EXAMINING MATHEMATICAL JUSTIFICATION LEVELS OF 7TH GRADE STUDENTS

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Abstract: Reasoning and justification are among the core skills that mathematics education aims for students to acquire. Studies indicate that students having these skills are able to formulate meaningful geometric ideas. It is also important that students know the significance underlying the operations and mathematical ideas employed or, in other words, that they be able to justify their solutions while learning geometry. This study analyses the justification skills of 7th-grade students using the survey research pattern. A total of 254 middle school students constitute the research sample. The study made use of a descriptive analysis of the justifications given to four open-ended questions in the geometry knowledge test to examine the data collected using the mixed method incorporating quantitative and qualitative methods. The answers given to the questions were coded as follows: complete and convincing justification, incomplete justification, incorrect justification, and no justification. The findings of the study show that most of the student solutions provided for the said questions requiring justification can be categorized under the title of incorrect justification. Furthermore, the number of complete and cogent justifications is significantly lower. The study concludes that the ability of the participants to come up with complete and cogent justifications is limited.

Key words: Mathematics education, Justification, Geometric reasoning, Middle school students

1. Introduction

In the present day, there is a need for individuals able to use their mind, to think fast yet comprehensively, and to come up with creative and novel ideas. Therefore, developing one's reasoning, problem-solving, and questioning skills has become one of the principal objectives of education. The importance attached to justification consisting of questioning and reasoning within the scope of mathematics education also increases gradually. The National Council of Teachers of Mathematics (NCTM, 2000) indicates that mathematical justification is an effective way of developing and expressing ideas on various events.

The same argument also applies to geometry, a sub-branch of mathematics (Jones, 2000). Geometry education is crucial for both making students able to use certain knowledge in daily life and helping them develop many mathematical concepts and skills (Clements et al., 2004). Geometry education is a process aiming for students to acquire geometric knowledge and reach geometric generalizations by analyzing elements constituting geometric objects and shapes (sides, angles etc.) and their qualities (parallels, right angles etc.) on tangible objects and models (Dai et al., 215). In addition to having the knowledge outlined above, geometry education allows one to interpret and intervene in their surroundings; it is also useful as a means of conducting studies on other subjects like mathematics and sciences. Besides, the classification of geometric shapes and understanding of their features contributes to the solution of real-life and other mathematical (measurements, algebra, and numbers) problems (NCTM, 2000).

One of the higher-order thinking skills, reasoning is about the development of skills such as geometric thinking and generalization, and the formation of geometric ideas in a meaningful way (Driscoll et al., 2007). In the development of these skills, students should be able to justify the results they reach in their solutions (Cai, 2003). It is important for students to be able to answer the "Why" question while making mathematical justification (Sandborg, 1998). Additionally, a good mathematical justification must address the same question. Doing so would reveal the background of the information that

students have. Stylianides (2009) stated that the students' background in their thoughts by justifying them will be revealed with problems that include reasoning and proving activities. Many reform curricula place an emphasis on open-ended problems for which students are expected to provide justification for their solutions (Knuth, 2002). In this context, Cai (2003) revealed that students' mathematical justifications can be determined with the help of open-ended problems and revealed a theoretical framework with four categories. The theoretical framework of Cai (2003) was used in the students 'answers to open-ended questions in this study, and it was tried to determine students' justification skills.

2. Theoretical Framework

Mathematical justification is useful in interpreting the world one lives in, synthesising available information, making decisions, and asking questions (Goggins, 2001; Wilkins, 2000). According to Fitzgerald (1996), mathematical justification is a process of obtaining one kind of information from another. People who are able to justify in any given subject have adequate knowledge in the subject in question and are able to examine and explore a novel situation taking every aspect into account, to make logical estimations and assumptions, to provide justifications for their opinions, to reach certain conclusions, and to explain and defend these conclusions (Umay, 2003).

Reasoning and justification are among the basic skills that should be acquired by students within the scope of geometry teaching (NCTM, 1989; MEB, 2018). Many studies have shown that students do not have difficulty in making rules and operations in mathematics, but do not know the underlying meanings of their operations and mathematical ideas (Hadas, Hershkovittz, & Schwarz, 2000; Toluk Uçar, 2011). Generally, students have less difficulty in doing routine operations in problems compared to situations that require geometric thinking (Kinach, 2002; Yeşildere & Türnüklü, 2007). Similarly, it has been revealed in many studies that students have difficulty in explaining their solutions and ways of thinking (Karakoca, 2011). It is seen that the subject of justification is addressed in the literature in the form of "practical and mathematical explanations, the relationship of explanations with proof, explanations in terms of conceptual and operational aspects, and explanations as a cogent justification" (Hanna, 2000a; Hanna, 2000b; Yackel, 2001; Raman, 2002; Cai, 2003; Chick, 2003; Levenson, Tirosh & Tsamir, 2006; D'Amore & Pinilla, 2006; Türnüklü & Yeşildere, 2007; Hauk & Isom, 2009; Ball, Charalambous & Hill, 2011; Toluk Uçar, 2011; Matteson et al., 2012; Staples, Bartlo & Thanheiser, 2012;). The differing points of view can be listed as practical and mathematical explanations, the relationship between explanations and proving, conceptual and operational explanations, and explanations as cogent justifications.

Studies in the literature reveal that justification from various perspectives. For instance, Levenson and colleagues (Levenson, Tirosh & Tsamir, 2006; Levenson, 2010; Levenson, Tsamir & Tirosh, 2010) studied practical and mathematical explanations. Additionally, there are also other research studies in which explanations are examined on the basis of their relationship with proofs (Arslan, 2007; Hadas, Hershkowitz & Schwarz, 2000; Hanna, 2000; Sandborg, 1998), based on its conceptual and operational aspects (Ball, Charalambous & Hill, 2011; Kinach, 2002; Toluk Uçar, 2011), or from the aspect of being a cogent justification (Cai, 2003; Türnüklü & Yeşildere, 2007; Chick, 2003). An evaluation of existing studies reveals recommendations regarding the further development of justification among prospective teachers. The studies suggest that classroom environments in which prospective teachers can provide mathematical justifications improving their proving skills or propose cogent justifications should be established. There are a few studies conducted with middle school students in Turkey (Duatepe, Çıkla & Kayhan, 2005; Türnüklü & Yeşildere, 2007), but these studies do not deal with the academic consequences of the lack of justification skills among students. The number of similar studies on this subject is quite limited. Thus, the present study will handle justification skill levels of 7th-grade students in the learning domain of geometry.

In this study, within the framework of Cai (2003), middle school students' justification skills related to some geometry concepts were examined. For this reason, the aim of the study is to determine the justification skill levels of middle school students in the field of learning geometry with the geometry knowledge test. In accordance with this purpose; "What are the levels of justification skills of middle

school 7th grade students in the geometry sub-learning area?" The answer to the question will be sought.

3. Material and Method

The present study was conducted with the objective of assessing the justification skills of 7th-grade students. In this respect, the method of descriptive surveys, a qualitative research pattern, was employed.
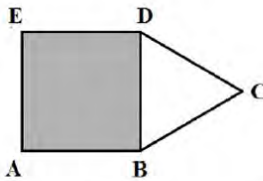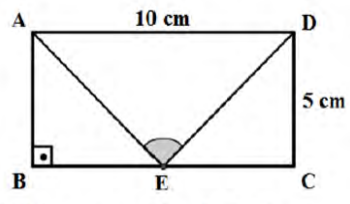
3.1. Student participants

The participants in this study were 254 (50% male, 50% female) 7th grade middle school students from a socio-economically diverse school district in the south Turkey, Gaziantep region. Two classes of 7th-grade students from each middle school were selected randomly and the students in these classes participated in the study.
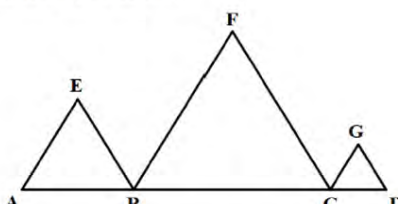
3.2. Data collection tool

Data collection tool consists of 7th grade geometry student knowledge test. The test consists of two parts. The first section is the achievement test composed of 10 multiple-choice questions whereas the second includes 4 open-ended questions testing the justification skills of students. While the open-ended questions aimed to evaluate the justification skills, the multiple-choice questions were designed to assess the academic success of the students.

To determine the justification skill levels of the 7th-grade students, the four open-ended questions were considered. The students' justification skills were derived from their answers to these open-ended questions. The questions were named with the subject they were related to: the square, the angle, the triangle and the circle. The problems and the solution steps expected from the participants are presented in Table 1.

Table 1. *The problems (square, angle, triangle and circle) and expected participants solutions*

In figure given below ABCD is a square and BCD is a equilateral triangle

If perimeter of the polygon ABCDE is 40 cm, find the area of shaded region.

A) 64   B) 49   C) 36   D) 32

As to persuade someone else explain your solution and its reasons

In the problem, the perimeter of a polygon consisting of equilateral triangle and square is given. It is required to find the area of the square. Participants are expected to use the knowledge that all sides of the equilateral triangle and the square are of equal length, then find one side of the polygon whose perimeter is given, and then calculate the area of the square whose side is known.

ABCD is a rectangle and E is a middle point of BC.

So find the measure of angle AED

A) 30   B) 60   C) 90   D) 120

As to persuade someone else explain your solution and its reasons

In the question, the participants were asked to justify the operations they did while finding the angle inside the figure with the help of the midpoint in a rectangle. Participants are expected to use the midpoint feature on the geometric shape. While doing this, they need to discover other geometric shapes that emerge and justify the solution of the problem based on the relationship between the shapes they find. Here, students are expected to explain their way of going to the conclusion they have reached.

In figure given below the three diferent equilateral triangles lined up on AD.

If the length of side AD is 9 cm find perimeter of the whole figure.

A) 18   B) 27   C) 36   D) 45

As to persuade someone else explain your solution and its reasons

Three different equilateral triangles are given. The side lengths of the triangles are not known, but they are given as 9 cm, which is the sum of the sides of the triangles. The participants are asked to calculate the perimeter of the geometric shape formed by these three equilateral triangles. In this question, participants are expected to apply equilateral triangle properties. It is expected from the participants to discover the relationship between the objects under the given conditions and to justify the solution of the problem based on the relationship they find.

A line can intersect a circle in at most two points.

According to this situation how many points at most will a quadrilateral cut a circle

A) 10   B) 8   C) 6   D) 4

As to persuade someone else explain your solution and its reasons

In the circle problem, the students were given preliminary knowledge that "a line can cut a circle at two points". It is asked at how many points a quadrilateral can intersect a circle. Students are expected to discover the relationships between objects under the given conditions and justify the solution of the problem based on the relationship they find. Students are expected to use drawing/visual representations or to explain the solution stages using mathematical language.

After the assessments concluded that the problems were suitable for the mathematics curriculum for 5th to 8th grade, the pilot study was conducted with 7th-grade students. A randomly selected class of 38 7th-grade students participated in the pilot study. The pilot study revealed no problems stemming from the questions nor the application and came into the conclusion that a duration of 1 class hour was sufficient for the study. The problems included in the data collection tool allows the student to reason geometrically and review other ways of thinking as well. The teachers are not required to tell anything to the students except for the fact that the study will last one class hour.

3.3. Data analysis

Three point types were calculated for each student, one for multiple-choice questions and two for open-ended questions. The first was used as a measure of student achievement and the other two were used as the student's justification. In terms of justification, student answers were assessed based on a framework consisting of four thematic categorizations. Based this framework, answers given to each open-ended question were evaluated within the context of qualitative analyses. The justification framework proposed by Cai (2003) was used for students answers. The themes of Cai's (2003) framework on justification skills are shown in Table 2.

The themes were coded as complete and convincing justification (3), incomplete justification (2), incorrect justification (1), and no justification (0). Using this framework, the answers given by the participants to each question were assessed and at the end of this assessment, a score for justification skills was given to each participant. Table 3 shows how the theoretical framework is used for (the square problem) one of the four open-ended problems.
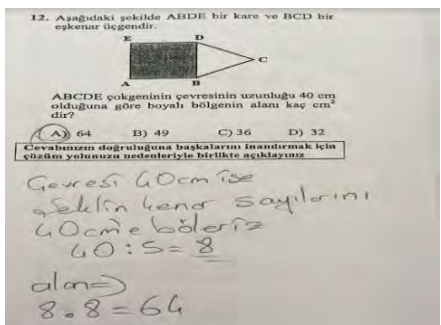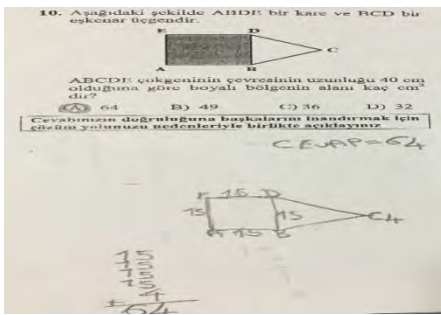
One of the strengths of the analysis process is inter-coder reliability (inter-coder consistency). The initial step to determine the level of reliability of the codes used in the analyses is to ask two researchers to code 18 randomly selected answer sheets each along with the framework used to assess justification skills. Inter-coder reliability was then tested by comparing the coding made by the researcher and by the mathematics expert. The Kappa statistic was used to determine inter-coder reliability. The Kappa coefficients, i.e. an indicator of inter-coder consistency, were calculated as K = 1.00 for the area problem, and K = 1.00 for the perimeter problem. As all these values are greater than 0.4, the Kappa statistic showed that the reliability of scoring is sufficient (Sim& Wright, 2005).

Table 2. *The framework used in the analysis of the open-ended questions in the data collection tool*

Themes	Definition
Complete and Convincing Justification (3)	Answers supporting the correct solution of the question with correct mathematical justifications will be included in this code
Incomplete Justification (2)	Answers not including a sufficient amount of symbols and indications and showing mathematical justifications with vague statements even if the question is solved (partially) correctly will be included in this code.
Incorrect Justification (1)	Answers including incorrect solutions, having operational or conceptual errors, and showing unrelated statements and explanations that do not provide the required solution will be included in this code.
No Justification (0)	Answers with no mathematical indications nor statements or unanswered questions will be included in this code.

For the distinctiveness of the multiple-choice questions in the knowledge test, the students were ranked in order of scores obtained from the 10-question geometry knowledge test, from the highest score to the lowest, and upper and lower groups of 27% were defined. After defining these upper and lower groups, the 4 extreme values from each group were excluded. As there were n=254 students in total, the lower and higher groups consisted of 68 participants each.

Table 3. *Exemplary analyses based on Cai's (2003) justification framework (the square problem)*

Type of Justification	Definition	Student Example
Complete and Convincing Justification	Answers supported by correct mathematical justifications in addition to the correct solution itself will be included in this code.	If the perimeter is 40 cm, we divide the number of sides of the shape by 40 cm. 40 : 5 = 8 Area => 8 · 8 = 64
Incomplete Justification	Answers not including a sufficient amount of symbols and indications and showing mathematical justifications with vague statements even if the question is solved (partially) correctly will be included in this code.	

Incorrect Justification	Answers including incorrect solutions, having operational or conceptual errors, and showing unrelated statements and explanations that do not provide the required solution will be included in this code.	
No Justification	Answers with no mathematical indications nor statements or unanswered questions will be included in this code.	

Table 4. *Distinctiveness results of the justification questions included in the geometry knowledge test*

	Higher group (n = 68) Mean ± sd	Lower group (n = 68) Mean ± sd	t- statistic	p-value
The square problem	1.68 ± .72	.53 ± .53	10.78	< .001
The angle problem	1.41 ± .58	.73 ± .51	7.34	< .001
The triangle problem	1.35 ± .56	.54 ± .56	8.47	< .001
The circle problem	1.78 ± .70	.51 ± .53	11.99	< .001

4. Results

The distribution of the area and the perimeter questions to the four codes are shown in the table below. Table 5 shows that 15.2% of the answers given to open-ended questions include no justifications while 65.5% has incorrect justifications, 17.1% gives incomplete justifications and 2.2% provides complete and convincing justifications. To help the reader understand these distributions better, the following sub-sections provide details for each question.

Table 5. *Descriptive statistics of justification levels based on the explanations provided*

	No Justification n = 154 (%*)	Incorrect Justification n = 666 (%*)	Incomplete Justification n = 174 (%*)	Complete and Convincing Justification n = 22 (%*)
The square problem	43 (16.9)	156 (61.4)	48 (18.9)	7 (2.8)
The angle problem	30 (11.8)	190 (74.8I	32 (12.6)	2 (0.8)
The triangle problem	39 (15.4)	143 (56.3)	61 (24.0)	11 (4.3)
The circle problem	42 (16.5)	177 (69.7)	33 (13.0)	2 (0.8)
Total	154 (15.2)	666 (65.5)	174 (17.1)	22 (2.2)

*Values are percent of rows.
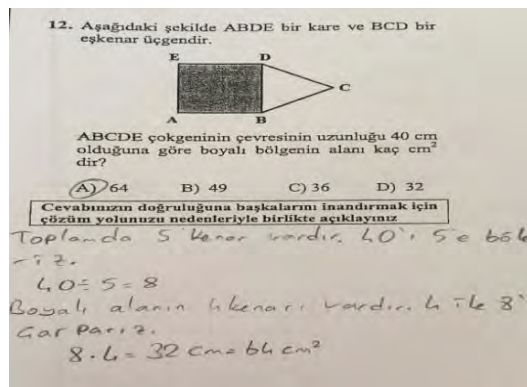
4.1. The square problem

Based on the answers provided for the question, the frequencies among participants in terms of justification levels are shown in Table 5. The survey showed that 43 participants provided no justification while 156 gave incorrect justifications, 48 provided incomplete justifications, and only 7

of 254 gave complete and convincing justifications. In the square problem, the participants were asked to calculate the area of the square using the perimeter of a pentagon formed by a square and an equilateral triangle and to justify the correctness of their answers. 41 out of 43 participants that provided no justification did not give any explanation while the remaining 2 gave an answer without any justification.

Figure 1 shows one of the answers categorised under the title of incorrect justification. As seen in the figure, the participant #31, found the length of one side of the polygon and the perimeter of the square correctly, but then expressed the equation of 32 cm=64 cm2. There are no justifications of the reason why the participant came into the conclusion that 32 cm=64 cm2. This finding also indicates that the participant confuses the concepts of perimeter and area, leading to a conceptual mistake. The substantial majority of the participants whose answers were categorised under the title of incorrect justification (90%) made conceptual mistakes. 48 participants whose answers were categorised under the title of incomplete justifications had certain vague statements. 7 of the answers given by participants to the square problem included explanations supported by correct mathematical justifications.
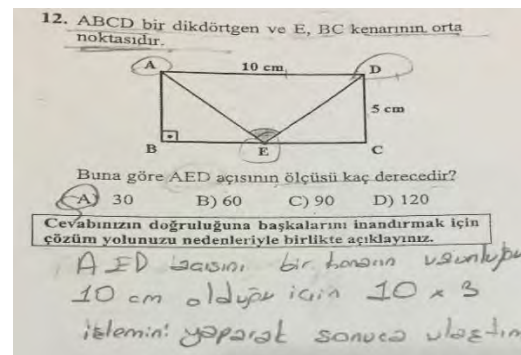
4.2. The Angle Problem

The examination of the answers given to the question by participants showed that only 2 of them were able to give complete and cogent justifications (Table 5). 30 participants did not provide any justification while a majority of the participants (75%) gave incorrect justifications for the angle problem. The question required students to realize that the lengths of the different sides of a rectangle with a ratio of 1:2 were not coincidental, to draw two congruent right-angled triangles using the center of the polygon, and to explain this with mathematical terms. At the end of these steps, the angle m(AÊD) is created. The problem expects the students to calculate the degree of this angle and justify their answers. The assessment of the answers shows that 27 out of the 30 participants who did not provide justifications to their answers did not give any explanation while the remaining 3 just gave the answer without justification.

There are 5 edges in total. We divide 40 by 5.
$$40 : 5 = 8$$
The painted area has 4 edges. We multiply 4 by 8.
$$8 . 4 = 32 \text{ cm} = 64 \text{ cm}^2$$

Figure 1. *Answer given to the square problem by #31*

Since the length of one side is 10 cm, I got the result by doing the 10 × 3 operation to calculate the AED angle.

Figure 2. *Answer given to the angle problem by participant no. #53*

The explanations provided by the participant (#53) includes a statement claiming that "10x3 as the length of one side is 10 cm with the angle AED" (Fig. 2). The problem asks the participants to calculate the degree of the angle AED. However, upon examining the explanation, it was observed that the participant made a conceptual mistake by confusing the concept of angles with the concept of length. The answer was categorized under the title of "incorrect justifications" as there were false explanations. 188 out of 190 participants whose answers were categorized under the title of incorrect justifications made conceptual mistakes while the remaining 2 participants made operational errors. 32

participants who provided incomplete justifications had vague and unclear statements in their explanations. The answers given by 2 of the participants categorized under the title of complete and cogent justifications consisted of explanations supporting the answer with correct mathematical justifications.

4.3. The triangle problem

The frequencies among participants in terms of justification levels are shown in Table 5. The analysis of the answers given to the triangle problem showed that only 4.3% of the participants were able to provide complete and convincing justifications. Furthermore, more than half of the answers included incorrect justifications, 24% were based on incomplete arguments, and 15% did not include any justification at all. The problem gave the student 3 equilateral triangles and a length value obtained by the combination of these triangles. This length is the sum of one side from each of the three separate triangles. The participants were asked to calculate the perimeter of the shape given in the question and to write the justifications for their answers. 30 out of 39 participants categorised under the title of no justification provided no explanation while the remaining 9 students gave the answer without any justification.

Cuts at 4 points. A quadrilateral with 4 sides cuts a circle at a maximum of 4 points.
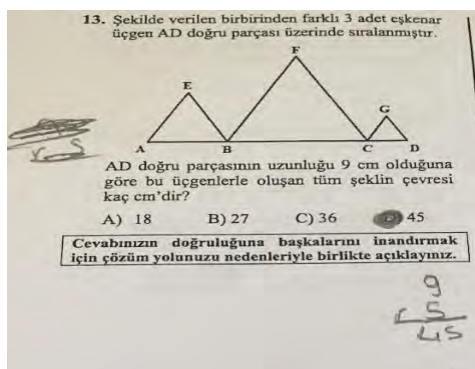
Figure 3. *Answer given to the triangle problem by #172*
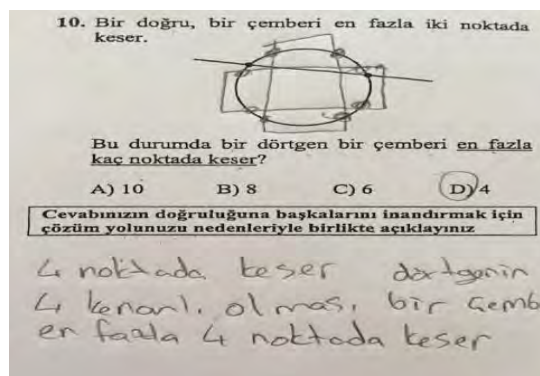
Figure 4. *Answer given to the circle problem by #97*

Figure 3 shows the answer given by participant #172 to the triangle problem, categorised under the title of incorrect justifications. The examination of the answer given by the participant in question (#172) show that they multiplied the statement |AD|=9 cm by 5 and no other notation, operation, nor justification is present in the solution. This answer was categorised as "incorrect justification". 117 out of 143 participants whose answers were categorised under the title of incorrect justifications made conceptual mistakes while the remaining 26 participants made operational errors. The answers of 41 participants giving incomplete justification consisted of certain vaguely expressed statements as explanations for the answer. The answers given by 11 of the participants categorised under the title of complete and convincing justifications consisted of explanations supporting the answer with correct mathematical justifications.

4.4. The Circle Problem

The examination of the answers given to the circle problem by participants showed that only 2 of them were able to give complete and cogent justifications. Additionally, 70% of the answers included incorrect arguments and 13% were based on incomplete justifications while 17% did not include any justification at all. The question required the participant to calculate the maximum number of points in which a tetragon can intercept a circle and show written or pictorial/visual justifications for the accuracy of their answers. 26 out of 42 participants categorised under the title of no justification provided no explanation while the remaining 16 students gave an answer without any justification.

The answer given by the participant shown in Figure 4 (#97) argues that the number of sides of the tetragon also gives the number of points in which it intercepts the circle. This was categorised under the title of "incorrect justification" as the reasoning is not accurate. 149 out of 177 participants whose answers were categorized under the title of incorrect justifications made conceptual mistakes while the remaining 28 participants made operational errors. The answers of 33 participants giving incomplete justification consisted of certain vaguely expressed statements as explanations for the answer.

5. Discussion and Conclusions

The four questions included in the achievement test given to 7th-grade students that assess the justification levels of the participants were analysed qualitatively based on Cai's (2003) justification framework; at the end, the justification levels were ranked. In the achievement test, a significant majority of the student solutions were categorised under the title of incorrect justifications (65.5%) while the rate of complete and convincing arguments (2.2%) was rather low. In definitions of mathematical studies given within the scope of certain studies in the academic literature, the top 10% comparison point was described as the ability of students to arrange the information given to them, make generalisations, and explain their strategies while solving problems (Yeşildere & Türnüklü, 2007). However, only 2.2% of students were able to display these capacities. This might stem from the characteristics of the selected student group. Nevertheless, the data provided here indicate that these students are way below the general average. The following paragraphs discuss the findings regarding the justification skills among students based on the answers given to each problems.

In the square problem, the students were asked to calculate the area of the square but in most of the answers, they calculated the perimeter of the polygon. This finding suggests the existence of conceptual mistakes among the students. A review of studies in the literature concerning measurement learning and evaluation reveals that students generally encounter difficulties while understanding concepts related to measurements, making associations, and including these concepts into the problem-solving process, trying to reach a solution only through memorised formulae without grasping the logic behind concepts such as area, perimeter, and volume (Chappell & Thompson, 1999; Grant & Kline, 2003; Martin & Steutchens, 2000). Besides, other studies in the literature state that the concepts of area and perimeter are the subjects in which students make the most mistakes and have the most difficult time understanding (Chappell & Thompson, 1999; Woodward & Byrd, 1983). In addition to the difficulties experienced regarding these concepts, students made mistakes and had conceptual fallacies, particularly in questions concerning perimeters, areas, and volumes, when they tried to solve a problem exclusively with memorised formulae without knowing the definitions of these concepts (Gough, 2008; Kidman & Cooper, 1997; Moreira & Contente, 1997). Similarly to the findings recorded in this paper, the study by Kidman and Cooper (1997) argued that without any class distinctions, students expressed the sum of the lengths of all the sides of the rectangle as its area. The qualitative analysis of the data collected showed that students calculated the perimeter of the square, making incorrect justifications. This situation observed within the scope of several studies in the literature is an indicator of the confusion regarding the concepts among students. This also points to the necessity of more in-depth discussions on the concepts in the classroom environment because the lack of discussion opportunities might result in the insistence on misconceptions among students. Students must be given the opportunity to make mathematical explanations, reason on their own, and react to their peers' opinions through discussions. In such settings, students would assume the responsibility of self-learning and structure the knowledge accurately.

As far as the angle problem is concerned, the rate of students whose answers were categorized under the title of incorrect justification was higher than any other category. The underlying reason for this high rate was revealed to be the lack of understanding regarding the concept of mid-point. Another finding was the confusion of the angle value with the perimeter of the rectangle. However, the question required the students to find the angle value using the properties of the rectangle and the concept of mid-point. In this case, as mentioned in the study of Bütüner and Filiz (2018), students have difficulties using angles in geometric subjects related to angles such as triangles, tetragons, and polygons. A concept internalized by students in a prior, more limited context leads to frequent conceptual mistakes when the scope of the concept is widened afterwards (İç & Demirkol 2008). This

might be the reason underlying the confusion between the perimeter and angle values, resulting in conceptual mistakes. The conceptual mistakes stemming from a lack of due comprehension of the concepts among students arise from this situation. The reason for this might be, as explained in the study conducted by Simon, Tzur, Heinz, and Kinzel (2004), the inability of students to discuss the concepts in the classroom setting. Therefore, an environment encouraging debates must be established in the classroom, allowing the students to discuss concepts like area, perimeter, square, and rectangle. This would allow educators to take precautions by designating the subjects about which students might potentially be confused. Furthermore, exact and clear definitions of the concepts must be provided at the end of these discussions.

In the triangle problem, the participants were expected to use the characteristics of the equilateral triangle, i.e. a triangle with three equal sides. The solution of the question required them to formulate an algebraic equation using the equality of all sides. Naming each equilateral triangle using different algebraic terms, they were expected to find that the total was 9 units and the answer was 27 units as the result would be the total of these algebraic terms multiplied by 3. Students who were able to demonstrate this in their solutions were categorized under the title of complete and cogent arguments. The reason why the rate of students who were able to provide complete and cogent justifications for their answers was higher than other questions might be the fact that knowing only one feature of equilateral triangles is sufficient to solve the problem. One might reach this conclusion since some studies in the literature (Bütüner & Filiz, 2018; İç & Demirkol, 2008) display that students have a harder time in questions for which they need to associate multiple concepts when compared with those requiring a single concept. Additionally, the rate of students providing incorrect or incomplete arguments while solving other questions testing justification skills might be higher as these questions require the use of multiple concepts. This leads to certain difficulties as more than one geometric concept is included in the solution because students are taught one concept after another without complete comprehension. Therefore, conceptual confusion arises among students.

The rate of students able to provide complete and cogent justifications was low for the circle problem. The existence of multiple concepts and the inability of students to internalize these concepts underlie these low rates of complete and cogent arguments. Özsoy and Kemankaşlı (2004) describe the knowledge of a concept as the sum of meaningful associations made within one's mind. Besides, owing to conceptual knowledge, one can structure new information in their mind using existing knowledge and integrates it with novel knowledge, contributing to the internalization of concepts (Ülgen, 2004; Özsoy & Kemankaşlı, 2004). Students make conceptual mistakes because they do not know the features of a concept or how to use the definitions given to them to solve the question in different stages. This might be prevented by the inclusion of questions involving multiple concepts or definitions by instructors.

When assessed considering all the students participating in the study, the achievement test revealed that students have a hard time expressing themselves and the operations they made to solve the problem and that they do not know exactly how they found the solution. Furthermore, students providing incorrect arguments in many questions made certain conceptual mistakes. The study by Özsoy and Kemankaşlı (2004) revealed that a potential reason for mistakes and conceptual fallacies among students might be the incapacity to use axiomatic structures and features of geometric shapes while making geometric explanations, defined as the characteristics of the fourth level within Van Hiele's framework. Therefore, the prevalent conceptual mistakes among students might be explained by the inability of employing axiomatic structures and features of geometric shapes while providing geometric justifications. Additionally, the relationships between concepts must be explained in detail in order for students to develop geometric thinking skills. Instructors can do so by enabling students to explore geometric rules using well-planned activities and suitable tools and supplements while preventing conceptual fallacies by teaching them to defend their geometric arguments. In addition to enabling the student to interpret their actions, providing explanations and justifications while solving geometric and mathematical problems also make them aware of their intentions. Therefore, conceptual fallacies and mistakes students adopt and make unconsciously might be revealed while being instructed on a subject. For this reason, it might be a good idea to ask students to provide explanations

and justifications in a way that would convince their listeners about the subject discovered in class or a problem solved.

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