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Research Article

Pre-Service Mathematics Teachers' Web of Knowledge Recalled for Mathematically Rich and Contextually Realistic Problems

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ABSTRACT

This study aimed to elicit middle school preservice mathematics teachers' self-reported web of knowledge recalled in generating mathematically rich and contextually realistic problems. We designed this study as multi-tier design research incorporated into two teacher education courses in which 40 preservice teachers enrolled in total. Preservice teachers worked in small groups and recorded the characteristics of mathematically rich and contextually realistic problems. They were also asked to produce webs of knowledge recalled in this process. Preservice teachers' individual reflection papers, audio records of their group discussions and interviews were analyzed to understand how different types of knowledge in their web of knowledge function in relation to the mathematical richness and contextual realness aspects of the problems. The findings indicated that preservice teachers could identify various characteristics for mathematical richness and contextual realistic aspects of the problems. In relation to those characteristics, the preservice teachers' self-reported web of knowledge produced three core knowledge types for ensuring mathematical richness (i.e., knowledge of content, curriculum, and pedagogy) and two aspects of realistic contexts (i.e., real life knowledge and interdisciplinary knowledge). Furthermore, although they included common knowledge types, the webs of knowledge were in different shapes and indicated various relationships (i.e., hierarchical, categorical, influential, and holistic.). Considering the various relations indicated by the webs of knowledge, we claimed that teachers needed an interconnected knowledge base for mathematically rich and contextually realistic problems, the implications of which we discussed for mathematics teacher education.

Keywords: teacher knowledge, mathematically rich problems, contextual mathematics problems, multi-tier design research

INTRODUCTION

People deal with many problems in our daily lives but may not be fully aware of the problem-solving process. In real-life problems, they interpret situations and develop solutions that may even entail mathematical and logical thinking. This observation of the real-life problem-solving process provoked a genre of research such as Realistic Mathematics Education (RME) (Treffers, 1987) and Models-and-Modeling Perspective (MMP) (Lesh & Doerr, 2003). Both RME and MMP suggested a problem-solving perspective to connect mathematics and real-life situations. Specifically, the RME emphasized progressive mathematization of realistic contexts through transformations among mathematical ideas and between mathematical concepts and real-life situations (Gravemeijer & Doorman, 1999; Treffers, 1987). As for the MMP, in contrast

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to the traditional problem-solving process, problem-solving is not linear but a cyclic and iterative modeling process (Lesh, 2006; Lesh & Doerr, 2003; Lesh & Lehrer, 2003). In the modeling process, problem-solvers (i) describe the problematic situation, (ii) express the anticipated solutions, (iii) test the solutions, and then (iv) revise or refine them until the solutions provide a sufficient response to the problem situation (Lesh & Lehrer, 2003). The MMP researchers argued that this modeling cycle would not be possible with traditional textbook math problems and therefore proposed a genre of modeling activities based on reality and meaningfulness principle (Lesh et al., 2000). For the RME researchers, the problems of progressive mathematization also involved a realistic aspect, with the help of which students mathematized the problem situation (Gravemeijer & Doorman, 1999).

Realistic and contextual mathematics problems are important not only for students' mathematics learning but also for teacher development (Brown, 2019; Sevinc, 2022; Sevinc & Lesh, 2021; Verschaffel et al., 2000). Therefore, understanding teachers' and prospective teachers' conceptions of realistic and contextual mathematics problems took researchers' attention (e.g., Bonotto, 2007; Gainsburg, 2008; Leavy & Hourigan, 2020; Lee, 2012; Sevinc & Lesh, 2021). Some researchers were particularly interested in teachers' conceptions of modeling activities (e.g., Greefrath et al., 2022; Shahbari, 2018). The modeling perspective also merits professional development for improving teachers' noticing of students' mathematical thinking (Bas-Ader et al., 2021) and their competencies in writing realistic mathematics problems (Sevinc & Lesh, 2018).

Whilst the importance of teachers' conceptions and beliefs about realistic mathematics problems, teacher education research also pointed out the construct of teacher knowledge. The literature presented a noteworthy role of the teacher knowledge in teachers' conceptions of the instructional tasks such as word problems (Chapman, 2013; Csíkos & Szitányi, 2020), and their problem-posing practices (Lee et al., 2018; Sevinc, 2022). Therefore, this study focused on the types of teacher knowledge that preservice teachers self-reported as accompanying the process of generating mathematically rich and contextually realistic problems and addressed the following research questions:

1. What types of teacher knowledge do preservice mathematics teachers self-report as recalled for the mathematically rich and contextually realistic problems?
2. How do preservice teachers depict the relation in their web of knowledge regarding the mathematical richness and contextual reality of the problems?

The web of knowledge referred in this manuscript indicated a (concept) map of different types of teacher knowledge, disclosing different relationships between different types of knowledge. Albeit limited to preservice teachers' self-reports, eliciting their web of knowledge is important because this would uncover the internal process of generating mathematically rich and contextually realistic problems, at least to a certain extent. To accomplish this purpose and investigate these research questions, we had a theoretical frame around the teacher knowledge categorization and a methodological frame of the multi-tier modeling research, explained in the following sections.

THEORETICAL FRAMEWORK

Our focus construct in this study is the teacher knowledge, not assessed but self-reported by the preservice teachers. Therefore, teacher knowledge categorizations constituted main theoretical framework.

Teacher Knowledge Categorizations

Over the last half-century, understanding what teachers need to know has been one of the most important concerns of teacher education researchers (Cross & Lepareur, 2015; Greefrath et al., 2022; Hill et al., 2004; Park & Oliver, 2008; Shulman, 1986, 1987). Grossman and Richert (1988) defined teacher knowledge as "a body of professional knowledge that encompasses both knowledge of general pedagogical principles and skills and knowledge of the subject matter to be taught" (p. 54). This body of knowledge has different characteristics for different disciplines and for the same subject at different grade levels. For instance, Hill et al. (2004) argued that teacher knowledge for teaching mathematics was multidimensional in terms of the variation in mathematical topics (e.g., geometry, algebra, fractions) and variations in zones of interest (e.g., content, context, learners, and teaching).

Teacher education research produced various frameworks to understand the nature of knowledge that teachers possess for effective instruction. One of the earliest frameworks was Shulman's categorization of teachers' content knowledge: subject matter content knowledge, pedagogical content knowledge (PCK), and curricular content knowledge (1986). Following his work, Grossman (1990) identified four aspects of the PCK:

1. Knowledge and beliefs about the purposes for teaching a subject at different grade levels.
2. Knowledge of students' understanding, conceptions, and misconceptions of particular topics in a subject matter.
3. Knowledge of curriculum materials available for teaching particular subject matter, as well as knowledge about both horizontal and vertical curricula for a subject.
4. Knowledge of instructional strategies and representations for teaching particular topics (pp. 8-9).

These four aspects indicated that teachers need knowledge of content, curriculum, pedagogy, and context for teaching a particular topic of a particular subject matter at a particular grade level. In another study, Tamir (1991) identified four components of teachers' professional knowledge: subject matter knowledge, general pedagogical knowledge, subject matter specific pedagogical knowledge, and teacher education pedagogical knowledge. Among these, subject matter specific pedagogical knowledge and teacher education pedagogical knowledge could be seen as revisions of the PCK.

Taking a specific subject matter (i.e., mathematics) perspective, Simon (1997) identified the following eight facets of teacher knowledge for teaching mathematics:

1. *Knowledge of and about mathematics* … [that] defines what is worth learning and the activities that are appropriate to the enterprise;
2. *A personally meaningful model of mathematics learning* … [that includes] teacher's concepts of how mathematical knowledge is developed;
3. *Knowledge of students' development of relevant concepts* … [that is related to] the teacher's ability to make sense of the students' mathematics;
4. *Relationship to students' mathematics* … [that] encompasses commitment to understanding his or her students' mathematics, ability to elicit and probe their mathematics, and ability to analyze students' mathematical activity and form useful models of their knowledge;
5. *A personally meaningful model of mathematics teaching* … [that] is itself recursively embedded … [that] guides and constrains instructional decision making and defines the teacher's role in relation to students' learning … [and that] is constantly in the state of construction and renovation;
6. *Ability to define appropriate learning goals for students* … [and] longer range goals as well as goals for spontaneous interventions [that requires] the ability to identify key mathematical ideas in the mathematics being considered, as well as to make use of knowledge of students' concept development and knowledge of that teacher's own students' mathematics;
7. *Ability to anticipate how students' learning might ensue* … [which suggests that] development is necessary in the teacher's ability to modify the hypothetical learning trajectory in response to new insights; and
8. *Ability to construct lessons consistent with one's model of teaching* … [which] means constituting new forms of practice both by adapting new techniques and by rethinking the role of familiar ones (p. 81-82).

These aspects of mathematics teacher knowledge highlighted that mathematics teachers need to possess professional knowledge (i.e., knowledge of mathematics, mathematics learning, and mathematics teaching) and personal knowledge (i.e., knowledge of students' mathematics and knowledge about how mathematical ideas are evolved).

Similarly, from a mathematics teaching perspective but adding an international lens, Ma (1999) compared U.S. and Chinese elementary teachers' subject matter knowledge. She proposed "profound understanding of fundamental mathematics" as a new category of teacher knowledge: "By profound understanding I mean an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough. Although the term *profound* is often considered to mean intellectual depth, its three connotations, *deep*, *vast*, and *thorough*, are interconnected" (p. 120). Lastly, Ball et al. (2008) proposed a framework of Mathematical Knowledge for

Teaching (MKT). The fundamental premise of MKT is that being able to use mathematics content knowledge in teaching is more crucial than just knowing the mathematics (Hill et al., 2008):

> By "mathematical knowledge for teaching," we mean not only the mathematical knowledge common to individuals working in diverse professions, but also the subject matter knowledge that supports that teaching, for example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and the types of errors students are likely to make with particular content. By "mathematical quality of instruction" we mean a composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson, including the presence or absence of mathematical errors, mathematical explanation and justification, mathematical representation, and related observables (p. 431).

MKT posits two types of content knowledge: *specialized knowledge of content* and *common knowledge of content*, derived from Shulman's subject matter content knowledge (Ball et al., 2008). Hill and Ball (2004) illustrated specialized knowledge of content with the following teacher behaviors: (i) explaining concepts, procedures, and algorithms and why they work, (ii) understanding students' solution methods for a problem, (iii) appreciating students' using novel methods, and (iv) determining whether students' methods are generalizable to other problems. These behaviors characterize knowledge specific to mathematics teachers who engage with students while teaching. Common knowledge of content, on the other hand, is not unique to the individual who is teaching mathematics and includes being able to solve mathematics problems correctly, recognizing correct and incorrect solutions to a problem, and using mathematical language and notations precisely (Hill & Ball, 2004). Other elements of MKT are *knowledge of content and students* and *knowledge of content and teaching* (Ball et al., 2008), which unfold Shulman's PCK. Those two types of knowledge are as important as the types of content knowledge because the work of teaching requires both specific mathematical understanding and an understanding of pedagogical issues affecting students' learning and their interactions with each other (Ball et al., 2008).

As seen in the abovementioned research, Shulman's categorization of teacher knowledge launched a research area in teacher education and constituted a theoretical framework in a vast amount of mathematics education research (e. g., Cross & Lepareur, 2015; Even, 1993; Greefrath et al., 2022; Park & Oliver, 2008; Rossouw & Smith, 1998). Some of those studies addressed the relationship between subject matter content knowledge and the PCK (e. g., Agathangelou & Charalambous, 2021; Even, 1993; Kind & Chan, 2019; Norton, 2019), and some studied the role of instruction in teachers' development of the PCK (e.g., Cross & Lepareur, 2015; Rossouw & Smith,1998). Some studies also investigated the role of reflection in eliciting and consolidating the PCK (e.g., Park & Oliver, 2008). As seen, those studies and others revealed different aspects of the PCK and highlighted the role of the PCK in effective mathematics instruction; therefore, Shulman's categorization deserves a closer look to comprehend the teacher knowledge framework of this study.

Shulman's Categorization of Content Knowledge

In the mid-1980s, Shulman proposed a teacher knowledge categorization based on a subject matter understanding: "He or she, *the teacher* [italics added], must understand the structures of subject matter, the principles of conceptual organization, and the principles of inquiry" (1987, p. 9).

As mentioned earlier, his categorization suggested three main types of content knowledge: (1) subject matter content knowledge, (2) pedagogical content knowledge, and (3) curricular knowledge. Shulman (1986) argued that knowing the subject matter should not be reduced to knowing a collection of facts, rules, or concepts but rather include understanding the structures of the subject matter and deep reasoning about the concepts and procedures. The second knowledge type was his widely known notion of the PCK, which has been extensively studied since then. This notion extended the knowledge beyond the possession of subject matter knowledge to "how subject matter was transformed from the knowledge of the teacher into the content of instruction" (Shulman, 1986, p. 6). To identify what the PCK covers, Shulman (1986) stated:

> Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful

> analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others (p. 9).

The third knowledge type in this categorization, curricular knowledge, included the sequence of the topics in the curriculum and various instructional strategies and tools for different grade levels. The curricular knowledge encompassed not only the sequence of topics in the curriculum of a subject area (e.g., mathematics) across grade levels but also the relationship between the curricula of different subjects (e.g., mathematics and science).

Not only the PCK particularly took researchers' attention (e.g., Cross & Lepareur, 2015; Park & Oliver, 2008; Rossouw & Smith,1998) but also the relationship between the content knowledge and the PCK has been explored (Agathangelou & Charalambous, 2021; Even, 1993; Kind & Chan, 2019; Norton, 2019). Those studies demonstrated how these core knowledge types of teachers feed each other. In this sense, Agathangelou and Charalambous (2021) argued that the content knowledge was observed as pre-requisite of the PCK albeit it was found to indicate more complicated nature than the content knowledge. Kind and Chan (2019) claimed that while the PCK rests on the content knowledge, it has more direct pedagogical implications and stressed that "providing teachers with support to create PCK from baseline knowledge and facilitating its deployment in a teacher's classroom to ensure quality instruction and positively impact student learning outcomes seems essential" (p. 975). Confirming the interaction between content knowledge and the PCK, Norton (2019) suggested teacher education courses addressing both knowledge "in tandem" rather than independently in mathematics content courses and mathematics teaching methods courses. Furthermore, researchers asserted that both the knowledge of content and the PCK played role in teachers and/or preservice teachers' problem posing (Lee et al., 2018; Sevinc, 2022) and teaching word problem-solving (Csíkos & Szitányi, 2020). In other words, developing an appropriate and qualified problem and addressing word problem solving strategies in the instruction recalled activation of the content understandings and integrating the content for particular pedagogical purposes. Similarly content knowledge and the PCK constitute a foundation for word problems (Csíkos & Szitányi, 2020) and modeling problems involving realistic context (Greefrath et al., 2022; Sevinc, 2022).

As seen, Shulman's categorization of teacher knowledge opened the door for abovementioned investigations in mathematics education and other education fields such as science education (e.g., Carlson & Daehler, 2019; Kind & Chan, 2019). Therefore, in this study, we considered Shulman's knowledge categorization as a theoretical frame for analyzing the preservice teachers' reports on their knowledge recalled for mathematically rich and contextually realistic problems. Although our investigation is in the scope of mathematics education, it provides a research design on how Shulman's knowledge categorization framework can be utilized in other education fields or in different topics of mathematics where the teacher knowledge is at the center of the investigation. With this research, we aimed to understand what types of teacher knowledge the preservice teachers self-reported and what kind of relation between different knowledge types they depicted. Therefore, the pre-service mathematics teachers were asked to create webs of knowledge, but not based on predetermined relation between knowledge types such as the ones mentioned above (e.g., content knowledge being pre-requisite of the PCK). In this respect, this study presents a potential contribution to the literature by eliciting different relationships between teacher knowledge types based on the preservice teachers' self-reports in the context of mathematically rich and contextually realistic problems.

METHODS

This study presents a smaller portion of a larger dissertation study[1] that was designed as a multi-tier design research (Koellner-Clark & Lesh,, 2003). Multi-tier design research is derived from multi-tier professional development and posits that there are tiers of modeling experiences. Each tier includes

[1] We want to note that although some data and pieces of texts included in this manuscript might have been found also in the larger dissertation study (see Sevis, 2016), we re-organized and re-articulated the theoretical frame and discussion of the findings considering the results presented here in this study. Hence, this manuscript was produced from a larger dissertation study.

individuals developing models through a series of modeling cycles in which the ideas are expressed, tested, revised, or refined (Chamberlin, 2005); hereafter, it is called multi-tier modeling research.

Since the modeling perspective also highlighted the nature of mathematics problems that involve realistic and meaningful context and target mathematically important ideas (Lesh & Doerr, 2003; Lesh & Lehrer, 2003), multi-tier modeling research constituted our methodological frame on which we designed the data collection and analysis processes. The tiers involved in the larger study were Tier 1 – Preservice teachers, Tier 2 – A group of researchers, and Tier 3 – The principal investigator. In this study, we only presented the analysis of a portion of the data collected in the preservice teachers' tier due to two main reasons: (1) this portion of data collected in Tier 1 involved pre-service teachers' self-reports on the knowledge types whereas the data from Tiers 2 and 3 were the researchers' articulations, and (2) the webs of knowledge developed by the pre-service teachers that this study focused on were collected in Tier 1.

Participants

As mentioned earlier, multi-tier design modeling research rested on a professional development that took place at a large public university in Turkey. The participants of the professional development involved junior and senior preservice mathematics teachers enrolled in a mathematics teaching methods course and a field-experience course in two subsequent semesters. The professional development sessions were integrated into the last two weeks of these two courses.

There were 34 preservice teachers (31 female and three male) enrolled in the first course and six preservice teachers (five female and one male) in the second course. The participant preservice teachers were at the age of 21–22 and, after their graduation, would be granted to teach mathematics in middle schools (i.e., grades 5-8).

Although the professional development sessions were incorporated into two teacher education courses, course objectives were not directly related to mathematically rich and contextually realistic problems. Still, they improved preservice teachers' understanding of various mathematical topics and their pedagogical insights. Specifically, in the mathematics teaching methods course, the preservice teachers explored various elementary (1st-4th grades) and middle school (5th-8th grades) mathematics concepts, learned the ideas behind the standard algorithms, rules and formulas, and gained insight into teaching particular mathematical topics. In the field experience course, they were placed in field experience schools for twelve-week teaching practice.

Data Collection and Analysis Procedures

The larger study was designed as multi-tier modeling research involving three tiers. The modeling perspective guided the data collection and data analysis of these tiers as a methodological framework. Hence, the data collection and analysis processes were intertwined within each tier and across the tiers. A summary of the data collection and analysis procedure is given in **Figure 1**.

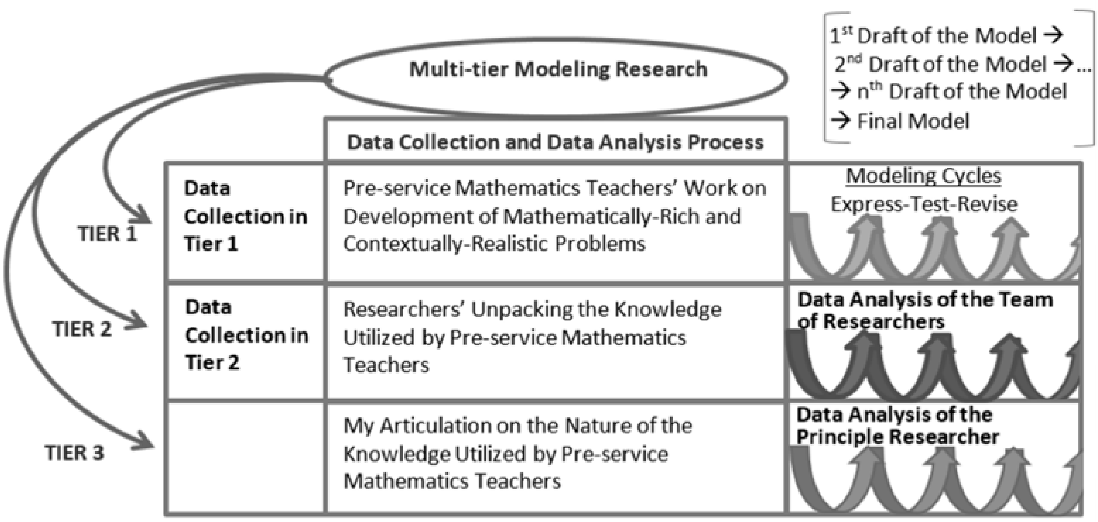

Figure 1. Summary of data collection and data analysis processes in three tiers

In Tier 1, the preservice teachers worked in small groups on developing mathematically rich and contextually realistic problems. There were 11 groups in total, and each group worked on problem production and identified the characteristics of such problems. The preservice teachers were also given opportunities to record their work (i.e., the characteristics of mathematically rich and contextually realistic problems) and reflect on the knowledge that they recalled during this process. In other words, as they wrote mathematically rich and contextually realistic problems, they needed to be awake for the types of knowledge activated in this process, which was not easy to do. Therefore, they were asked to write individual reflection papers as a response to some guiding questions (see Appendix A for the guiding questions of the reflections). In addition, we provided a set of graphic organizers, each of which indicated a different relationship, and asked the preservice teachers to produce a web of knowledge representing the knowledge types recalled for writing mathematically rich and contextually realistic problems (see Appendix B for the graphic organizers and the relationship each indicated). Those graphic organizers were selected from SmartArt Graphics of Microsoft Office PowerPoint 2013, and the relationships that they posit were identified by the group of four researchers worked in Tier 2 of the larger study. Hence, the classification of the relationships such as categorical, hierarchical, and continuous was not theory-driven, rather they were identified and validated by a research team. In addition to those written data, preservice teachers' small group discussions and interviews that were conducted with each group at the end of each course (i.e., each semester) were audio-recorded. The semi-structured group interview questions were similar to the guiding questions of the individual reflections with additional questions regarding the web of knowledge they developed such as "Why did you choose this graphic organizer?" and "Can you please describe your map of knowledge?" Hence, the data set included both written data (i.e., tables of characteristics of the problems, web of knowledge with graphic organizers, and individual reflection papers) and audio data (i.e., group discussion and group interviews) that were then transcribed for the data analysis.

The written and audio data transcriptions were analyzed in three coding cycles suggested by the constructivist grounded theory (Charmaz, 2006). Using a qualitative data analysis software called MAXQDA, we carried out the three coding cycles: (1) initial coding, (2) focused coding, and (3) theoretical coding (Charmaz, 2006; Thornberg & Charmaz, 2012). While the initial coding involved incident-by-incident coding, the focused coding entailed synthesizing and conceptualizing larger segments of data (Thornberg & Charmaz, 2012). In the theoretical coding cycle, we aimed to relate the codes and categories, which resulted in the web of knowledge including the types of teacher knowledge recalled for mathematically rich and contextually realistic problems. We also employed a constant comparative method for making systematic comparisons of data with data, data with codes, codes with codes, and codes with categories to refine the properties of each theoretical category (Thornberg & Charmaz, 2012). In contrast to the classical grounded theory, constructivist grounded theory acknowledges the researchers' role in developing the theoretical codes and theoretical relations (Charmaz, 2006). Therefore, during the entire research process, the researchers kept a research journal including analytical memos, which also help to assure trustworthiness of the study (Lincoln & Guba, 1985).

As mentioned earlier, a group of researchers in Tier 2 worked as a research team in analyzing preservice teachers' data. More specifically, guiding questions for the individual reflection papers, interview questions, and graphic organizers utilized in creating webs of knowledge were checked by the research team for validation of the data collection tools. In the coding process, the open codes in cycle 1 and focused codes in cycle 2 were triangulated by the research team to ensure the trustworthiness of the results. The theoretical coding in cycle 3 was conducted by the first author and triangulated by the second author. Furthermore, the findings were triangulated by different researchers in the research team and by different types of data (i.e., written data versus audio interview data and individual reflection data versus group discussions/interview data) (Lincoln & Guba, 1985).

RESULTS

The preservice mathematics teachers worked in small groups to write mathematically rich and contextually realistic problems. As they generated the problems, they also recorded the aspects that provoked mathematical richness and contextual realness of the problems. Although the problems they developed were also important, they were outside the scope of this article. Here in this study, we presented preservice teachers' self-reported web of knowledge accompanying this process.

To make sense of the knowledge types included in preservice teachers' web of knowledge, we first provided a summary of the characteristics of mathematically rich and contextually realistic problems from their perspectives and then presented preservice teachers' web of knowledge in relation to these characteristics. We also want to note that due to the space limitation, all preservice teachers' work could not be provided but selected the ones that would represent the variety in all preservice teachers' web of knowledge, and pseudonyms were used to refer the preservice teachers whose exemplifying work were presented in the findings below.

Characteristics of Mathematically Rich and Contextually Realistic Problems

Among the groups of preservice teachers, we observed a tendency of starting to write a problem after determining a general math idea such as geometry, proportional reasoning, probability, or statistics, rather than thinking about specific mathematical big ideas. While revising and refining their problems, some preservice teachers' thinking has changed. For instance, Filiz from Group 11 stated that those problems should be different than standard textbook problems and, therefore, should support conceptual mathematical thinking more than other problems do:

> These problems should not be procedure-oriented but concept-oriented. In other words, they should require mathematical reasoning, not just an application of a procedure or algorithm, and so intend to be higher level of cognitive demands. ... For our problems, we should have a clear answer to the question of "what are we measuring with this problem?" or "what mathematical concept are we aiming to have our students develop?"

While such comments indicated that their thinking was changing, they still had difficulty identifying big ideas within a mathematics topic. One of the reasons for this difficulty could be balancing context-reality and mathematical-richness. Ayla's statement below indicates the difficulty of maintaining this balance between context and content:

> Mathematical ideas should be our priority to consider while writing our problems. The realistic nature of the context should not mask the mathematical content of the problem... It should not be just a story. Students should recognize that it is a mathematics problem.

Although the preservice teachers had different perspectives on whether a story would be helpful in writing contextually realistic mathematics problems, some thought that the stories were welcoming in the sense that they invited students to engage in and motivate them to solve.

As seen above, the preservice teachers focused on two aspects of the problem: mathematical richness and contextual reality. They recorded the characteristics that they thought as supporting these two aspects. Group 1 produced the list of characteristics[2] shown in **Figure 2**.

[2] Some of the characteristics were already written in English, but some were in Turkish and translated by first author.

For Being Mathematically-Rich		For Being Contextually-Realistic	
M1	It must be challenging.	R1	1. The problem should be related with other lessons, it makes sense for students.
M2	It shouldn't be a drill. It requires students to think even deeply and different ways to solve question when confront.	R2	*We may consider local needs and differences.*
M3	It should be enjoyable. When student is solve the question s/he should enjoy even if when s/he stucks with the problem, due to the enjoyable effect of it, s/he insist on solve it.	R3	*We may use everyday life situations such as shopping.*
M4	The solution of the problem should be curiously. (merak edilen)	R4	*We may let students explore the math ideas using what they observe, such as proportion in body parts.*
M5	The problem should be placed immediately after the subject told. Instead of this, while students start to solve problem, they should think which subject the problem is about/related.	R5	*We may let students use the tools around them.*
M6	*The numbers involved in the problems should be carefully.*	R6	*We may let students collect data around them, particularly in statistics or probability.*
M7	*There could be multiple mathematical ideas involved in a single problem.*	R7	

Figure 2. Group 1's list of characteristics for mathematical richness and contextual reality

The list of characteristics indicated that, for this group of preservice teachers, mathematical richness was related to the problem's involving various mathematical ideas and different subjects. Therefore, preservice teachers thought that such problems would be challenging, not straightforward. The contextual aspect, on the other hand, was related to connections with other lessons, with everyday life situations, or with cultural/local needs. So, this group of preservice teachers considered that if the problem involved a closer context to students' lives, students would find it more realistic. Other groups also had similar concerns. For instance, Group 11 produced the following list of characteristics (**Figure 3**).

As seen in **Figure 3**, Group 11 also thought that, in solving such problems, students would experience a challenge, and this challenge might have been produced by not signaling a particular mathematical procedure with particular words used in the problem text (e.g., the word "sum" signals addition operation). Moreover, for this group, mathematical richness was derived from the links that was set based on students' prior knowledge. Preservice teachers also thought that mathematically rich problems would allow students to develop more than one solution method and different ways of modeling the problem situation. The contextual reality aspect of the problem was associated with interesting, motivating, and meaningful situations, which was a common feature across the groups of preservice teachers' lists. In addition, preservice teachers in Group 11 emphasized everyday life situations that students would find realistic. The other items in their list indicated the potential role of realistic context in mathematics education.

Although we presented two groups' lists of characteristics here, the other nine groups also addressed similar concerns about mathematically rich and contextually realistic problems. In short, preservice teachers identified the problem's involving more than one mathematical idea, multiple solutions, challenging content, and connections with prior knowledge as indicators of a problem's being mathematically rich. They also highlighted everyday life practices and closeness of the contexts to students' lives regarding local and cultural appropriateness and needs as indicators of a problem's being contextually realistic. While preservice teachers worked on developing mathematically rich and contextually realistic problems, they were also asked to trace what types of knowledge they recalled in this process, which are presented next.

For Being Mathematically-Rich		For Being Contextually-Realistic	
M1	Being challenging	R1	Interesting and related to everyday life
M2	Including four operations	R2	Being meaningful
M3	Developing mathematically thinking	R3	Feeling a need for how to use it in real-life
M4	Providing relational thinking	R4	Need to be related and appropriate to local lifestyles
M5	Also using [illegible]	R5	Real-life connections improve positive attitude toward math
M6	Including prerequisite knowledge	R6	Real-life connections help students to make sense of abstract math ideas
M7	-Having alternative solutions	R7	Real-life connections take students' interests and increase motivation
M8	Solving by using modeling	R8	Real-life connections may improve students' understanding of math
M9		R9	More than one math topics can be related to each other

Figure 3. Group 11's list of characteristics for mathematical richness and contextual reality

Web of Teacher Knowledge Recalled for Mathematically Rich and Contextually Realistic Problems

During the small group discussions and group interviews at the end of each semester, the preservice mathematics teachers were asked to introspectively and retrospectively articulate their web of knowledge recalled for mathematically rich and contextually realistic problems. From the examination of 11 groups of preservice mathematics teachers' lists of characteristics and their web of knowledge five common types of knowledge were identified:

(1) Mathematical content knowledge,

(2) Pedagogical knowledge,

(3) Knowledge of students,

(4) Curriculum knowledge, and

(5) Real-life knowledge.

Those types of knowledge were related to the characteristics that the preservice teachers identified for mathematical richness and contextual reality of the problems. To understand how they constructed their web of knowledge, we analyzed their self-reports in their individual reflection papers, group discussions, and interviews. The sample web of knowledge provided here are representative and illuminating for those of other preservice teachers.

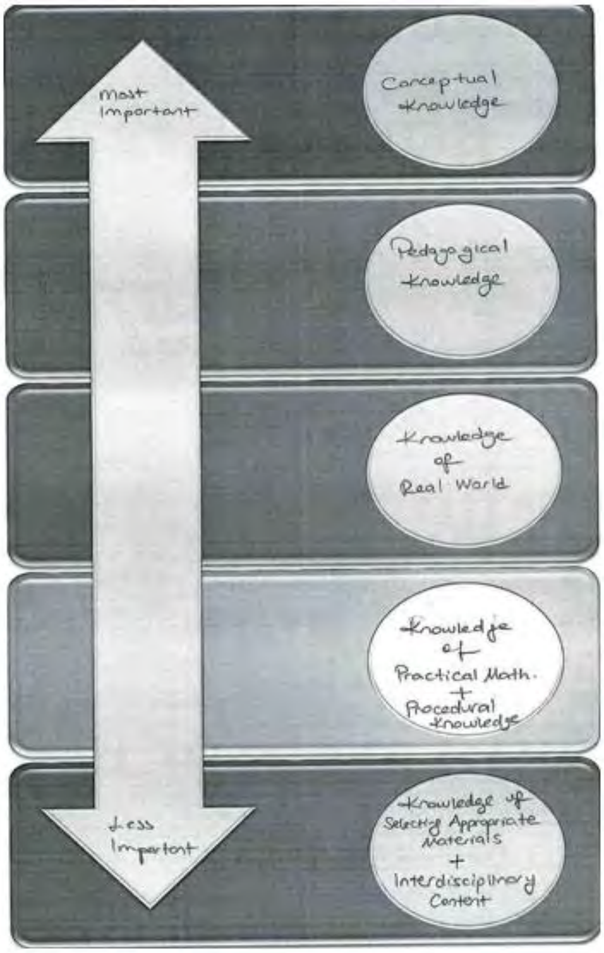

Figure 4. Filiz's web of knowledge recalled for mathematically rich and contextually realistic problems

Knowledge webs indicating hierarchical relation between knowledge types

Filiz, from Group 11, selected the graphic organizer shown in **Figure 4** (see **Appendix B** for GO #7) that the research team classified as presenting a categorical relationship. Filiz used it in a vertical orientation and demonstrated a hierarchical relation as well as a categorical relation between knowledge types.

Filiz stated that she selected this graphic organizer because it implied a hierarchy to her, which was how she thought her web of knowledge was. The hierarchy that she set up was based on levels of importance, which she explained as follows:

> The first [top] one is conceptual content knowledge because it is the most important one. We had difficulty in writing a good mathematically rich and contextually realistic problem because we lacked conceptual knowledge, we could not connect the arithmetic mean with other mathematical ideas. The next one is pedagogical knowledge. Even though we know the content, it won't work unless we make it understandable at the students' level. Then real-world knowledge, then practical mathematics knowledge for which students could invent their own strategies and which did not restrict or limit them to standard algorithms. Procedural knowledge is also as important as practical mathematics knowledge. Then, knowledge of selecting appropriate materials and interdisciplinary content come to play.

As seen, Filiz emphasized knowledge of content, knowledge of students, and knowledge of real-life as central knowledge types in her web of knowledge. Filiz additionally pointed out that conceptual content knowledge and procedural knowledge were not at the same level of importance. Filiz's web of knowledge also indicated that knowledge of selecting appropriate materials and interdisciplinary knowledge played significant role in writing a mathematically rich and contextually realistic problem. Furthermore, Filiz's web of

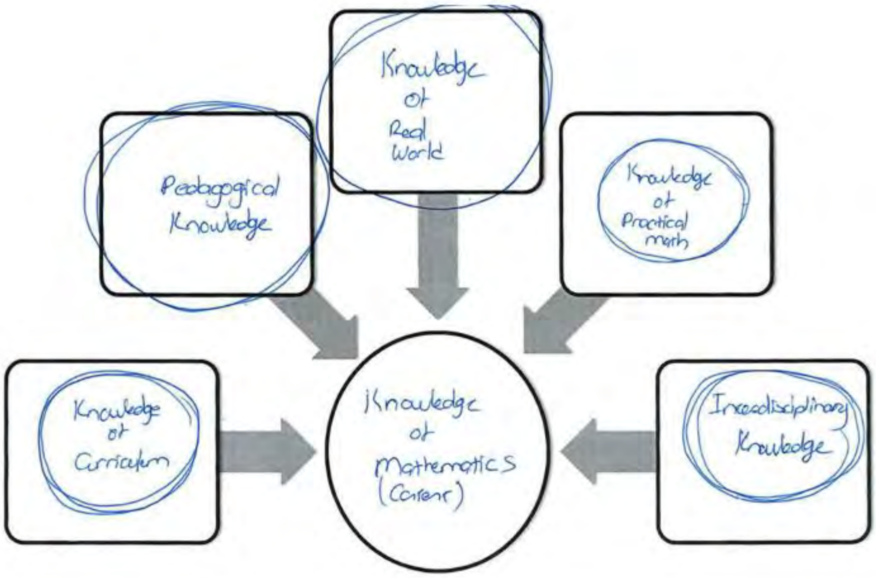

Figure 5. Cigdem's web of knowledge recalled for mathematically rich and contextually realistic problems

knowledge encompassed most of the knowledge types that her groupmates, Doga, and Oya, also identified in their webs.

Knowledge webs indicating categorical and influential relations between knowledge types

Cigdem from Group 9 selected a graphic organizer that the research team classified as presenting categorical and causal relationships (see **Appendix B** for GO #15). Like Filiz, Cigdem also expressed that she saw a different relationship than a causal one. She stated that there were categories, but the organization of the categories allowed her to show the central knowledge type and peripheral knowledge types (see **Figure 5**).

As seen in her web of knowledge, she identified the knowledge of mathematics (content) as the central knowledge type with five peripheral knowledge: knowledge of curriculum, pedagogy, real-world, practical mathematics, and interdisciplinary subjects. During the interview, Cigdem explained that the knowledge of mathematics at the center of her web of knowledge had two aspects: (1) her own knowledge of mathematics as a teacher and (2) her knowledge of students' mathematics. She further stated: "Without considering students' mathematics knowledge, it was impossible to write a mathematically challenging problem." Moreover, Cigdem drew circles inside the rectangles that surrounded the circle in the center. She explained that the size of these circles represented the level of importance; a bigger circle represented a more important knowledge type. When we asked her how this web of knowledge worked out, she stated,

> All these knowledge types at the periphery are connected to the knowledge of content. First, you need to connect your mathematics knowledge with real-world knowledge. You need to look for the ways in which your problem will allow students to make a connection between their mathematics knowledge and the real world. Secondly, I used my pedagogical knowledge to anticipate students' solution methods. Thirdly, my knowledge of curriculum helped me to create the problem which would be based on their prerequisite knowledge. Fourth, practical mathematics knowledge helped me think about possible solution strategies. The fifth was interdisciplinary knowledge, which was behind the scene but helped me to find appropriate context.

As seen, Cigdem mentioned the knowledge types similar to the ones that Filiz and most of the other preservice teachers also identified. However, different from Filiz, who organized her web of knowledge hierarchically based on the level of importance, Cigdem considered both the level of importance and the connections between them.

Another graphic organizer that the research team classified as categorical (see **Figure 6**) was selected by Tuna from Group 1.

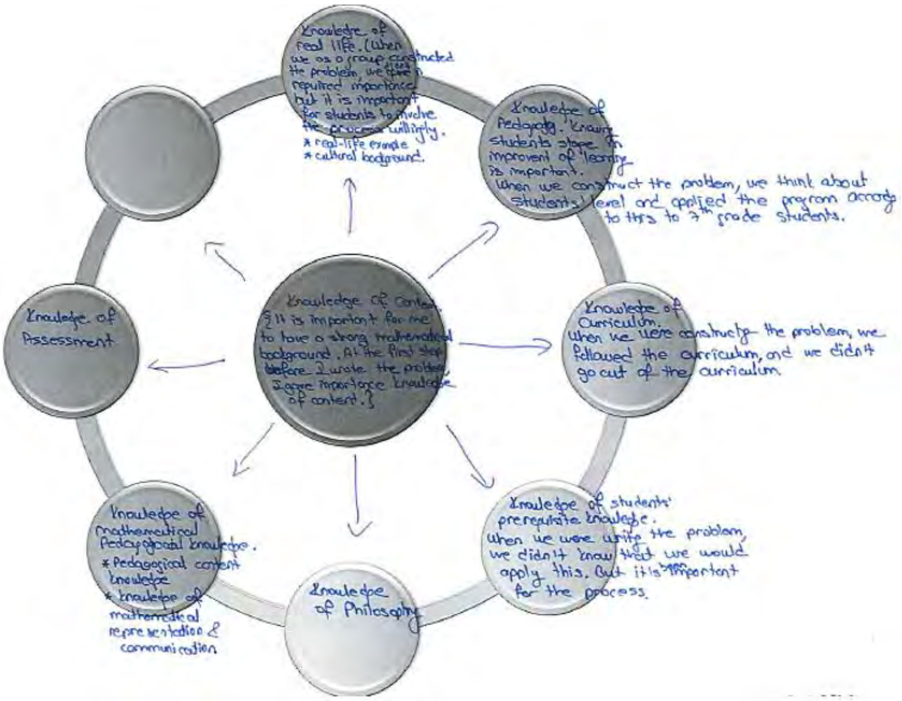

Figure 6. Tuna's web of knowledge recalled for mathematically rich and contextually realistic problems

As seen in **Figure 6**, Tuna also put the knowledge of content at the center of her web of knowledge and considered knowledge of pedagogy, curriculum, and rea-life as connected with the content knowledge for creating mathematically rich and contextually realistic problems. In this sense, Cigdem's and Tuna's web of knowledge were similar. However, the direction of the arrows was reverse; while the arrows were from center to periphery in Tuna's web, they were toward the center in Cigdem's web. In fact, there was no arrow given in the graphic organizer that Tuna selected, but he drew those arrows particularly in that direction to indicate the relationship between the content knowledge and other knowledge types. When asked him, he explained as follows:

> Knowledge of content affects all others. For example, if we do not know mathematics, we won't be able to understand the mathematics school curriculum, so knowledge of curriculum depends on knowledge of content. Others are the same: if we do not know the content, we cannot know how to assess it, or we won't be able to determine students' prerequisite knowledge to learn something new in math. That's why I drew the arrows like this.

Like many other preservice teachers, Tuna and Cigdem placed mathematical content knowledge and real-life knowledge as the knowledge types interconnected in their webs of knowledge. Furthermore, while knowledge of students' levels, dispositions, prior knowledge, and possible conceptions are included under the knowledge of pedagogy in Cigdem's web, Tuna distinguished knowledge of students' prior knowledge from knowledge of pedagogy. When we asked him how he distinguished them, he stated: "They are related, but I wanted to write students' prerequisite knowledge separately because it is important to consider when you write a mathematically rich and contextually realistic problem."

Different from Tuna's web of knowledge, Cigdem – and also Filiz – included interdisciplinary knowledge in their webs of knowledge in addition to the knowledge of real-life. Lastly, both Tuna and Cigdem's web of knowledge included curriculum knowledge. In this regard, Tuna stated: "We cannot ask students a problem that is beyond the curriculum of their grade level, so we need to consider the curriculum." On the other hand, Cigdem stated: "We need to learn what students learned previously." As seen, how curriculum knowledge functioned in the construction of mathematically rich and contextually realistic problems was different in Tuna's and Cigdem's web of knowledge. In Tuna's web, it functioned as determining the border of mathematical content in the problem whereas, in Cigdem's web, it helped to make related mathematical ideas visible to ensure the mathematical richness of the problem.

Besides, Tuna's web of knowledge included two other knowledge types: knowledge of philosophy and mathematical pedagogical knowledge (i.e., pedagogical content knowledge). Regarding the former one, Tuna stated: "I meant the problem you write will be different if you're a constructivist teacher or a behaviorist teacher." Thus, with this knowledge type, he addressed the teaching and learning philosophies the teacher might hold. For the latter one, he wrote pedagogical content knowledge but specifically for mathematics, and therefore, he called it "mathematical pedagogical knowledge" that involved the knowledge of mathematical language (i.e., communication) and mathematical representations. When he was particularly asked about this type of knowledge, he stated: "I need to decide which mathematical representation, table or graph, will be useful and appropriate in the mathematically rich and contextually realistic problem I wrote." So, he implied that knowing the ways of mathematical communication was a part of the pedagogical content knowledge recalled for mathematically rich and contextually realistic problems.

Knowledge webs indicating causal and holistic relations between knowledge types

The webs of knowledge presented above indicated certain relationships and implied a working system. Nevertheless, there were some preservice teachers selected a graphic organizer (see **Appendix B** for GO #4) specifically to show that all knowledge types work together in a mechanism to create a mathematically rich and contextually realistic problem. In this graphic organizer, smaller and larger gears were turning toward a specific direction, and they all made each other work; that is, the whole system would stop without one. Among the preservice teachers who selected this graphic organizer, we presented Akin's web (from Group 3) and Ruya's web (from Group 6), respectively, in **Figure 7** and **Figure 8**.

In both preservice teachers' webs of knowledge, three core knowledge types were distinguishable: knowledge of content, knowledge of curriculum, and knowledge of real life. The knowledge types in those webs of knowledge were not, in fact, different from the abovementioned ones. Furthermore, similar to Cigdem and Filiz, Akin and Ruya included interdisciplinary knowledge connected with the content knowledge. Unlike others, Akin specified culture-dependent content knowledge as connected with real-life knowledge. In the list of characteristics that preservice teachers identified, they had mentioned that the context needed to be culturally relevant to students' lives. Akin indicated this characteristic with this knowledge type:

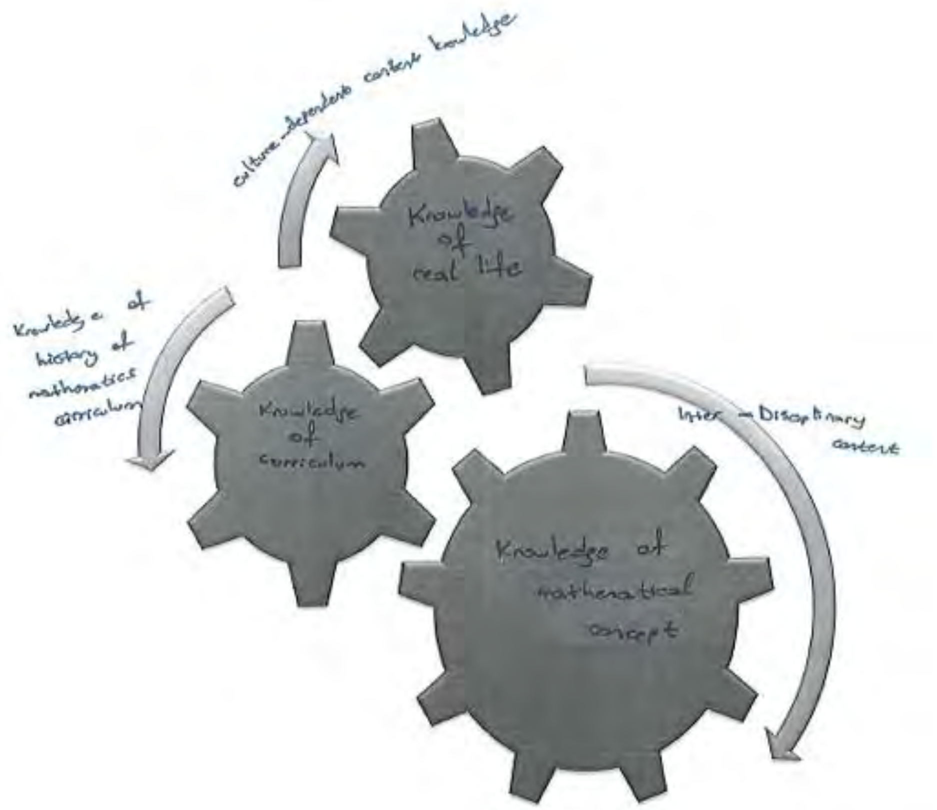

Figure 7. Akin's web of knowledge recalled for mathematically rich and contextually realistic problems

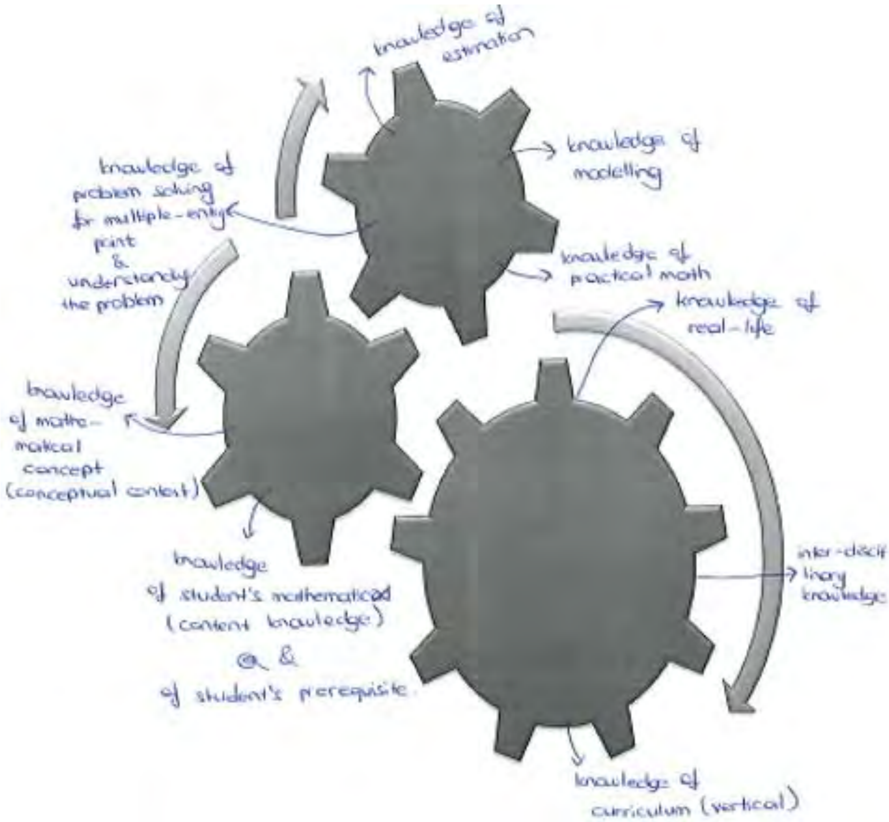

Figure 8. Ruya's web of knowledge recalled for mathematically rich and contextually realistic problems

> For instance, if we mention pizza in the problem, we need to be sure that students know what pizza looks like. For example, in some cultures, pita bread may be preferable to use in the problem instead of a pizza; it is because students are more familiar with it, and they can imagine its shape, etc.

Particularly, in Ruya's web of knowledge, the medium and large gears included knowledge of content, curriculum, real-life, and interdisciplinary content. Those knowledge types were placed in the larger gears because she considered them as more in the foreground while writing a mathematically rich and contextually realistic problem. She also specified students' prerequisite knowledge and curriculum knowledge as particularly important. The smallest gear in her web involved some aspects of the knowledge of pedagogy from her perspective:

> Knowledge of problem-solving, knowledge of modeling, and knowledge of strategies – this is what I mean by knowledge of practical math – are also useful to create a mathematically rich and contextually realistic problem. They are like knowledge of pedagogy altogether but pedagogy specific to mathematics problem solving.

Thus, Ruya listed "pedagogies of mathematics problem solving" that involved these specific types of knowledge.

Other preservice teachers' web of knowledge mostly included a combination of the knowledge types that were exemplified above and their hierarchical (i.e., level of importance or how often they were recalled), categorical, and influential relationship with each other. Hence, we observed that preservice teachers mainly recalled their content knowledge and curriculum knowledge to support the mathematical richness of the problems, and real-life knowledge and interdisciplinary knowledge to ensure contextually realistic aspect of the problems. There were other detailed knowledge types that accompanied this process, such as knowledge of students' prior knowledge, knowledge of problem-solving, or knowledge of representations, which were, in fact, a part of the PCK that was explicitly mentioned by Tuna among the given webs of knowledge.

CONCLUSION AND DISCUSSION

This study aimed to uncover preservice mathematics teachers' web of knowledge recalled in the process of generating mathematically rich and contextually realistic problems. While the first research question addressed the teacher knowledge types (i.e., What types of teacher knowledge do preservice mathematics teachers self-report as recalled for the mathematically rich and contextually realistic problems?), the second research question focused on the relations depicted by the knowledge webs created by the pre-service teachers (i.e., How do preservice teachers depict the relation in their web of knowledge regarding the mathematical richness and contextual reality of the problems?). The findings are discussed in this section under the subheadings associated with each research question.

Knowledge Recalled for Mathematically Rich and Contextually Realistic Problems

As presented above, the preservice teachers considered rich mathematical content knowledge – various mathematical ideas – and the connection with prior knowledge as core indicators of mathematical richness. These characteristics were also supported by other researchers (Hill & Ball, 2004; Shulman, 1986, 1987) and the findings of other studies (e.g., Sevinc & Lesh, 2021). Regarding the contextual reality aspect of the problems, preservice teachers emphasized the everyday life situations that students would find related to their own lives and could easily make sense. This consideration was parallel with the RME and MMP approaches in that both suggested students work on meaningful situations and mathematize the problem given in a real-life situation (Gravemeijer & Doorman, 1999; Lesh & Lehrer, 2003). Furthermore, the MMP researchers have emphasized that everyday life problems in the 21st century possess different characteristics that teachers need to be aware of (English & Sriraman, 2010; Stillman & Brown, 2011; Zawojewski et al., 2003). The preservice teachers also identified that mathematical richness and realistic context would challenge students and involve higher-level thinking, which corroborates Brown's suggestion that "researchers must acknowledge that such tasks [*tasks involving real-life context*] involve higher order thinking and are necessarily more challenging and demanding of learners." (2019, p. 76).

Aligned with the lists of characteristics, the preservice teachers contemplated about their webs of knowledge functioned in this process, which elicited the core teacher knowledge types. The findings indicated that mathematical richness of the problem was encircled by three core knowledge types proposed by Shulman (1986, 1987). Regarding the mathematical content knowledge, preservice teachers highlighted that mathematically rich and contextually realistic problems involved more than one mathematical big ideas. Therefore, knowing these mathematical big ideas and their connections with students' prior knowledge was one of the knowledge recalled by the preservice teachers. Teacher education researchers also had an affirming claim that teachers' content knowledge should be broad and deep (Agathangelou & Charalambous, 2021; Grossman, 1990; Hill et al., 2008; Ma, 1999; Shulman, 1986) so they could identify mathematical big ideas and peripheral ideas in mathematics problems (Sevinc, 2022). In addition, this type of knowledge was addressed by Simon (1997) as one of the eight facets of teacher knowledge for teaching mathematics. Specifically, he argued that identifying key mathematical ideas constituted a critical step in determining appropriate learning goals for students. Ma (1999) also addressed the connection between mathematical ideas as she set three connotations for a "profound understanding of fundamental mathematics." These three connotations were "deep, vast, and thorough" (p. 120) and achieved through the interconnection between mathematical concepts at the zone of a particular grade or various grade levels. In addition, for the mathematical richness of the problems, preservice teachers (e.g., Filiz, Cigdem, and Ruya) identified practical mathematics knowledge and expressed that it covered the knowledge of strategies. From their perspectives, this knowledge type was related to the content knowledge because the strategies they considered were related to the math content. The knowledge of strategies also supported the deepness of the mathematical content knowledge (Ma, 1999).

Another core knowledge type that enclosed the mathematical richness aspect was the knowledge of curriculum, stated by all preservice teachers. They reported that this knowledge was recalled for determining whether the problem was appropriate for the targeted grade level and whether the mathematics involved in the problem built on prior knowledge of the students. The third knowledge type was Shulman's well-known notion of the PCK. Although not all preservice teachers explicitly called the PCK in their knowledge webs, the specific knowledge types, such as knowledge of students' conceptions and prior math, knowledge of representations, and knowledge of modeling, signified the PCK. These specific types of knowledge were also consistent with Grossman's (1990, p. 8-9) "knowledge of students' understanding, conceptions, and misconceptions of particular topics in a subject matter" and "knowledge of instructional strategies and representations for teaching particular topics." Still, there were preservice teachers (e.g., Tuna) explicitly stated the PCK. Tuna also called this knowledge type "mathematical pedagogical knowledge" to indicate the pedagogies specific to mathematics. In this regard, Ruya mentioned pedagogies specific to mathematics problem solving. These preservice teachers' emphasis on the subject matter was consistent with Tamir's identification of "subject matter specific pedagogical knowledge" as a part of teachers' professional knowledge (Tamir, 1991). Hence, preservice teachers' web of self-reported knowledge involved the PCK.

Their inclusion of the PCK into their knowledge webs indicated that they recalled their PCK in identifying the mathematical richness and contextually realistic aspects of the problems and integrating those aspects in generating their problems. This was also consistent with the findings of the related studies in the literature which found the PCK important for problem posing skills of teachers (Lee et al., 2018; Sevinc, 2022), word problem solving (Csíkos & Szitányi, 2020), and modeling (Greefrath et al., 2022). Those studies revealed that word problems that especially involved realistic context call the particular aspects of the PCK (Greefrath et al., 2022; Sevinc, 2022). Some of those particular aspects included knowledge of students, knowledge of representations, knowledge of modeling, and knowledge of problem solving, as the preservice teachers self-reported in this study.

For the contextual reality aspect of the problem, preservice teachers recalled real-life knowledge to identify the situations that students might experience in their daily lives and so would find the context realistic. This finding suggests teachers observe real life situations with a mathematical focus to identify a context to bring into the classroom. In this regard, albeit indirectly, the present study supports Brown's claim about "ascertaining ways in which teachers should be aspiring to support learners in knowing more about the world in which they live and analysing how the real-world contexts support student learning of mathematics and maintaining the high cognitive demand of such tasks." (2019, p. 76) The preservice teachers also recalled the interdisciplinary knowledge to set up a context that was connected to mathematics. Interdisciplinary nature, in fact, matched the changing needs of 21st-century problems (English & Sriraman, 2010).

To summarize, preservice teachers' self-reported web of knowledge produced three core knowledge for ensuring mathematical richness and two kinds of connections for identifying realistic contexts. We acknowledge that those knowledge types were not novel but consolidated that the knowledge of content, curriculum, and pedagogy were core knowledge types in teachers' professional knowledge base.

Knowledge Webs Eliciting the Relations between Teacher Knowledge Types

The analysis of preservice teachers' webs of knowledge revealed how these core knowledge types functioned – from the preservice teachers' perspective – in creating a particular type of problem, mathematical rich and contextually realistic. The knowledge webs they constructed as representations of their internal process were in different shapes and indicated various relationships (i.e., hierarchical, categorical, influential, holistic, etc.) but included common types of knowledge.

Some preservice teachers considered a hierarchy between knowledge types in terms of the level of importance that they assigned. For instance, Filiz created her web of knowledge considering the conceptual knowledge and pedagogical knowledge more important than the knowledge of real life and interdisciplinary knowledge. This indicated that either she put more emphasis to mathematical richness aspect of the problem than contextually realistic aspect, or she struggled more on ensuring the mathematical richness of the problem and so paid more attention to those knowledge types. In both cases, this web of knowledge demonstrated that the mathematical richness and contextual realistic aspects were not balanced for the preservice teacher. This result was parallel to the findings of the study that traced preservice teachers' realistic problems and indicated how challenging it was to keep the balance between mathematics and realistic context (Sevinc & Lesh, 2018).

The present study also showed that all pre-service teachers placed the content knowledge at the center of their knowledge webs. This result was evident especially in the knowledge webs indicating an influential relation. As presented in the previous section, Tuna and Cigdem's knowledge webs presented an influential relation but involved opposite directions. While Tuna's web showed that knowledge of content affects all other knowledge types, Cigdem's web presented that other knowledge types are dependent on content knowledge. The studies investigating the relationship between knowledge types particularly focused on the relationship between the content knowledge and the PCK (e.g., Agathangelou & Charalambous, 2021; Grossman, 1990; Hill et al., 2008). Regarding the relation between these two knowledge types, Agathangelou and Charalambous (2021) stressed a pre-requisite relationship; that is, the content knowledge was considered as a pre-requite of the PCK. Both this study and the preservice teachers' webs of knowledge in the present study indicated how crucial the content knowledge is in teachers' knowledge base. In this regard, the preservice teachers' self-reports were parallel with the findings of the related studies in the literature (e.g., Agathangelou & Charalambous, 2021; Grossman, 1990; Hill et al., 2008; Sevinc, 2022). Another relationship that the preservice teachers' knowledge webs demonstrated was holistic relation. To demonstrate that all types of knowledge function together in a system and each cause another to work appropriately, some preservice teachers (e.g., Akin and Ruya) selected a gear model among the graphic organizers. The knowledge in the gear model functioned as a whole system and allowed preservice teachers to identify the appropriateness of the mathematically rich and contextually realistic problem.

The preservice teachers' depicting the PCK as one of the core knowledge types in their knowledge webs also supports the claim of Kind and Chan (2019) on PCK's pedagogical implications. In other words, identifying mathematical richness and contextual realistic characteristics of the problems and generating such problems involved a series of pedagogical decisions such as identifying what is a challenging content for particular students, appropriateness based on the pre-requisite knowledge, and the context that students would find interesting and meaningful. Hence, the PCK triggered pedagogical insights in relation to posing mathematically rich and contextually realistic, which was consistent with the findings of Lee et al.'s study (2018). Furthermore, considering the various relations indicated by different webs of knowledge, we argue that teachers needed to activate an interconnected knowledge base (Ma, 1999) in generating mathematically rich and contextually realistic problems.

Limitations and Recommendation

The major limitation of the study was that the knowledge types articulated here were based on preservice teachers' self-reports. On the other hand, this study indicated that reflecting on the internal process played an important role in the multi-tier modeling research to elicit teachers' web of knowledge. In this sense, the present study corroborated Park and Oliver's claim about the role of reflection in provoking the PCK (2008) and other types of knowledge. Still, we recommend investigating teachers' or preservice teachers' knowledge base with different measures such as written assessment and observations, which would allow teacher education researchers to compare with the self-reported webs of knowledge.

Lastly, we suggest teacher educators include either induvial or small group practices in the professional development activities since this study showed that such practices (i.e., individual reflection papers, graphic organizers, and group discussions) have a potential to awake preservice teachers for monitoring their internal process. As a final remark, considering that "[*e*]ngagement by learners with such tasks [*tasks involving real-life context*] is a critical part of mathematics for all learners at all levels of schooling and beyond" (Brown, 2019, p. 76), teachers' and preservice teachers' contemplation about the characteristics of mathematically rich and contextually realistic problems and self-actualizing what knowledge they recalled in the process of generating such problems constitute important practices in the long term for ensuring effective mathematics instruction.

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Data availability: Data generated or analyzed during this study are available from the authors on request.

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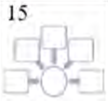
APPENDIX A-INDIVIDUAL REFLECTION QUESTIONS

Please reflect on your thoughts and experiences about the following questions.

1. What were your thoughts about “mathematically-rich and contextually-realistic” problems at the beginning?
2. What do you think now? What changed your mind? What influenced this change?
3. Which ideas did not change at all? What might be the reasons for thoughts that did not change?
4. What types of knowledge did you utilize while producing “mathematically-rich and contextually-realistic” problems?
5. Is there any new type of knowledge you developed while producing “mathematically-rich and contextually-realistic” problems? If yes, can you describe those new types of knowledge?
6. What do you think about the types of knowledge that you utilized/developed throughout these experiences?
7. What is most interesting about writing “mathematically-rich and contextually-realistic” problems? Think about an aspect of writing the problems you have never thought about before.
8. What do you think about the role of “mathematically-rich and contextually-realistic” problems in bridging in- and outside-school mathematics?

APPENDIX B-RESEARCH TEAM'S CATEGORIZATION OF GRAPHIC ORGANIZERS

GO #	Categorical	Holistic	Continuous	Evolutionary	Fragmented	Mechanic	Hierarchical	Causal	Accumulative	Thematic
1 	X									
2 			X	X				X		
3 	X				X	X	X			
4 			X			X	X	X		
5 	X									
6 		X					X		X	
7 	X									
8 		X					X		X	
9 	X	X	X				X	X		
10 	X		X					X		

GO #	Categorical	Holistic	Continuous	Evolutionary	Fragmented	Mechanic	Hierarchical	Causal	Accumulative	Thematic
11 	X		X					X		
12 	X									X
13 	X	X								
14 	X	X	X							
15 	X							X		

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