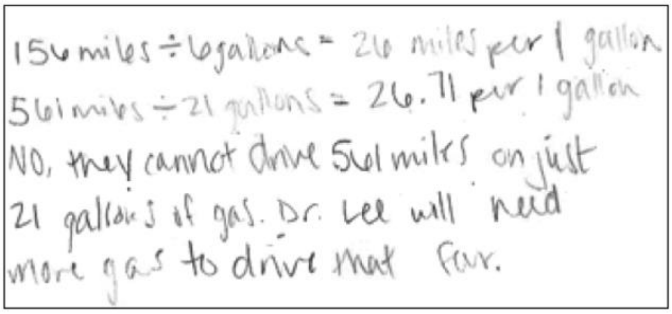

Figure 4. An example of the strategy "finding and interpreting efficiency rates"

Strategy 2: Finding and interpreting mileage

Thirty-six PSTs chose to focus on mileage by

a) calculating the miles the car could go on a full tank,

b) comparing that mileage to the desired mileage of the trip, and

c) concluding that the car could not complete the trip on a full tank.

The crucial real-world information is that the length of the desired trip must be smaller than the mileage possible on a full tank for the trip to be possible. **Figure 3** shows an example of a response using this strategy, in which the PST used the *scale factor strategy* by dividing 156 miles by 6 gallons to finding the unit rate (26 miles per gallon) and multiplying it by 21 gallons to calculate how far that many gallons would take the car, an amount that fell short of the actual distance to be travelled.

Strategy 3: Finding and interpreting efficiency rates

Fourteen PSTs chose to focus on efficiency rates by

a) calculating the efficiency rate of gallons per mile of the known trip and the hypothetical efficiency if the desired trip used exactly a full tank,

b) comparing the two efficiency rates, and

c) concluding that the car could not complete the trip on a full tank.

The crucial real-world information is that the gas efficiency of the projected trip must be larger than the known efficiency of the car. An example of a PSTs response is shown in **Figure 4**.

In this example, the PST found the efficiency rates for two pairs of information by dividing 156 miles by 6 gallons and 561 miles by 21 gallons and concluded that more gas would be needed to drive 561 miles because the first efficiency rate (26 miles per gallon) is less than the second efficiency rate (26.71 miles per gallon).

Challenges Demonstrated in PSTs' Interpretations of Their Calculations

In total, 46 PSTs, almost a quarter (23.1%) of the prospective teachers in our sample, struggled to find the appropriate quantitative meanings of their own calculations. Among the difficulties they exhibited, we found several themes, as follows:

1. inappropriate rounding,
2. switching meanings of calculations,
3. failure to give a verbal answer,
4. unrealistic modeling of gas consumption, and
5. meaningless calculations.

Each of these can be attributed to a lack of attention to the real-world context of the task and/or the question.

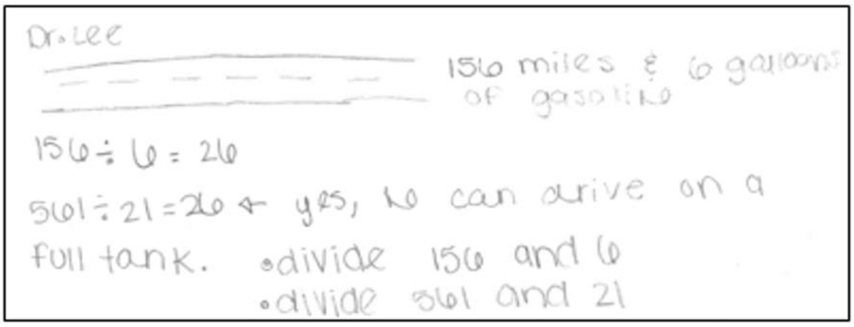

Figure 5. An example of the assumption "efficiency rates are all whole numbers"

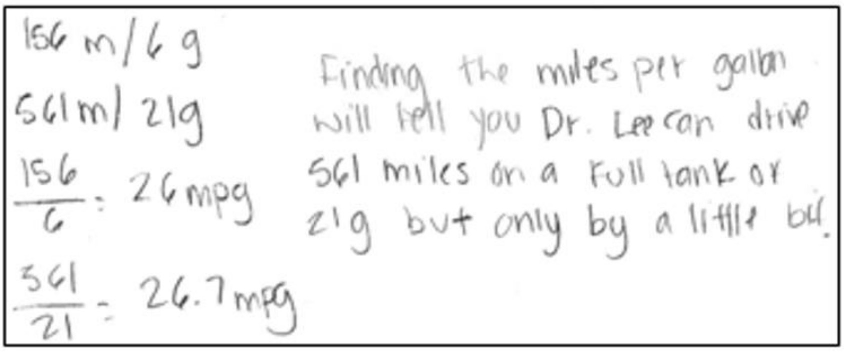

Figure 6. An example of the assumption "only whole number parts of efficiency rates matter"

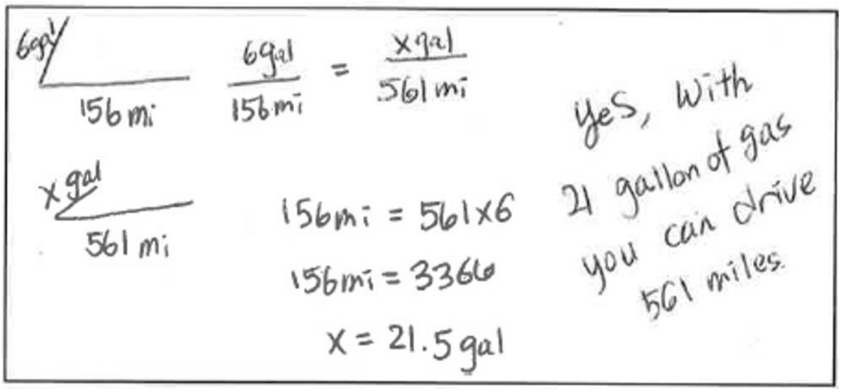

Figure 7. An example of the assumption "only whole number parts of gas volume matter"

Whole number bias: Rounding inappropriately

The most common problematic interpretations revolved around a strong bias towards either calculating only whole numbers or reasoning with only the whole numbers in their calculations. Of the 46 PSTs who struggled to interpret their own work, 18 did so at least in part because they chose not to reason with decimal numbers. We hypothesize that these teachers looked for similarities in the results of their calculations without attending to the measures that they represented.

Five of these 18 PSTs used the *finding and interpreting gas efficiency method*, rounded both gas efficiency rates to 26 miles per gallon (mpg), and concluded that Dr. Lee could complete his/her trip (**Figure 5**) even though it is clearly relevant that 26.7 mpg is not the same as 26 mpg. Therefore, PSTs who rounded to 26 mph made a problematic interpretive decision when transferring their work from calculator to paper, or when deciding to end their long division after finding the whole number part of their response.

Eight of these 18 PSTs used the *finding and interpreting gas efficiency method* to conclude that the gas efficiency rates of both trips were approximately the same, so they concluded that Dr. Lee could complete his/her trip, although four qualified their answers by saying that Dr. Lee could just barely make it (**Figure 6**). In doing so they neglected to keep track of the meaning and significance of their own calculations. The equality of both efficiency rates is irrelevant; rather, it matters whether the needed gas efficiency of the hypothetical trip is less than or equal to the known efficiency of the car.

Five of these 18 PSTs used the *finding and interpreting gas usage method* to find that it would require approximately 21.5 gallons for Dr. Lee to complete his/her trip, and nevertheless concluded that his/her trip was possible (**Figure 7**). In doing so, they neglected to recognize that what matters is whether the gas the trip requires is less than or exactly equal to the gas that Dr. Lee has in a full tank, not whether they are approximately equal.

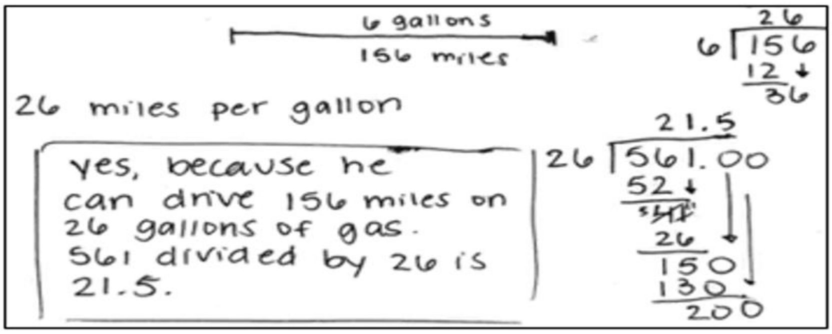

Figure 8. An example of the assumption “efficiency calculations become gas usage results”

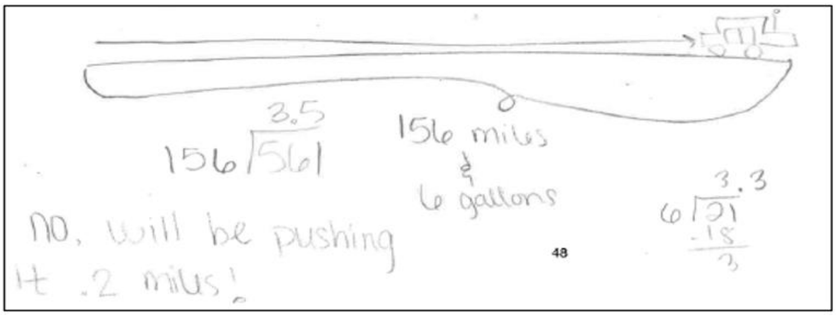

Figure 9. An example of the assumption “efficiency calculations become mileage results”

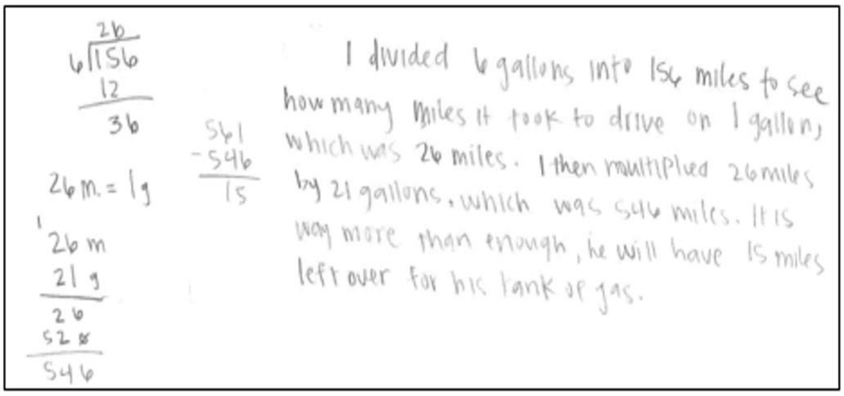

Figure 10. An example showing “directionless difference”

Mixed up quantities: Switching meanings of calculations

The next most common cause of this mistake was that PSTs (n=16) did not keep track of the quantitative meaning of their results. If we start with the presupposition that every calculation should have meaning to the person doing the calculating, these mistakes cannot simply be dismissed as writing down the wrong unit. We hypothesize that these PSTs had difficulty interpreting their own work at least in part because they carried out operations on numbers instead of quantities (Thompson, 2011). For example, nine out of 16 PSTs calculated values and then ascribed the wrong quantitative meaning to them. As seen in **Figure 8**, they calculated the gallons needed to complete a trip of 561 miles (21.5 gallons) but then merely wrote down 21.5 miles, and calculated the gas efficiency of the shorter trip (26 miles per gallon) but then merely wrote down 26 gallons without further operating on these quantities.

Another PST calculated the relative sizes of the trips in both gallons and miles (561 miles is 3.5 times as large as 156 miles, and 21 gallons is 3.3 times as large as 6 gallons) and then concluded that the difference meant that “Dr. Lee will be pushing [the car] 0.2 miles.” That is, the PST used the “*between ratio (functional) strategy*” to find ratios between contexts and could have proceeded by concluding that the relative size for the gas must be greater than or equal to the relative size of miles in order for Dr. Lee to complete the trip. Instead, the PST interpreted the difference of two scale factors as a measurement of miles (**Figure 9**).

Six out of the 16 PSTs found values that would enable them to answer the question and interpreted the quantitative meaning of these values correctly, but then did not keep track of the meaning of the difference. Four PSTs correctly set up a proportion to find a value of 546, but then said that this means Dr. Lee can make the longer trip; two of these four PSTs interpreted the result of the calculation 561-546=15 to mean that Dr. Lee could drive an extra 15 miles (**Figure 10**). They did not keep track of the meaning of the difference as the miles that Dr. Lee could not drive the full distance on a full tank. Two PSTs calculated other differences (in the gas efficiencies of each trip and gallons of gas used in each trip) and also concluded that Dr. Lee could complete the trip.

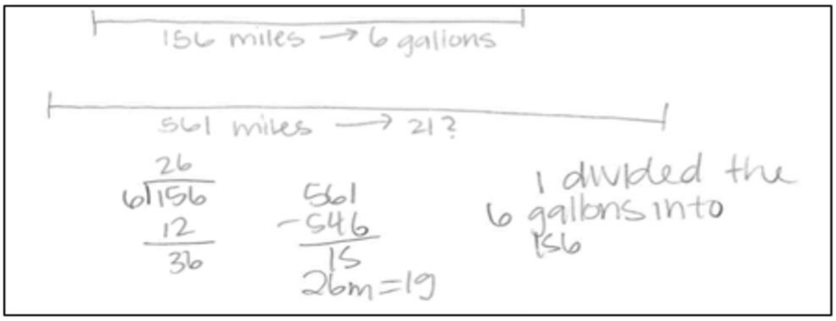

Figure 11. An example of failure to provide the answer

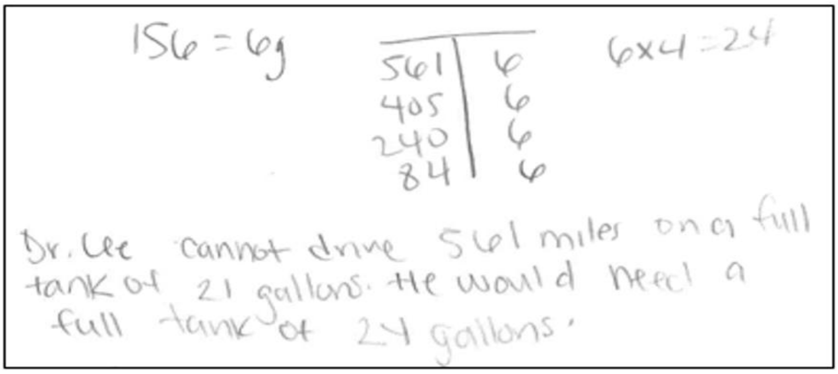

Figure 12. An example of calculating gas consumption in six-gallon chunks

Values are answers, no meaning ascribed: Failure to provide verbal answers

Five of the 46 PSTs reached the values they needed for either the number of miles Dr. Lee could travel on 21 gallons or the gas efficiency of both trips, but simply did not answer the question (**Figure 11**). While this may be an oversight, it may also illustrate a belief that mathematics is only about finding numerical answers or that solving problems involving proportions equates to finding a missing value. This may come from their dominant experiences with solving missing-value problems in their school mathematics courses, in which students are usually required to find the missing value when three of four values are given in the proportion (a/b=c/d). According to Tjoe and de la Torre (2014), the use of ratio or proportion to solve missing value problems is the most common task in the US mathematics curricula.

Chunking gas amounts: Unrealistic modeling of gas consumption

Four of the 46 PSTs reasoned only in chunks of six gallons. For example, in **Figure 12**, the PST attempted to find how many reiterations of 26 fit into 561 (which would provide a calculation of needed gas volume for Dr. Lee's trip), but only by working with chunks of six reiterations of 26 (=156) at a time. After repeatedly subtracting 156 from Dr Lee's trip length left a non-zero amount, the PST concluded that 24 more reiterations would be needed. Such a chunking strategy works for solving some proportional problems and is often used by students who are in transition from being additive thinkers to being multiplicative thinker (van Dooren et al., 2010). According to prior studies, such chunking is commonly seen in more abstract contexts like linear equations, slope, and accumulation in calculus (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017), but we were surprised to find it in such a concrete example.

Equality is everything: Meaningless calculations

Three of the 46 PSTs used cross-multiplying and looked for the equality of both sides. For example, in **Figure 13**, a PST used the strategy of cross-multiplying and checking for the equality of both sides although the resulting numbers (3,276 and 3,366) do not have clear quantitative meanings, as can be seen from their incoherent units of "dollar-gallons." This strategy is appropriate when checking whether two fractions are equal, but not an appropriate strategy for determining whether one-unit rate or gas efficiency is greater than or equal to another. This issue has often been raised by prior studies of students' errors derived from dependence on the cross-multiplying formula in solving proportional reasoning problems (Arican, 2019; Lamon, 2007; Livy & Herbert, 2013). That is, students tended to simply use the cross-multiplying formula without reference to the meaning of the problem context in solving a word problem about proportion.

CONCLUSIONS

One of the enduring problems of mathematics education research is how to improve mathematics education in a way that benefits students both inside and outside the classroom. The days when a human calculator was valued and useful are over; now,

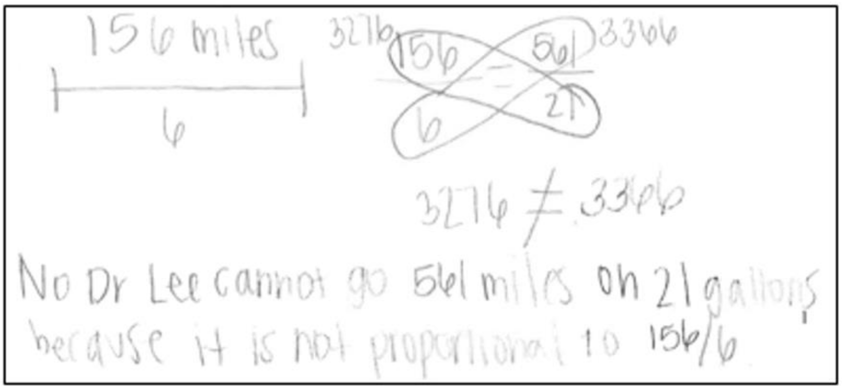

Figure 13. An example of the assumption that equality is everything

as technological tools have developed, a person's ability to interpret the significance of mathematical outputs has been emphasized more than simple calculation. In the common core, the ability to interpret the meaning of one's answer is central to at least five of the eight mathematical practices: make sense of problems, reason quantitatively, construct viable arguments, model with mathematics, and attend to precision (National Governors Association Center for Best Practices, 2010).

When considering this trend, it is worth noting that almost a fourth of the PSTs surveyed over the course of three semesters at a large university's teacher education program struggled to interpret the meaning of their own calculations on a sixth-grade level task. Teachers that struggle to assign meaning to their calculations will certainly also struggle to impart that skill to students, but there are also further concerns. Such a teacher is also likely to avoid an area he/she feels weak in, which can be reflected in the problems he/she chooses to assign to students or to do with students in class. Prior research shows how crucial it is to student achievement for teachers to have a deep understanding of the meanings underlying the procedures they carry out (Hill et al., 2005; Hilton & Hilton, 2019; Lee, 2017a, 2017b, 2021). Additionally, teachers' choices can also convey beliefs to students about what mathematics entails or what kinds of problems should be encountered in a mathematics class (Lee & Francis, 2018). According to prior studies (Giorgi et al., 2013; Son & Lee, 2021), teachers' beliefs and conceptions of problem solving influence their way of teaching and their students' achievement.

Implications

The only foreseeable solution to the problem discussed above lies in making teachers' quantitative reasoning a core focus of both preservice teacher education and in-service professional development. This quantitative reasoning is applicable not only to real world or story problems (e.g., Cross et al., 2012) but also to a conceptual understanding of mathematics; for example, students need to reason about the abstract quantities represented by the independent and dependent variables in order to make sense of functions. Such a focus on quantitative reasoning would be best implemented by refining the structure and content of current methods courses and professional development opportunities; a separate course might imply that quantitative reasoning is a special and separate topic applicable only to a narrow section of mathematics rather than a pervasive feature of mathematical thinking and real-world problem solving.

Here we have four specific recommendations for teacher educators derived from our experiences in analyzing this data set. First, teacher educators need to discontinue the use of the cross-multiplying procedure (where the intermediate results have no clear quantitative meaning and may produce incoherent units such as dollar-gallons) in favor of finding unit rates, or finding the relative size of one pair of measurements and then applying it to the other. This suggestion is related to PSTs' errors due to unrealistic modeling of gas consumption and meaningless calculations. Second, teacher educators need to focus explicitly on reasoning with non-integer measurements of quantities, which is related to PSTs' inappropriate rounding error. Third, teacher educators need to provide tasks necessitating proportional reasoning along with tasks that sound similar but do not involve proportional quantities, so that PSTs and teachers need to genuinely assess whether proportional reasoning is appropriate for each task. Although this issue was not specifically addressed in this study, the PSTs' errors with switching meanings of calculations and unrealistic modeling of gas consumption may be related to this suggestion. Lastly and most significantly, teacher educators need to devise tasks that require PSTs and teachers to ascribe meaning to their own calculations as the norm in their classrooms, without exception. This suggestion is especially derived from some PSTs' failure to recognize the need for verbal answer.

This study has some limitations in that it used only one problem and relied only on a written assessment without human interaction such as interviews. Thus, follow-up studies should include multiple problems requiring quantitative reasoning in real-life contexts and different research tools such as focal interviews. But despite such methodological limitations, this study warrants a closer examination of the problem that PSTs are adept at carrying out calculations without having coherent meanings for their own results. When considering that a teacher who cannot make sense of his or her own calculations has no chance of helping students to understand theirs, this issue should be noted and addressed through appropriate interventions in teacher education programs.

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