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Grade 10 teachers' example selection, sequencing and variation during functions lessons

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Examples that teachers choose and use are fundamental to what mathematics is taught and learned, and what opportunities for learning are created in mathematics classrooms. This qualitative multiple case study, using Sfard's commognitive theory, draws attention to mathematics teachers' classroom practices during functions lessons which is unexamined in the South African context. In this article, data sets include unstructured non-participant classroom observations on functions, which were videorecorded. Sfard's commognitive theory served as an appropriate lens in interpreting and analysing teachers' discourses and giving meaning to teachers' classroom practices during functions lessons. The findings demonstrate that the example selection and sequences teachers used during functions lessons either constrained or enabled the development of endorsed narratives about the effect of parameters on the different families of functions.

Keywords: discourse; functions; examples; commognition; mathematics; teaching.

Introduction

The concept of functions has received attention within the field of mathematics education (Kabael, 2011; Trigueros & Martinez-Planell, 2010). Very few protracted studies have been conducted in South Africa in respect of functions (Moalosi, 2014; Moeti, 2015; Mudaly & Mpofu, 2019; Roberts, 2016), and the existing studies were conducted with learners to explore the difficulties they experience when learning the topic. Felix Klein in 1908 viewed functions as 'the soul of mathematics', and this notion has since been discussed by various researchers (Hansson, 2006; Mudaly & Mpofu, 2019). In relation to this, Sierpinska (1992, p. 32) stated that 'functional thinking should pervade all mathematics, and at school, students should be brought up to functional thinking'. This statement resonates with Eisenberg's (1992, p. 153) iteration that developing learners' sense of functions 'should be one of the main goals of the school and collegiate curriculum'. While the statements focus on the learners, they indirectly address the importance of teachers' content knowledge on functions, to ensure learners' development of knowledge related to the topic. Lloyd, Beckmann, Zbiek and Cooney (2010) posit that functions are one of the key topics in secondary school mathematics, because of their relatedness with other topics within the mathematics curriculum such as finance and growth, algebra, and equations, as well as patterns and sequences. Thus, functions can be considered a meta-discourse of algebra internationally (Sfard, 2012).

Within the functions concept in school mathematics, both its importance and problems relating to its learning have been researched and documented in mathematics education research (Moalosi, 2014; Mpofu & Pournara, 2018). Swarthout, Jones, Klespis and Cory (2009) posit that functions are a very important topic in the mathematics curriculum, because of the role that the topic is often seen to play as a unifying concept in mathematics. This makes it essential for learners to develop good conceptual understanding of the topic. While this is the case, Moalosi's (2014) study with Grade 11 learners demonstrated that functions are a topic that learners find difficult to understand, because of the over-reliance on procedures in learning the topic. Another difficulty in the learning of functions is learners' ability to observe change between the given variables and identify the relationships between them (Moeti, 2015). Sierpinska (1992) also suggested that teachers should introduce functions as models of relationships drawing from real-life situations, and in turn view functions as tools for representing a system in another system. The rationale:

> 'for the motivation of mathematical concepts by using concrete examples in the teaching of mathematics stems from the commonly accepted notion that, nowadays, students are interested in the study of the subject matter if they are confident in the applicability of the material they are about to learn.' (Abramovich & Leonov, 2009, p. 2)

The interest and confidence are influenced by the quality of a teacher because learners observe and do what they observe the interlocutor does during teaching and learning in the classroom (Sfard, 2008, 2012). Of concern is that research on Grade 10 teachers' discourses and approaches during functions lessons is scarce within the South African context. In the current article, I attribute the challenges concerning functions to the examples that teachers select and use during teaching. Examples that teachers select and use, and their sequence are fundamental to what functional contents are taught and learned, and the opportunities for learning that are created by teachers in mathematics classrooms (Pillay, 2013). Accordingly, the current study intended to answer the following research questions:

1. How do teachers select and use examples while teaching functions?
2. How does the selection and use of examples facilitate or limit the development of learners' knowledge of functions?

My contention is that for learners to understand the different properties of the concept of functions, teachers should ensure that learners are taught to undertake the following actions: interpretation and construction of functions to help them to comprehend them. Interpretation refers to 'action by which student makes sense or gains meaning from a graph (or a portion of a graph), a functional equation, or a situation' (Leinhardt et al., 1990, p. 8). This statement includes but is also not limited to actions such as describing changes brought by the changes in the values of parameters in a graph or table of values as well as reading off the values of x and y from sketched graphs, for example determining the values of x for which $f(x) = g(x)$. The following section focuses on the review of previous studies on the concept of functions.

Literature review

The constitutive elements of a function

Earlier, Anderson (1978, p. 23) stated that the constitutive elements of a function refer to the 'raw material, a rule or a process … and an end product'. Of importance to note is that not every function has a rule or process; for example, a set of ordered pairs could constitute a function. That is, merely thinking of a function as a rule or process is dangerous for learners' conceptual development as they then easily fall into the trap of only associating formulae or equations with the existence of functions (see Vinner & Dreyfus, 1989). In addressing these concerns, Sierpinska (1992, p. 30) asserted that the constitutive elements of a function should be viewed as 'worlds' and the teaching of the concept should focus on three worlds: world of changes or changing objects, world of relationships and world of rules, regularity and laws. Mathematics has discourses and teachers are expected to use mathematical rules, deeds and interactions that are part of the subject's discourse and legitimise certain forms of mathematising both orally and from learners' written work (Sfard, 2008).

Firstly, the world of changes entails an identification of 'what' is changing in given relationships and 'how' the change is taking place. In this sense, teachers should teach learners how to work with the idea of 'transformation' in functions, and pay attention to the appearance, displacement and orientation of functions (Chimhande, 2013; Mudaly & Mpofu, 2019). For Sierpinska (1992), teachers must emphasise to the learners the need to move from viewing x and y as knowns and unknowns, to conceiving them as variables and constants for meaningful understanding of functions. It is essential to note that the formation of a new function from an old function can be viewed in two ways: numerically as magnitude changes in number operations and graphically as transformations in terms of reflection, rotation, translation and enlargement. Considering that this is expected of teachers, it is assumed that teachers have adequate mathematics knowledge for teaching (MKT) relating to the teaching of functions (Ball, Phelps, & Thames, 2008). This MKT is important, especially when the understanding that teachers with a stronger knowledge base are more responsive to learners' mathematical learning needs appropriately and effectively is seriously considered (Ball et al., 2008). Also, when teachers possess stronger MKT, they are more likely to make fewer language and mathematical errors during teaching and learning and select and use examples effectively to bring the mathematical concepts to the fore.

Secondly, teachers should teach the learners how to observe change between the given variables and identify the relationships between them.

Accordingly, the nature of teachers' exemplification during functions lessons plays a crucial role in promoting or hindering learners' understanding of the topic. In this article, a critical examination of the examples that five participating teachers selected and used while introducing functions, as well as their sequencing of such examples, enables me to unearth the effectiveness of their teaching of the topic.

In addition to the above discussion, for Sierpinska (1992), functions should be viewed 'as tools of description and prediction' (p. 32) of how variables are related to each other, making functions models of patterns in real-life phenomena. This resonate with Euler's and Dirichlet's definitions of what a function is: that functional relationships can be expressed in terms of covariation or by using a rule of correspondence between variables (Bazzoni, 2015; Blanton, 2008; Wilkie, 2020). In describing covariational relationships, Borba and Confrey (1996) stated:

> 'One quantity changes in a predictable or recognisable pattern, the other also changes, typically in a differing pattern. Thus, if one can describe how x_1 changes to x_2 and how y_1 changes to y_2 then one has described a functional relationship between x and y.' (p. 323)

This is not sufficient to constitute a function. According to Bazzoni (2015), learners need to be taught that the association between the two variables can be understood as fixed points on a Cartesian plane and are usually represented by a set of ordered pairs as coined by Bourbaki in the form $(x; f(x))$. Of importance to note is that the enablement of learners' understanding of functional relationships depends on the quality of the examples that a teacher selects and uses to guide learners towards generality about specific functions concepts (Essien, 2021).

Thirdly, a function is considered a rule that governs the relationship between variables (Sierpinska, 1992). According to Van de Walle (2004, p. 436), a function can be viewed as a rule 'that uniquely defines how the first or independent variable affects the second or dependent variable'. What should be noted is that rules, patterns and laws refer to well-defined relationships, a reason for a strong link between this concept and the one discussed above. DeMarois and Tall (1996) argued that the development of the function concept is very complex and that change, relationships and rules are not mutually exclusive pockets of knowledge about the concept. This means that the world of changes or changing objects, world of relationships and world of rules, regularity and laws discussed above are interdependent on each other and the topic should be taught likewise. For example, when learners are observing the change in the values of the independent variable, they should be able to observe how such change influences the values of the dependent variable to construe a rule that signifies a functional relationship. Thus, the foregoing necessitates that as teachers teach functions to their learners, all three conceptions should be developed if enabling learners' fluency in working with functional problems is seriously considered. In view of the above, it becomes clear that change, relationships and rules are to be seen as components of a complex association in understanding the mathematical concept of function. The set of examples teachers introduced across the different lessons can be described in terms of the following patterns of invariance and variation: in all the examples we find an equation in the form '$y = f(x)$', together with an equation in the form '$y = a.f(x)+q$', as stipulated in the Curriculum and Assessment Policy Statements (CAPS). According to Resnick (1997), mathematics is 'a science of pattern' in which there is an emerging invariant structure when a phenomenon is undergoing variation. The following section presents the espoused theoretical framing for the current study and details how the components of Sfard's (2008) commognitive theory are used in analysing and making sense of the teachers' teaching of functions.

Theoretical framing: Commognitive theory

Sfard's (2008) commognitive theoretical framework is a lens to analyse and interpret teachers' communication during functions lessons, and to understand the intricacies and elements of the discourses from what is or is not endorsed by the mathematics discourse community. The commognitive theoretical framework is influenced by Ludwig Wittgenstein and Lev Vygotsky who emphasise the 'inseparability of thought and its expression, either verbal or not' (Sfard, 2015, p. 132), which means thinking in mathematics is a well-defined form of communication, and mathematics teaching is participating in a discourse (Roberts, 2016). Of importance to note is that the effectiveness of teachers' communication of mathematical contents during teaching depends on their content knowledge. Participants in the mathematical discourse show their internal communication through what they say, write, draw or sketch; therefore, communication is seen in both talk and action (Mudaly & Mpofu, 2019). The teaching of functions requires teachers to communicate different concepts, processes and rules explicitly and effectively, which is an expression of mathematics at an intrapersonal (cognition) and interpersonal (communication) level (Vygotsky, 1987). The framework helped with avoiding oversimplified views of teaching. The commognitive theory also allowed for rich descriptions and discussion of teachers' ways of teaching functions, through its focus on the contextual, cultural, dialogical and dynamic nature of participants' discourses in mathematics. By doing so, commognitive theory was used to account for the differences in individual teachers' thinking and teaching methods during the lessons on functions.

According to Sfard (2008), mathematics is 'autopoietic' because it is 'a system that contains the objects of talk along with the talk itself' (p. 129), a feature that makes school mathematics difficult to teach and learn. Thus, familiarity with 'what the discourse is all about' (Sfard, 2008, p. 130) is needed for participation in the discourse, but paradoxically this familiarity only comes through participation by mathematics teachers and learners. Even though this study focuses on teachers, the nature of classroom teaching involves learners who should actively partake in the lesson for familiarity with the mathematical object and to develop mathematical discourse. This does not take away the fact that it is the job of the teacher to ensure that every learner learns to work with functions and to create the right environment that encourages meaningful mathematics classroom discourse. An effective environment for learning functions is one in which learners are allowed and encouraged to engage in investigative processes and where teachers create opportunities to explore particular cases, create conjectures and prove them to make generalisations (Moeti, 2015).

The word 'discourse' implies the use of words and symbols in a way that is generally endorsed by members of a community (Sfard, 2015, p. 45). Accordingly, mathematical discourse communicates mathematical ideas that are ratified by the body of theorems, proofs and laws that govern mathematics (Sfard, 2012). The unit of analysis for Sfard (2008) is discourse, which is considered a special type of communication, made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions.

According to Sfard (2008, 2012), four components can be employed to understand and describe different mathematical discourses during teaching and learning:

- **Words and their use:** Words specific to mathematical discourse which teachers and learners use in discourse-specific ways during mathematics teaching and learning.
- **Visual mediators:** Visual objects that teachers and learners operate upon during discursive activities in the classroom; examples could be graphs, tables and special symbols that are used during mathematical communication.
- **Narratives:** Teachers' and learners' utterance sequences as they speak about mathematical objects, relationships between objects and mathematical processes upon the objects, which are subject to rejection or endorsement within the mathematics community.
- **Routines:** Teachers' and learners' repetitive patterns during mathematics processes and communication about mathematical objects, for example mathematical conventions and performing calculations.

In this article, I use the commognitive theory to describe discourses of functions presented by five teachers in the study, to reveal the opportunities as well as constraints for learning offered by the examples that the five teachers selected and used during teaching. The data from which this article emanates revealed that the dominant narratives were the presentations of functions as formulas, while there were limited opportunities for learners to make conjectures and engage in proof activities. The prevalent routines included the sketching of graphs from functions presented in symbolic form. The following section details the research methodology for the study, to highlight the nature of data generation I used.

Research methodology

The empirical data in the current article consist mainly of videotaped lessons presented by five mathematics teachers at five different school sites in Mpumalanga province of South Africa, representing multiple cases. As reported in this article, a qualitative research approach was espoused (Creswell, 2013). The qualitative approach entails 'a systematic subjective approach used to describe life experiences and situations to give them meaning' (Burns & Grove, 2003, p. 19). This approach allowed me to gain insight into teachers' teaching practices in their uniqueness. To understand the teachers' lived experiences, I immersed myself into the lives of the five participating teachers to explore and understand the teaching of functions as experienced by teachers.

The current study used a multiple case study design. This design enabled me to understand the nature of mathematics teaching, specifically the teaching of functions within a bounded context and bounded activity (Creswell, 2013). For the current study, the bounded context is schools in Acornhoek and mathematics classrooms in the schools, and the bounded activity is the teaching of functions at Grade 10 level. The region is classified as rural as there is dominance of residents that work on farms to sustain themselves and their families, poor transportation services and isolation from the national and provincial government offices. The study was conducted with five (5) Grade 10 mathematics teachers at five (5) secondary schools in rural Acornhoek, Mpumalanga province of South Africa, forming multiple cases. The schools and participating teachers were selected purposively, based on their participation in the Wits Rural Teaching Experience (WRTE) project. Also, teachers needed to possess experience and knowledge of teaching Grade 10 mathematics. Table 1 presents participating teachers' biographical information. To conceal and protect teachers' true identities, I use pseudonyms, as shown in Table 1.

TABLE 1: Teachers' biographical information.

Pseudonym	Gender	Mathematics education qualifications	Number of years teaching	Institution trained at to become a teacher
Zelda	Female	Bachelor of Education	5 years	North-West University, South Africa
Mafada	Male	Honours in Mathematics Education	20 years	Giyani College of Education, South Africa
Tinyiko	Female	Bachelor of Education	5 years	University of Venda, South Africa
Mutsakisi	Female	Bachelor of Education	30 years	University of Zimbabwe
Jaden	Male	Bachelor of Education	17 years	College of Education, India

The empirical data in the current study were generated by means of unstructured non-participatory classroom videotaped observations. Johnson and Christensen (2008, p. 206) defined observation technique to refer to 'the watching of behavioural patterns of people in certain situations to obtain information about the phenomenon of interest'. In the context of the current study, the definition suggests that classroom observations can be used to explore and generate in-depth understanding of the nature of teachers' classroom practices related to the teaching of functions (Guthrie, 2011). The nature of my participation in the observations was non-participatory, I adopted a 'passive, non-intrusive role' during teaching in all the classrooms that I observed (Cohen, Manion, & Morrison, 2011, p. 459). One way of ensuring the trustworthiness of data was through peer scrutiny of the research processes. During the study, I welcomed scrutiny of the project by peers, colleagues and academics at conferences, which allowed me to address biases and assumptions relating to my interpretations of teachers' classroom practices during the lessons.

Data analysis

According to Nieuwenhuis (2007, pp. 99–100), 'qualitative data analysis tends to be an ongoing and iterative process, implying that data collection, processing, analysis, and reporting are intertwined, and not necessarily a successive process'. In the current study, the analysis of observed lessons commenced during the process of data collection and units of analysis were created through ascribing codes to the teachers' observed practices during teaching (Muir & Beswick, 2007). After transcription, the recorded lessons were analysed with the purpose of segmenting and distinguishing the discursive activities characterising the teachers' respective discourses of functions. I firstly analysed each lesson for individual teachers separately, paying attention to repetitive patterns and characteristics of the use of different modalities of mathematical representations and narratives. I then compared the different lessons, searching for similarities and differences and using the identified nuances to inform and reshape my analyses of the separate lessons. I have intentionally adopted an outsider position as an attempt to view the discourses unfolding from the different teachers' teaching in as unbiased a way as possible.

In addition to the above discussion, analysing teachers' classroom discourses while teaching the topic under study required an approach that allowed me to look within and across the different teachers' lessons, to create a picture of the quality of each teacher's teaching. Thus, I overlaid into the tenets of the commognitive theory for both structure and generality about the teachers' discourses during the observed lessons. I initially chunked lessons into episodes based on what activities were set and their related examples for each lesson. Within the episodes, I then noted the nature of teachers' mathematical discourse as framed by the four components of commognitive mathematical discourse. Tables 2–6 depict teachers' discourses of teaching functions across the different episodes.

Findings and discussion

This section addresses participating teachers' selection of examples during algebraic function lessons, and whether and how they facilitated or constrained the learning of functions' critical features during teaching. According to Renkl (2017), it is important that teachers choose appropriate examples to facilitate and deepen the learning and understanding of the concepts and knowledge for the content. The discussion focuses specifically on how teachers worked with the examples during the lessons to help learners understand the critical features for linear functions, parabolic functions, hyperbolic functions and exponential functions. The South African CAPS curriculum recommends that teachers teach critical features for the different families of algebraic functions, such as the effect of different parameters, domain, range, intercepts and turning points (DBE, 2011). The curriculum further asks teachers to provide learners with opportunities to make conjectures, and prove them, to formulate generalisations, especially with the effect of different parameters for different functions. There are two categories in the teachers' systems of variation of parameters: teachers who sequenced the examples showing the effect of one parameter while keeping the other invariant (Jaden, Mutsakisi and Zelda), and teachers whose set of examples in the lessons simultaneously varied both parameters (Mafada and Tinyiko). The two categories are discussed as two sub-themes: *varying parameters simultaneously* and *varying one parameter while keeping the other invariant*. Thus, intuitively we can say that always varying both a and q simultaneously does not seem as optimal in the learning process.

TABLE 2: Discourses in Mafada's teaching episodes.

Sfard's commognitive theory				
Episodes and observable actions	**Visual mediator**	**Words used**	**Endorsed narratives**	**Routines**
1. Introducing the four families of functions and showing learners what a coordinate is in the form of coordinate pairs (x, y). Explaining the mathematical convention that x-values represent the independent variable and y-values represent dependent variables. He showed the learners how to assign values of the independent variable on the table of values, stressing mathematical convention of input versus output values. Explaining how to plot points on the Cartesian plane using the linear function in the form $y = x$	Symbolic mediators: written functions are: $y = x$; $y = x^2$; $y = \frac{1}{x}$ $y = x^2$ Iconic: sketching a graph to depict a coordinate Symbolic: written function is: $y = 2x$. Coordinate pair in the form (x, y)	Functions; parabola; hyperbola; straight-line graph; variable; coordinates; dependent variable; independent variable; point; x-coordinate; y-coordinate	Object-level narrative: "*We said y is a variable* (writing on the board), *we said y is a variable, we said x is also a variable*"	Clarifying. Ritual to find a coordinate. Ritual to sketch a graph. Ritual to compute a coordinate pair on a Cartesian plane.
2. Introduction of the parabolic function in the form $y = x^2$. Demonstrating the change in representations, from algebraic representation of the parabolic function $y = x^2$ to the table of values. Substituting specific values of x into the equation to generate corresponding values of y in the table. Revising mathematical rules associated with performing arithmetic calculations. Asking, then telling learners why $y = x^2$ is a function.	Symbolic: using the function $y = x^2$ to compute the table of values Iconic: table of values	Function; x-value; y-value	Object-level narrative: "*That's an equation, it's a function, that function represents the graph; let us now see the shape that this function is giving us*"	Rituals to determine the output values for given input values, completing the table of values and plotting and drawing the parabola.
3. Summarising the steps needed to draw graphs of functions. Illustrating to the learners that the shape of the parabolic function does not always give the shape given by the function $y = x^2$.	Iconic: graph of a function $y = x^2$ and sketches of 'other' parabolic functions	Plot; hyperbola; subject of the formula; sign of a function; face up	Meta-level narrative: "*I said we first set up the table neh, because without setting up the table, we will not be able to proceed, then the second one we said, you must use that function that you are given aniri* (isn't), *we substitute by the values of x* (pointing at the table) *that we have set up them at the table*" Object-level narrative: "*Those things*" (referring to arrows showing continuation of graphs). This is not an endorsed narrative.	Memorisation ritual on how to draw the graphs of parabolic functions.
4. Demonstrating to learners how to determine the intercepts for $y = 3x^2 - 3$ and $y = x^2 - 1$ and in turn sketching their graphs.	Symbolic mediators: written functions are: $y = x^2 - 1$; $y = \pm ax^2 \pm q$ and $y = 3x^2 - 3$ Iconic: sketching a graph of a parabola and the graphs of $y = 3x^2 - 3$ and $y = x^2 - 1$	Value of q; value of a; positive; value of x; value of y; y-intercepts; negative; x-axis; point; compare; turning point	Object-level narrative about the effect of parameter q: "*And here we said our q, let us talk about our q, our q on the first one it is negative one; our q on the second one is negative three. If you look here at point A and point B, hatwanana?* (am I clear?), *here we said our q is negative one and here we said our q is negative three, so that is the turning point, and that's what I wanted you to see*"	Rituals to determine the x- and y-intercepts Exploration of the effect parameter q.

Mafada viewed the equations as merely producing a result of calculating, resulting in seeing the different functions as recipes to apply to numbers, then remaining unchanged across numbers.

TABLE 3: Discourses in Mutsakisi's teaching episodes.

Sfard's commognitive theory				
Episodes and observable actions	**Visual mediator**	**Words used**	**Endorsed narratives**	**Routines**
1. Using the general equation for linear functions in the form $y = mx + c$ as an archetype to symbolically mediate learners' identification of what the letters in the equation represent.	Symbolic mediators: written functions are: $y = mx + c$; $y = x - 3$ and $f(x) = x - 3$	Variable; dependent variable; independent variable; x-coordinate; y-intercept; gradient;	Object-level narratives, identifying y, m, x and c from the equation $y = mx + c$ Describing the use of the notation $f(x)$ "*y and f(x) can be used interchangeably; we can use y or we can use f(x)*"	Clarifying. Memorisation to identify the values of m and c from linear equations.
2. Using the notion of intercepts to draw the graph of the function in the form $y = x - 1$.	Symbolic: using the function $f(x) = x - 1$ t o compute the table of values Iconic: table of values and graphic visual mediator Symbolic: (0, 1) – y-intercept Symbolic: $0 = x - 1$ – determining the x-intercept	Intercepts; positive; x-intercepts; slants; greater than; y-intercepts; transpose	Describing the notion of intercepts: "*when we are talking about the intercepts, we are talking about the point where our graph touches the line*" The effect of parameter m on linear graphs: "*If m is positive, the graph slants to your right, which means the graph that we are going to have will slant to your right*"	Ritual to complete the table of values from an algebraic equation. Ritual to find the y-intercept and x-intercept for linear functions. Ritual for plotting the straight-line graph.
3. Introduction of a new family of functions (parabolic) to juxtapose the structural differences between linear and parabolic functions in terms of their symbolic appearances.	Symbolic mediators: written functions are: $y = x$; $y = x^2$; $y = 2x + 3$; $y = x^2 + 2$; $y = x^2 + 1$. Iconic: table of values	Quadratic functions; straight line; linear functions; output values; domain; input values; range; x-values; y-values	Distinguishing linear and parabolic functions in terms of their symbolic representations: "*Quadratic functions have a power of 2 whereas linear functions have a power of 1*" Saming domain with input values and output values with range: "*The values that we put into the equation, they are the input values, they are the domain, outputs are the values that we get, and that output is the y-value, which is our range*" This is not an endorsed narrative.	Ritual to complete the table of values from an algebraic equation.
4. Using the parabolic function in the form $y = 2x^2 + 1$ to show learners how to complete the table of values and in turn draw the graph. From the graph, she identified the turning point and the y-intercept.	Symbolic mediators: written functions are: $y = ax^2 + q$; $y = 2x^2 + 1$, $y = x^2$; $y = 2x^2$ and $y = 3x^2$	Turning points; greater than; positive; smile; faces up	Narrative about turning point Describing the effect of parameter a: "*Now, with the parabola, if a is positive the graph faces up*"	Ritual to complete the table of values from an algebraic equation. Ritual for plotting the graph of a parabola.

The observable action that is prevalent across Mutsakisi's lessons was drawing the graphs of functions, which was characterised by Mutsakisi demonstrating the drawing of graphs of functions. That is, functions given in symbolic form were represented in tables of values, whereby the teacher demonstrated to the learners the algebraic calculations to find y-values for given x-values.

TABLE 4: Discourses in Tinyiko's teaching episodes.

Sfard's commognitive theory				
Episodes and observable actions	**Visual mediator**	**Words used**	**Endorsed narratives**	**Routines**
1. Recapping on the features of linear functions to set the scene for parabolic functions. Substitution and calculations and completion of table of values to draw the parabolas of given functions.	Symbolic mediators: $y = x^2$; $y = x^2 + 1$; $y = x^2 - 1$ Iconic visual mediators: function in the form $y = x^2$ depicted in the table of values.	Parabola; linear; functions; straight; linear graphs; dual-intercept method; intercepts; gradient; turning point; y-intercept; output values; x-intercepts; positive; face up; table method	The effect of parameter a: "*because you were given the function as y = x² + 1 the coefficient of your x squared is positive, it simply tells you that your graph will go up*"	Clarifying.
2. Showing learners how to use the dual-intercept method and table method to determine the output values for chosen inputs. Engaging in non-mathematical memorisation of 'stealing' procedure to determine the intercepts.	Symbolic syntactic mediators: $y = x^2 - 1$; $y = x^2$; $y = x^2 + 1$; $y = ax^2 \pm q$ and $y = ax^2$ to perform mathematical calculations Iconic visual mediators: graphical representations for $y = x^2$ and $y = -x^2$	Substitute; formula; stealing; x-intercepts	The object-level narrative about calculations: "*the reason why whenever we substitute these numbers, whether we substitute the negative or a positive we always get a positive answer*"	Rituals to use the dual-intercept method and table method to determine the output values and drawing graphs of functions.
3. The teacher intended to introduce hyperbolic functions, but the example she introduced was for exponential functions. Although she realised this after engaging in mathematical calculations, she continued performing rituals to demonstrate to the learners how to substitute and calculate for output values.	Symbolic syntactic mediators: $y = x^2$; $y = x^2 + 1$ Iconic visual mediators: table of values and graphical representations for $y = \frac{1}{x}$	General formula; hyperbola graph; x-intercepts; y-intercept; exponential graph; asymptote; asim-touch; x-asymptote; y-asymptote	Describing the notion of asymptote: "*But there is a unique thing that we must always have is the asymptote, asymptote meaning 'asim-touch', we don't touch, the line which this graph will never touch. We have two asymptotes, the x-asymptote and the y-asymptote*"	Rituals to use the dual-intercept method and table method to determine the output values and drawing the graph of the function.
4. Substitution and calculation of intercepts for $y = -\frac{1}{x}$.	Symbolic syntactic mediators: $y = -\frac{1}{x}$; $y = -\frac{1}{x}$	General equation; linear graphs; parabola; cup; cave; concave; gradient; table method; y-intercept; x-intercept	The object-level narratives about intercepts: "*y is equals to zero, meaning that our y-intercept is equals to zero*"; "*So, the x-intercept is also zero*" – treating intercepts as numerical values instead of coordinate pairs	Rituals to use the dual-intercept method to determine the output values.
5. Using two examples $y = -\frac{1}{x}$; $y = \frac{1}{x}$ to generalise the effect of changing the sign of parameter a.	Symbolic syntactic mediators: $y = -\frac{1}{x}$ Iconic visual mediators: graphical representations: $y = -\frac{1}{x}$; $y = mx + c$.	Arrow; continuing; inputs; domain; range; points; facing up; output; positive; negative	The effect of parameter a: "*What makes them to be different? On this one, the coefficient of x squared is negative, that is why it is facing down and the first one the coefficient of x squared was positive, that's why the graph is facing up*"	Memorisation about the effect of parameter a in terms of its sign from graphical visual mediators.

Tinyiko's pedagogical actions in all the episodes reveal the prominent use of the rituals to draw or sketch the graphs of given functions in symbolic form. This makes the observable action of drawing or sketching the graphs the end goal of Tinyiko's teaching of functions. Focusing on the how of the routines resulted in Tinyiko's discourse of rituals rather than explorations, as she emphasised the following of rules without explication and understanding their applicability.

Development of parameters discourse within the function concept depends on the content teachers make available for learners to learn, the teaching approach they use to convey the notion of parameters to the learners as well as how they select and vary examples to develop learners' thinking about the effect of different parameters on the behaviour of the functions. Two of the five teachers in this study struggled to offer explanatory talk that would enable learners to make conjectures,

TABLE 5: Discourses in Zelda's teaching episodes.

Sfard's commognitive theory				
Episodes and observable actions	**Visual mediator**	**Words used**	**Endorsed narratives**	**Routines**
1. Introducing the functions: $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ in the table of values and engaging in the process of substituting and calculating the output values and completing the table of values.	Iconic visual mediators: functions are: $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ depicted in the same table of values Symbolic syntactic mediator: substitution and calculation process for output values	Same axes; compare; values of y	(none)	Ritual to substitute and calculate the values of y for chosen values of x and in turn completion of table of values.
2. Exploration of the effect of varying the values of parameter a on parabolic functions, both with the aid of the table of values and the graphical representation. The effect of parameter a is also explored in terms of the notion of turning point.	Iconic: graphs of the three functions $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ on the same set of axes Symbolic: $y = ax^2$; $y = ax^2 \pm q$ Iconic: The sketches of the graphs for $y = x^2$; $y = -x^2$ to juxtapose the direction of the graphs because of the sign of the value of a Symbolic: representing the coordinates of the turning points for the functions $y = x^2$; $y = x^2 + 1$ and $y = x^2 - 1$ in the form (0, 0); (0, 1) and (0, –1) respectively	Plot; highest value; y-values; x-values; standard equation; Cartesian plane; cup shape; mountain shape; turning point	The effects of parameters a and q on the parabola: "*a determines the shape of the graph and then q is the vertical shift of the graph. I said a determines the shape akere* (isn't)? *while q is the vertical shift of the graph*" Saming turning point and y-value: "*Turning point is where the graph turns ka* (at) *point ya* (for) *y, so meaning turning point is also a y value*" This is mathematically incorrect.	Exploration of the effect of parameter a in terms of its sign from the table of values and from graphical visual mediators.
3. Introduction of a new family of functions (hyperbolic functions) using two examples: $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ presented in the same table of values. The teacher and the learners complete the table of values and subsequently draw the graphs of the two functions on the same set of axes. The teacher guides learners to generalise the effect of varying the sign of parameter a.	Iconic visual mediators: functions are: $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ depicted in the same table of values Iconic: graphs of the two functions $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ on the same set of axes to juxtapose the direction of the graphs because of the sign of the value of a Symbolic: using inequalities $a < 0$ and $a > 0$. Iconic visual mediators: functions are: $y = \frac{2}{x}$ and $y = -\frac{2}{x}$ depicted in the same table of values	Undefined; Cartesian plane; increasing; decreasing	Describing the notion of asymptote: "*If it touches the line, it is not correct, but it must be close to the lines*" The effect of parameter m on hyperbola: "*The arcs will be in the second and fourth quadrant*" Direct and inverse relationship: "*If x is increasing, even the y is going to increase. This one if x is increasing, this one is decreasing*", "*greater than zero is one up*"	Exploration of the effect of parameter a in terms of its sign from the table of values and from graphical visual mediators.

What is starkly evident in Zelda's teaching is that she engaged her learners in interpretations of the given functions using an interactive communicative approach, highlighting the critical global features for the families of functions she focused on. In terms of discursive routines, Zelda's teaching appeals to the use of explorative routines to help learners to observe some critical features for each family of functions focusing mainly on applicability routines.

TABLE 6: Discourses in Jaden's teaching episodes.

Episodes and observable actions	Visual mediator	Words used	Endorsed narratives	Routines
1. Using the function machine approach to demonstrate to the learners the ritual to substitute and calculate output values. Determining the 'rules' for given relations using the patterns-oriented approach.	Iconic visual mediators: Using the function machine to calculate the output values Iconic visual mediator: Using table of values to determine missing values and rules for given relations	Output values; input values; function; difference; values of x; relation; values of y; corresponding	Meta-level narrative about the rule underpinning the relation: "*three is added to the values of x to get the values of y*"	Ritual to substitute and calculate the values of y for chosen values of x using the function machine.
2. Teaching learners how to substitute and calculate the output values using the function notation in the form $f(x)$.	Symbolic: Using the examples of functions: $y = 3x - 1$; $f(x) = 2x - 4$; $f(x) = 3x$; $f(x) = x^2 - 1$; $y = 2x$; $y = x$ and $y = 2x + 1$ to demonstrate to the learners how to use the function notation to determine the output values	f of; substitute; value; linear function; line	Meta-level narrative about how the function notation is used: "*instead of y, you have that f of x, function of x*"	Clarifying ritual to demonstrate to the learners how to use the function machine to determine the output values.
3. Using the examples of linear functions $y = x$; $y = 2x$ and $y = 3x$ to teach learners how to substitute and calculate output values, complete the table of values and draw the graphs. Once the graphs are drawn, Jaden engages in the action of interpretation, exploring the effect of varying the value of parameter a for linear functions. Providing learners with two examples ($y = 4x$ and $y = 5x$) asking them to make conjectures about where they think the graphs would be positioned compared to the other three.	Iconic visual mediators: functions $y = x$; $y = 2x$ and $y = 3x$ depicted in the same table of values and graphs drawn in the same set of axes Symbolic: $y = x$; $y = 2x + 1$; $y = x - 1$; $y = 2x$; $y = 3x$; $y = 4x$ and $y = 5x$; $y = -x$; $y = -2x$; $y = -3x$	Graph; linear function; exponent; line; substitute; increasing; steeper; gradient; y-axis; x-axis; negative signs; domain	Object-level narrative about the relationship between the symbolic mediators and the graphical mediator: "*The reason why these functions are linear functions is because the value of the exponent is one. It is also a linear function because when we draw the graphs of these functions, we get a line*" Object-level narrative about the effect of parameter m on linear functions: "*The graph of y equals to 3x is more steeper than 2x, that number 1, that number 2, that number 3* (pointing to the coefficients of x in the three functions), *it is because of that number that when it is increasing, the graphs are coming closer to the x-axis*"	Exploration of the effect of parameter a in terms of its magnitude from graphical visual mediators.

Jaden's teaching were rituals to translate the functions presented in symbolic form into the table of values and drawing of graphs. Across all Jaden's episodes, the teaching was dominated by his explanatory talk without providing learners with the learning opportunities to create mathematical meanings for themselves during the lessons.

prove them and make generalisations about the effect of the different parameters for the different families of functions they worked with in the classrooms. Analysing how the teachers selected and sequenced a set of examples in each lesson enabled a view of whether and how the examples accumulate to bring the object of learning in different lessons into focus for learners, and whether there is movement to achieve generality which is one of the curriculum objectives for Grade 10 level in South Africa (Adler & Ronda, 2015; DBE, 2011).

For the two sub-themes, the above statement means that teachers' systems of examples and their sequencing reveal whether there was movement towards generality relating to the parameters of functions. This relates to curriculum statement 3 for functions which expects learners to 'investigate the effect of a and q on the graphs defined by $y = a.f(x)+q$' (DBE, 2011, p. 24). This curriculum principle envisages that teachers vary parameter a while keeping q invariant or varying parameter q while keeping a invariant to ensure that learners make conjectures, prove them and construe generalisations relating to the effect of each parameter where $f(x)$ is defined by the following functions: x; x^2; $\frac{1}{x}$ and b^x. The focus here is on the role of examples and how they were sequenced to enable or constrain systematic learning, as symbolic mediators to support learners' knowledge building.

Varying parameters simultaneously

It is discernible that in the examples that Mafada and Tinyiko used in their lessons, they did not use patterns in which they vary one parameter while keeping the other one invariant. Table 7 illustrates the examples that the two teachers used and their sequences in selected lessons.

Mafada starts with $y = x^2$ and moves on to change both a and p simultaneously in the next two examples with respect to the first example, but then the next three examples change only a with respect to the first example. Then, with respect to the first example, an example is given where only p is different. Then the last example is again a change in both a and p. Thus, it could be said that putting examples 2 and 3 right after example 1 potentially limits the signifying of the changes that learners should observe during teaching. Tinyiko's example selection and sequencing demonstrate that she varied the examples in terms of the families of functions simultaneously and the simultaneity in the introduction of such functions limited observation opportunities for learners to observe the effect of parameters for the different functions.

While Mafada's and Tinyiko's sequencing of the examples moved from simplicity to complexity, as presented in Table 7, the system of variation did not however create opportunities for learners to observe what is changing as the teachers did not vary one parameter while keeping the other invariant. On this Moeti (2015) states that during the teaching of quadratic functions, the sequencing of examples 'moves from a parent function $f(x) = x^2$ where simple example is taken to complex ones' (p. 61). For Mafada and Tinyiko, the lack of an invariance-variance relationship to bring the world of changes to the fore in the example sets did not allow for systematic comparison of the different families of functions in terms of the effect of changing the values of a and q. Thus, it can be said that the example sequences teachers used constrained the development of endorsed narratives about the effect of parameters on the different families of functions. According to Martensson (2019, p. 7), 'rather than telling the students the critical aspects, the teacher must structure the critical aspects in terms of variation and invariance', to ensure that the effect of parameters is discerned and discriminated across examples.

TABLE 7: Sequences of examples in Mafada's and Tinyiko's lessons.

Teacher	Examples and their sequence
Mafada	$y = x^2$; $y = -2x^2 - 5$; $y = 2x^2 + 3$; $y = -4x^2$; $y = 2x^2$; $y = \frac{1}{4}x^2$; $y = x^2 - 1$; $y = 3x^2 - 3$
Tinyiko	$y = 3x^2$; $y + 4 = x^2$; $y + 16 = x^2$; $y = 3^x$; $y = 3^x + 2$

I argue that Mafada's and Tinyiko's examples across their lessons have constrained the discernment of the meaning and structure of the parameters of functions, because there was no systematicity in terms of what varies and what remains the same between two parameters. That is, the set of examples the teachers used did not demonstrate knowledge of what changes, what stays invariant and what the underlying meanings behind varying parameters a and q are. It could be said that the teachers' use of formulas as symbolic visual mediators did not enable effective visualisation of the effect of the parameters. According to Marton (2015), during teaching and learning, variation is a necessary component to enable learners to notice what they are expected to learn. The discernment of critical features related to the families of functions, particularly the effect of the parameters on the behaviour of the functions, did not occur, since there was no systematicity in terms of varying one parameter while the other remained invariant in the teachers' lessons. I therefore argue that the variation of one parameter while the other parameter remains invariant is a precondition for learners to develop a sense of structure and meanings of the parameters, to see what is changing and what remains unchanged and the related effects on different families of functions (Al-Murani, Kilhamn, Morgan, & Watson, 2019; Martensson, 2019).

In addition, the patterns of variation in Mafada's and Tinyiko's examples are contrary to Leung's (2012) postulation that:

> 'invariants are critical features that define or generalise a phenomenon … for a major aim of mathematical activity is to separate out invariant patterns while different mathematical entities are varying, and subsequently to generalise.' (p. 434)

The ways Mafada and Tinyiko varied the parameters during teaching did not bring about the discernment of structure in working with the different families of functions as well as generality about the effect of the parameters a and q as per curriculum standards. I argue that using parameters simultaneously without first exploring the effect of each parameter while the other remains invariant constrains learners' awareness of the effects of the parameters. Thus, the teachers' use of symbolic mediators in the form of algebraic equations was lacking what Sfard (2008, 2012) termed 'interpretive elaboration', because they did not offer learners elucidations about the behaviour of given functions when some variation is introduced to the parameters of functions.

This lack of interpretive elaborations and intellectual discussions with the learners about the effect of the parameters indicates that the teachers did not create a teaching and learning environment that facilitates learners' deep understanding of functions. The following extract exemplifies the ritualistic routines in Tinyiko's teaching:

1 Tinyiko: If for instance you are given y equals to x squared and I say draw a graph of that one, in other words, when I give you this there is an addition of zero (*see image 57 for symbolic mediator*), what is the y-intercept?

4 Learners: (*chorusing*) Zero!
5 Tinyiko: The y-intercept is zero. Why do you say zero? I said you can only use what? The table and the dual. By the way, how does the dual work?
7 Learner: We let x be zero.
8 Tinyiko: We said let x be equals to zero because you want to find the what?
9 Learners: (*chorusing*) To find the y-intercept.
10 Tinyiko: And let's remember that the x-intercepts are also the output values. What about the y-intercepts?
12 Learners: The outputs.
13 Tinyiko: Good, after you get all the values, all you need to do is draw the graphs.

The questions '*what is the y-intercept?*' (Line 3), '*because you want to find the what?*' (Line 8), '*I said you can only use what?*' (Line 5), '*by the way, how does the dual work?*' (Line 6) and '*What about the y-intercepts?*' (Lines 10–11), all represent an elicitation technique to check whether learners gained information from the previous lessons. The conversation above serves as an example of mathematical communication where the teacher used the words y-intercept and x-intercept as if they refer to outputs and inputs based on their relatedness, which Sfard called 'saming'. I noticed though that Tinyiko overlooked the idea that y-values are the output values and the x-values are the input values, but the notion of x-intercept entails a zero of a function where an input value produces an output of 0. Also, using the word y-intercept to signify synonymity with output values does not explain to the learners that a y-intercept is a point where the input value is 0 on a given function, which also addresses the commognitive construct of saming. Furthermore, the statement '*The y-intercept is zero*' (Line 5) reveals that intercepts are treated as a numerical value[1] rather than as coordinate pairs. This cannot be left unproblematised, considering that what teachers say and do during teaching shapes learners' development of correct mathematical word use to talk effectively about mathematical entities. According to Sfard (2019, p. 1), 'it is a common lore that teachers bear the main responsibility for what the students learn or fail to learn', suggesting their influence regarding learners' understanding or lack thereof for knowledge. Tinyiko's narrative in Line 13 was also concerning, because the choice of words – '*all you need to do is draw the graphs*' – illustrates the teachers' ritualisation when working with functions, and lack of interpretive elaborations about the critical features of the different families of functions.

The above discussion resonates with Mason's (2002) argument that worked-out examples might constrain learners' ability to generalise the nature of mathematical objects and the nature and effect of the parameters on different functions, as the teachers primarily focused on ensuring that learners recognise the syntactical template of the symbolic representation for functions. From Mafada's and Tinyiko's teaching approaches, it could be argued that learners could not notice what stayed the same and what varied, resulting in learners' inability to associate the patterns of variation with the different representations as well as the word use and narratives that go with them. Al-Murani et al. (2019) argued that learners' conceptualisation about the function concept:

> 'depends on discerning common and differing features among examples and experiences, generalising from these according to the scope of examples that are presented, and fusing these features into a concept.' (p. 8)

It is arguable that varying the two parameters simultaneously without first varying one while the other remains invariant makes it difficult for learners to experience the difference of their effect on the functions.

Varying one parameter while keeping the other invariant

Sieving out invariants in the parameters during the teaching of functions is an essence of experiencing the depth of the topic, and in turn developing conceptual understanding as this facilitates symbolic mediation for different functions (Chimhande, 2013; Moeti, 2015; Sfard, 2008). Table 8 depicts Zelda's, Jaden's and Mutsakisi's example sets.

Zelda's, Jaden's and Mutsakisi's patterns of variation in the selected lessons as presented in Table 8 demonstrated systematicity in terms of varying one parameter while the other stayed invariant to guide the learners about the effect of the parameter in focus. Although the degree of interpretive action differed for the three teachers, their selection and sequencing of examples demonstrated some intentionality to help learners move towards generality about the effect of the parameters on the different families of functions. For example, it can be said that Zelda varied the values of parameter q in terms of both magnitude and signs for the functions $y = x^2$, $y = x^2 + 1$ and $y = x^2 - 1$. Zelda's fourth example ($y = -x^2$) was introduced to mediate learners' thinking about the effect of changing the value of parameter a in terms of the sign and it was introduced after she was done with the effect of parameter q. Similarly, Jaden also varied the values of the gradient while the values of parameter q remained invariant, to bring the changes brought by changing the values of the gradient into focus.

For Mutsakisi, the pattern of variation also focused on varying the values of parameter q for parabolic functions $y = x^2$, $y = x^2 + 2$ and $y = x^2 + 1$ while parameter a remained invariant and the two linear functions $y = x$ and $y = 2x + 3$ were used as non-examples of the parabolic functions; the explanation she offered reinforced that the focus was on parabolic functions, and specifically the effect of parameter q. Mutsakisi is thus varying not just a parameter, but the type of

TABLE 8: Sequences of examples in Zelda's, Jaden's and Mutsakisi's lessons.

Teacher	Examples and their sequence
Zelda	$y = x^2$; $y = x^2 + 1$; $y = x^2 - 1$; $y = -x^2$
Jaden	$y = 2x$; $y = 2x + 1$; $y = x - 2$; $y = x$; $y = 2x$; $y = 3x$; $y = 4x$; $y = 5x$; $y = -x$; $y = -2x$; $y = -3x$
Mutsakisi	$y = x$; $y = x^2$; $y = 2x + 3$; $y = x^2 + 2$; $y = x^2 + 1$

1. This observable action was frequent also across the other episodes in this lesson and in other observed lessons for the same teacher.

function also. Accordingly, it could be argued that this can be confusing to learners trying to understand the variation in parameter q for the first time, but now having to consider two different family types simultaneously can be considered bad sequencing or variation. Also, Mutsakisi never used an example with a negative value for q, which is also not good practice. In addition, none of teachers used non-integer values for either a or q, which is also possibly bad for the learners' movement towards generality.

According to Lo and Marton (2012, p. 29), 'the learning of an object is not possible if we cannot first discern the object from its context'. The example sequences mediated the identification of 'what' is changing in given relationships, 'how' the change is taking place as well as how the changes were linked to the different parameters, thereby guiding learners towards the endorsed narratives about the parameters of quadratic and linear functions (Sfard, 2012). The following statements acted as endorsed narratives during teaching and are illustrative of how the system of variation enabled the commognitive elaborations for the three teachers:

> '*m* is equals to 1, *m* is greater than zero, in other words it means the value of *m* is positive … right, this has got a meaning, it has a very special impact on the graph that you are going to draw, *akere* [*isn't it?*], to the graph that you are going to draw. [*Writing on the board as she continues speaking*] If *m* is positive, the graph slants to your right, which means the graph that we are going to have will slant to your right, are we together? The graph will slant to the right because *m* is positive, *akere*, the general formula says *y* is equals to *mx* minus.' (Musakisi)

> '*a* determines the shape of the graph and then *q* is the vertical shift of the graph. I said *a* determines the shape *akere* [*isn't it?*] while *q* is the vertical shift of the graph. Now, let us check something here from our three graphs. I remember when I was introducing parabolic graphs, I drew this one and this one [*writing parabolic functions* $y = x^2$ *and* $y = -x^2$ *in symbolic form*] in the same Cartesian plane and then the shape of the graphs were not the same; the other one was like this and the other one was like this [*drawing sketches on the board*]. We had two different shapes, mountain shape and cup shape, so we need to know when we have this and when do we have a cup shape.' (Zelda)

> 'The graphs of *y* equals to 4*x* and *y* equals to 5*x* would be between the *y*-axis and the graph of *y* equals to 3*x* because the value is increasing. The graph of *y* equals to 3*x* is more steeper than 2*x*, that number 1, that number 2, that number 3 [*pointing to the coefficients of x in the three functions*], it is because of that number that when it is increasing, the graphs are coming closer to the *x*-axis. This is called the gradient; it is called the gradient of this line.' (Jaden)

Zelda, Jaden and Mutsakisi created opportunities through the system of variation and sequencing of examples to bring the idea of 'transformation' in functions, attention on the appearance (structure), displacement and orientation of functions, to the fore (Chimhande, 2013; Mpofu, 2018).

Conclusion

In this study, the teachers who did not vary one parameter while keeping the other invariant in the examples did not engage in interpretive actions about the effects of the parameters on the different families of functions, relating to the lack of explorative routines. Accordingly, this results in lack of formal word use and endorsed narratives related to quadratics and linear functions. This demonstrates that systematic variation, selection and sequencing of examples in symbolic form are the preconditions of productive communication about the behaviour of different parameters for families of functions in terms of formal words and endorsed narratives. That is, without systematic and sequential variation of parameters, teachers' communication becomes limited to rote steps to draw graphs, and nothing is revealed to the learners about the effect of the parameters. The teachers who selected and sequenced the examples showing the variance-invariance patterns of working with parameters for different families of functions engaged in interpretive actions about the effect of the parameters; as such, the variation patterns mediated both their communication about the effect of the parameters and created opportunities for learners to learn about the notion of parameters. Word use and endorsed narratives are enabled or constrained by the availability and systematicity of patterned variation or lack thereof.

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The author declares that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

H.W.M. is the sole author of this article.

Ethical considerations

Before the study could commence, ethical clearance was granted by the University of the Witwatersrand and access to the schools was permitted by the Mpumalanga Department of Education (certificate number 2018ECE006D). All teachers were informed of the purpose, confidentiality and voluntary nature of participation in the study before any data generation processes commenced and all participating teachers signed informed consents. I also adhered to the importance of ensuring that the identity of participants is protected, both in terms of keeping the information they provided confidential and by using pseudonyms to conceal their true identities as well as those of their respective schools. The assurances for confidentiality and anonymity in this study extended beyond protecting the teachers' names and those of their schools to also include the avoidance of using self-identifying statements and information.

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Data availability

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Disclaimer

The views and opinions expressed in this article are those of the author and do not necessarily reflect the official policy or position of any affiliated agency of the author.

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