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The role of teacher actions for students' productive interaction solving a linear function problem

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ABSTRACT

Many studies in mathematics education have emphasized the importance of attending to students' interactions, particularly, their mathematical reasoning when collaborating on solving problems. However, the question of how teachers can facilitate students' productive interactions for learning mathematics, is still a challenging one. This case study aims to provide detailed insights into opportunities and limitations related to teachers' actions for the productivity of students' interactional patterns solving a linear function problem together. Four student-pairs in the first year of upper secondary school (11th grade) serve as a background on students' interactional patterns, which in this study focused on three interactional aspects: collaborative processes, mathematical reasoning, and exercised agency. The student-pairs' three teachers provide insights on teacher actions observed as different funneling and focusing actions, which elucidated opportunities and limitations in several situations influencing the productivity of students' interactional patterns. The study used purposive sampling in selecting the particular school and three teachers, which were chosen based on acquaintances and willingness to participate in the study. The students' interaction when solving the mathematical problem and the teachers' interaction with the pairs were video recorded and observed by the researchers. The analysis method was a deductive analytical strategy, where specific events of interactions were identified, based on the three interactional aspects combined with teacher actions. Coding schemes on students' interactions were used, as well as on teacher actions. The findings indicate that teachers' actions and questions influenced students' interactions, but mainly their reasoning, and particularly the primary agent's reasoning. Moreover, students who were engaged in interactional patterns called bi-directional and one-directional did not change their ways of interacting after a teacher interaction. Thus, the teachers' actions did not impact students' collaborative processes and agencies in the same way as their reasoning. This study adds to the field of mathematics education by illuminating the importance of teachers being aware of students' roles when they work together, for facilitating a productive interaction for both students in dyads. The study highlights the importance of further research on teacher actions and teacher awareness for facilitating collaborative situations of bi-directional interactions for students' shared understanding of mathematical concepts and ideas.

Keywords: agency, collaboration, interactional pattern, reasoning, teacher actions

INTRODUCTION

A classroom environment focusing on students' interactions and problem-solving is important for building mathematical understanding (e.g., Hufferd-Ackles et al., 2004; Mueller et al., 2012; NCTM, 2014; Stockero et al., 2019). A vast amount of research highlights the importance of students' construction of their own mathematical knowledge for better understanding important mathematical ideas (e.g., Lithner, 2017; Mueller et al., 2012; Schoenfeld, 2013; Stockero et al., 2019). We argue that there are, at least, three central interactional elements important for making students' interactions productive for learning and understanding mathematics. These aspects are *collaborative processes, mathematical reasoning,* and *exercised agency.* This article uses the term *interactional patterns* to describe the intertwinement of these three aspects, and builds on findings from Hansen (2021), which studied students' interactional patterns with the same students as presented here. She found that whether students' interactions are productive or unproductive is connected to how students participate: choosing to engage or refrain from engaging through different types of agencies, including different ways of interacting in those roles. One implication of the study (Hansen, 2021) was the need for a better understanding of teacher actions in light of the findings. This is the aim of the present article, to increase our knowledge on ways teachers act to promote students' productive interactions through guidance and monitoring of mathematical reasoning, collaboration, and distribution of agency.

To promote opportunities for learning, the expected role of the teacher is no longer to be a "dispenser of knowledge", but to promote a learning environment where students actively engage in problem-solving and construction of their own understandings (Stein et al., 2008). Therefore, classroom activities should provide opportunities for sharing different thoughts, allowing students to respond to, negotiate, and build on each other's ideas (Norqvist et al., 2019). When students attempt to work together, teachers periodically check in on students' work or students ask for guidance. In those situations, teachers can engage with students' interactions, such as their mathematical justifications and suggestions, or with their lack of ideas that halts the progress in their problem-solving.

The literature provides many approaches to studying teachers' ways of acting to encourage students to engage with mathematics (e.g., Alrø & Skovsmose, 2004; Boaler & Brodie, 2004; Stein et al., 2008). Some approaches emphasize entire practices (Wood, 1998), whole-classroom discussions (Stein et al., 2008), small-group discussions (Webb, 2009), or teachers' questions more specifically (Boaler & Brodie, 2004). Altogether, there exists a diversity of models with the purpose of gaining insights into teachers' actions for facilitating mathematics discussions with both the whole classroom, smaller groups, and individual students. Several studies have developed analytical tools to categorize teaching practices. For instance, studies of teacher support in whole-classroom discussions (Staples, 2007; Stein et al., 2008), communicative features in teacher-student dialogues (Alrø & Skovsmose, 2004), or teacher actions in teacher-student interactions (Drageset, 2014).

Our case study involves three Norwegian mathematics teachers' interactions with dyads of first-year upper secondary students solving a linear function problem in their classrooms. Students discussed and used a dynamic software program, GeoGebra, in their problem-solving. Despite the growing knowledge about teacher actions for encouraging students to engage with mathematics, less is known about the connections between teacher actions and the three interactional aspects emphasized here: reasoning, collaboration, and agency. This article contributes to this area by studying in depth specific teacher actions and how those influence students' interactional patterns in situations where student pairs attempt to collaborate and reason mathematically, and students' interactions shape the teacher-student interaction.

Applying a fine-grained model to understand teacher actions in a teacher-student conversation, can give detailed insights to how teachers can facilitate students' reasoning and argumentation, as well as their collaboration and agency in those situations. In this study we have described teacher actions as suggested by Drageset (2014), outlined in the theoretical framework section. This article aims to give details of teachers' actions in teacher-student communication connected to students' interactions, focusing on collaborative processes, mathematical reasoning, and the students' agency related to linear functions. With this aim, and within the frame of this case, we ask the following research question: *What are the opportunities and limitations of teacher actions for the productivity of students' interactional patterns?*

Students' Interactional Patterns

Discussing mathematical ideas with a peer or with a teacher provides opportunities to defend and explain one's own ideas as well as ask questions regarding another point of view. Thus, sharing ideas and engaging in mathematical thinking may allow students to reason mathematically using arguments to justify ideas. Mathematical argumentation can be regarded as a prerequisite for learning mathematics and simultaneously as an outcome of students' math-talk (Krummheuer, 2007). Moreover, Krummheuer (2007) highlights students' arguments in their reasoning as an important interactional aspect that is important as a foundation for both learning mathematics and interacting with a peer in creating a mathematical classroom community. We find support in the perspective on students' talk as "interactional accomplishments and not as logical arguments" where the focus is on "what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered mathematically valid" (Yackel, 2001, p. 6).

In line with viewing mathematical argumentation as an interactional accomplishment, Lithner (2017) defines mathematical reasoning as not being restricted to only formal or logical proof. In Lithner's (2017) perspective on mathematical reasoning, students' building of arguments and exploration of mathematical connections concerns meaningful sequences of thoughts for the individual student. It does not matter whether the reasoning is simple or complex, correct or incorrect, nor the level of competence the student exhibits, as long as the student provides evidence to support the idea (Lithner, 2017).

Therefore, we see mathematical reasoning as "the explicit act of justifying choices and conclusions by mathematical arguments" (Boesen et al., 2014, p. 75). The explicit act concerns reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939). The latter statement about reasoning comes from a rather recently empirically developed framework of mathematical reasoning (Lithner, 2008, 2017). Students can take different reasoning paths to reach conclusions in problem-solving. Two different paths are identified by Lithner (2008) as two major types of reasoning: creative reasoning and imitative reasoning. The framework outlining these types of mathematical reasoning is called *creative mathematically founded reasoning*[1] (CMR). For students to exercise creative reasoning, their reasoning ought to fulfill three criteria:

1. *Creativity*: creating or re-creating a new solution method,
2. *Plausibility*: arguments supporting strategy choices and implementation, which explain why the conclusion is true, and
3. *Anchoring*: arguments based on mathematical concepts or relationships, which are intrinsic mathematical properties.

Together, these three aspects describe students' creation of solution methods called creative reasoning. Imitative reasoning, on the other hand, implies students' copying of procedures or recalling of facts.

[1] In line with Lithner (2008, 2017) and his colleagues studying creative mathematically founded reasoning, this study uses the wording creative reasoning or acronym CMR for linguistically simplicity.

A student group or a pair are frequently in classroom activities encouraged to work together. In literature this is referred to as collaboration or cooperation. We use the word collaboration, and this can simply put be two or more students in joint activity attempting to solve a problem or produce an agreed upon outcome. There are many definitions of collaboration with slightly different wordings and parameters to be fulfilled to identify an interaction as collaborative. Roschelle and Teasley (1995) suggest that "collaboration is a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (p. 70). In line with the same or similar definitions on collaboration used in other studies (e.g., Baker, 2015; Lai et al., 2017; OECD, 2013), this study views collaboration as defined by Roschelle and Teasley (1995), with focus on the collaborative processes enacted to "construct and maintain a shared conception" of a mathematical problem. Roschelle and Teasley (1995) state that students must build a shared conception by introducing and accepting knowledge, monitoring the on-going activity, and repair conflicting interpretations. We see the building, monitoring and repairing as three collaborative processes for collaboratively pooling knowledge together. Kuhn (2015) says that "it is essential to understand the underlying mechanisms" of learning when working together (p. 47). Aspects of students' interactions may therefore give important insights on underlying processes causing productive or unproductive collaboration. A productive collaboration involves engaging and negotiating with others, where all students are involved in constructing the arguments, and where statements and suggestions are challenged, counter-challenged and justified, and decisions are jointly made (Mercer, 2004; Powell, 2006).

Ways of participation when attempting to collaborate are exercised differently depending on several factors. For instance, relations to a collaborating peer and a given topic to solve may impact the likeliness of a fruitful conversation. Therefore, studying students' participation, seen as exercised agency (Gresalfi et al., 2009), in collaborative processes involving mathematical reasoning is another central social aspect of peer interactions. Hansen (2021) discusses students' roles in pairwise collaboration. She found that students were in a *bi-directional interaction* when mutually attempting to understand one another and when both were driving forces of the problem-solving process. Typical for a pair in a bi-directional interaction was a mutual effort where students co-constructed reasoning sequences with such shared agency (Hansen, 2021). Students showed mutuality in all aspects of their interaction: both chose to engage and thus participated with shared agency, they were equally driving the problem-solving process, thus both made plausible and anchored mathematical arguments, and engaged in turn-taking conversations in all stages of collaborative processes. In contrast, if students exercised different roles in a problem-solving process where the final outcome was expressed repeatedly by one of the students, they were in a *one-directional interaction*. In their interaction, both students were engaged, but they expressed agency differently. For instance, the co-working student's role was often to understand suggestions or explanations made by the primary agent. Moreover, questions were expressed by the student who exercised secondary agency and such input was either assimilated into the final outcome of the reasoning by the primary agent, or it was considered and refined or neglected by the primary agent. Consequentially, the primary agent was the main producer of plausible and anchored mathematical reasoning, whereas the secondary agent observed, questioned, or accepted the primary agent's ideas. Cobb (1995) says that a relevant distinction for learning opportunities involve students' explanations noticed as univocal and multivocal, which respectively relates to the concepts of one- and two-directional interaction. When the two interaction types are compared, Cobb (1995) emphasizes that univocal interactions rarely give rise to learning opportunities for either student, whereas multivocal interactions usually are more productive. Thus, learning opportunities are found in productive interactions such as two-directional interactions (Cobb, 1995; Hansen, 2021): depending on concepts that are "taken as shared" for a basis for the mathematical conversation where both students are equally involved. This aligns well with findings from Mercer (2004) and Powell (2006), who speak of exploratory talk and negatory discourse, respectively, as the interaction forms most productive for mathematical learning.

THEORETICAL FRAMEWORK

Considering the interactional aspect of reasoning connected to teacher actions, Maher et al. (2018) present an extensive list of research on teachers' attendance to students' development of mathematical reasoning for productive classrooms. They contend that "one of the most well-established research findings is that teachers' knowledge of students' reasoning is an essential component for student learning" (p. 3). However, the review does not reveal details of what teachers do when attending to students' reasoning. Nor any outline of teacher actions for promoting students' reasoning through teacher-student interaction. Likewise, Ayalon and Even (2016) point to a variety of roles a teacher has in promoting student argumentation, and highlight the importance of prompts, encouragements, and explaining for promoting students' argumentation. The existing advice on teacher guidance promoting students' engagement in CMR is to let students attempt their own construction of their own solutions. To support students' attempt, whether they fail or they to some degree succeed, the suggested teacher actions is general as well: diagnose students' difficulties with the particular task, and provide adapted feedback, but not a solution method (Lithner, 2017). Although there exists extensive research on the importance of students' mathematical reasoning, there is still a challenge to promote teachers' awareness of how classroom situations can facilitate mathematical reasoning (Maher et al., 2018). While it requires knowledge of student reasoning and validation of the quality (Maher et al., 2018), it is important for teachers and teacher educators to know how to act to promote mathematical reasoning.

Further, a teacher's role for students' collaborative work is similar to the role they have for students' reasoning since these aspects are tightly interwoven (Granberg & Olsson, 2015). Teachers' guidance for collaborative work is shaped by students' mathematical ideas and contributions, and their ways of participating (Staples, 2007). However, to successfully engage students in collaborative learning, Staples (2007) suggests that "this kind of teaching requires a deep understanding of mathematics, students' thinking, curricular materials, as well as how students reason and potentially develop proficiency with a mathematical domain and its practices" (p. 212). She outlines teachers' roles and specific teacher strategies to organize a whole-class collaborative inquiry in a model of three main components: supporting students in making contributions; establishing and monitoring a common ground; and guiding the mathematics. The findings emphasize that eliciting, scaffolding, and creating contributions, are important teacher actions in this setting.

In Howe et al.'s (2007) study on the role of the teacher for students' collaborative outcome, they highlight a teacher's optimal intervention through guidance and monitoring, and task design. Both aspects should facilitate proposal and explanation of ideas, and teachers are suggested to be relatively non-directive. Findings of another study, van de Pol et al. (2018), show that students in small groups using teacher support are likely to formulate answers that are more accurate. Moreover, useful teacher support was characterized as being more extensive initially and reducing over time. This is explained as support "provided at the right time when students need support, which can make further processing of new information easier for the students" (van de Pol et al., 2018, p. 4). Untimely support hindered students' abilities to make sense of teachers' feedback, but the use of teacher support proved beneficial to students' learning (van de Pol et al., 2018). Thus, van de Pol et al. (2018) stress the importance of teachers' awareness of how to use support for students in their problem-solving and for processes in group work.

The above-mentioned research contributes to our knowledge on teachers' role in students' collaborative work important for whole-classroom inquiry (Staples, 2007), and for students' collaborative *outcome* (Howe et al., 2007). However, more knowledge is needed on students' collaborative *processes* (our focus) in mathematics classroom (Seidouvy & Schindler, 2019), and there is a need for detailed insights on teacher's role for promoting students' collaborative processes in small-group interactions.

Facilitating reasoning and argumentation in mathematics classrooms is challenging, and there is a need for better understanding how to facilitate these aspects (e.g., Ayalon & Hershkowitz, 2018; Maher et al., 2018; van de Pol et al., 2018). How students share their thinking with one another, such as their reasoning processes (Lithner, 2008), and how they act or refrain from acting in the conversation–their *agency* (Gresalfi et al., 2009; Mueller et al., 2012), are such central aspects of the participants' interaction.

For students to exercise agency (the third interactional aspect in this study) it is not enough to ask students to work collaboratively on mathematical tasks for agency to automatically occur (Mueller et al., 2012). Students are afforded the agency to "author mathematical ideas" in cases where teachers distribute shared authority between students and teacher (Langer-Osuna et al., 2020). Teacher actions for sharing authority, so that students can exercise agency, is to offer students opportunities to address mathematics problems, and holding students accountable to their strategies, solutions, and ideas (Bell & Pape, 2012; Hamm & Perry, 2002 as referred to in Langer-Osuna, 2018). Classroom situations where students are afforded shared agency has the potential for conceptual agency (Cobb et al., 2009), which means students' opportunities for constructing their own meaning and methods (Mueller et al., 2012). Moreover, if students choose problem solving paths and connect mathematical ideas, a teacher is more likely to support students' mathematical learning through shared agency (Cobb et al., 2009). Classroom situations where teachers exercise authority, students are only afforded to exercise disciplinary agency (Mueller et al., 2012). Disciplinary agency is a concept posed by Pickering (1995), a complementary concept to conceptual agency, and explained as "…utilizing established procedures" (Mueller et al., 2012, p. 374). Consequentially, in teacher-student interactions with disciplinary agency, a teacher is responsible for determining the validity of student responses (Cobb et al., 2009).

A detailed framework of teacher actions for the purpose of studying in depth specific teacher actions and how those influence students' interactional patterns in situations of collaboration, reasoning, and exercised agency, however, not linked specifically to the interactional patterns in focus here, was developed by Drageset (2014). The framework, called *redirecting, progressing,* and *focusing actions*, is both empirically and theoretically built (Drageset, 2014). The three main categories, redirecting, progressing, and focusing, elucidate tools and techniques teachers use to make students' thoughts and strategies visible, help students progress in their problem-solving, or redirect students in an alternative direction (Drageset, 2014). The teacher interactions may facilitate different types of student responses (Drageset, 2015, 2019). The main teacher-action categories entail 13 sub-categories built on concepts from theories about mathematical discourse grounded in perspectives on student-centered versus teacher-centered classrooms.

In teacher-student interactions each utterance and every turn of speech and action depend upon the previous turn. Taking turns is a social practice and an important structure of a conversation (Sidnell, 2010). However, Drageset (2019) says that "looking at single comments, or turns, yield a very limited scope" (p. 2). Therefore, it is important to look at the interplay of turns of speech and actions together in conversation sequences. If the teacher actions components are detailed it can provide better understanding on how different turns of teacher actions influence students' interactions. Thus, different teacher actions influence students' interactions when they collaborate, discuss, reason mathematically, and take ownership of a problem. Although teacher actions influence students' collaborative work, teachers' actions are also shaped from interacting with students and from their ways of participating (Staples, 2007). This complex relation needs to be addressed by a fine-grained analytical model to give detailed insights on teacher's role for promoting students' collaborative interactions. This is possible to investigate by the framework of Drageset (2014), since it separates teacher's actions from student's talk and actions. Giving attention to specific students' interactional aspects, as emphasized above, and specific teacher actions provide opportunities to explore how teacher actions are related to students' interactional patterns.

Two overarching categories, funneling and focusing (Wood, 1998), organize the areas of teacher actions in the framework by Drageset (2014). If a teacher is *funneling* students' thinking, it means that "the student's thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself" (Wood, 1998, p. 172). Thus, mainly the teacher is doing the intellectual work. Redirecting and progressing actions are primarily categories of funneling actions where the teacher is the intellectual authority. Drageset (2014) explains *redirecting actions* as corrections exhibited implicitly or explicitly (Alrø & Skovsmose, 2002). Moreover, redirecting actions are categorized as a teacher's attempt to challenge the students (Drageset, 2014), which means "questioning already established knowledge" (Alrø & Skovsmose, 2004, p. 55).

Alternatively, to funneling is what Wood (1998) calls *focusing*, where the intellectual responsibility is with the students, hence to a larger degree than funneling actions are teacher actions for promoting productive interactions as reviewed in the first part of this section, zooming in on reasoning (Ayalon & Even, 2016; Lithner, 2017; Maher et al., 2018), collaboration (Howe et al., 2007; Staples, 2007; van de Pol et al., 2018), and agency (Langer-Osuna, 2018; Mueller et al., 2012).

Thus, we see the three main categories from Drageset (2014), redirecting, progressing, and focusing, as a useful tool to investigate opportunities and limitations of teacher actions for the productivity of students' interactional patterns. Using this framework, with its 13 subcategories, interpreted in light of theories on teacher actions for the emphasized interactional aspects, allows us to dig deeply and in detail into the question of how teacher's guidance for students' collaborative work and reasoning is shaped by students' mathematical ideas and social contributions to the conversations.

METHODS

This study seeks, through an instrumental case study (Stake, 2003), to generate more knowledge concerning the role of teacher actions in students' interactional patterns. The article aims at developing more inclusive theories on the issue at hand (Layder, 1998), based on the details and nuances from the particular case, which is explained in detail through the coding procedures.

Participants and Data Collection

The data was collected in 2017 at an upper secondary school in the eastern part of Norway. In this study, students had recently transitioned from lower secondary school (10th grade) to upper secondary school (11th grade). Children start grade 1 at the age of six and upper secondary starts at grade 11. The larger study, which this case study is a part of, followed three first-year theoretical mathematics classes (69 students in total, from 15-16 years old) and their three teachers over a time span of five months. Whole-class discussions were studied, as well as the collaboration and dialogue between pairs of students (six pairs in particular; two pairs in each class). In both instances, the role of the teacher in the interactions was emphasized. This article focuses on the dialogue between four of the six student-pairs—Emma and Hannah, Philip and Noah, Olivia, and Oscar, and Leah and Isaac—and their teachers. The teachers—Jacob, Lucas, and Sophie—were all engaged in the study, and contributed to the planning and evaluation of their teaching in light of the study's aim (Amiel & Reeves, 2008). The teachers were encouraged to help their students think together and to hold back on their guiding, and both prior to and after the planned lessons, the teachers and researcher discussed how to assist and interact with the students in order to encourage mathematical reasoning and collaborative work.

The study used purposive sampling (Bryman, 2016) in the selection of the particular school and the three teachers. The teachers and the school were chosen based on acquaintances and willingness to participate in the study. The six dyads for the in-depth study were chosen based on conversations with the teachers, and two aspects were particularly emphasized: (1) a high level of reasoning competence based on a test combined with average-to-high score levels for functional understanding and (2) the likeliness of a student-pair to be verbal and share thoughts with one another. Thus, the objects of study were students who willingly talked about mathematics with one another, were likely to reason about functions, and already had some knowledge about functions. The student pairs had previously to this session been presented with relevant concepts for talking about linear functions, such as the slope number and constant. From preceding school years, the students should have become familiar with the coordinate system and straight lines, but the topic of linear function was new to them.

The students' collaboration and the teachers' interactions with the pairs were video recorded and observed by the researchers. The teachers encouraged students to talk with one another, attempting justification of their thoughts and ideas, and use of relevant mathematical concepts. A microphone was placed on a desk in front of the students and connected to a video camera recording their talk and gestures. The laptop screen is not video recorded, but the recordings show when students used it to draw, write or point to the screen. Students' solving of the linear function problem lasted for approximately 45 minutes in each of the three classrooms.

The Linear Function Problem

A productive struggle with important mathematical ideas is central for effective mathematical problem-solving (Lithner, 2017). Lithner (2017) points out that "the focus is on the *particular type of struggle* when students construct task solutions instead of imitating them" (p. 938, italics added for emphasis by authors). The particular type of struggle should emphasize mathematical reasoning and non-routine solving of tasks, where the struggle should be more like a challenge to solve, rather than an obstacle (e.g., Hiebert & Grouws, 2007; Lithner, 2017; Stein et al., 2008). We see an obstacle as a too-difficult problem, which may be a stumbling block for students, whereas a challenge entails a better-adjusted problem for students to solve. Therefore, a fundamental aspect of a teaching design is choosing suitable tasks for facilitating a mathematical discourse to potentially strengthen the teacher-student and student-student interactions.

When designing a problem in this study, we wanted students to be presented with a challenge to connect function representations in order to not view linear function representations as "'topics' to be learned in isolation of the others" (Thompson, 1994, p. 24). Moreover, the linear function problem ought to facilitate an opportunity to discuss and share their own ideas, strategies, and knowledge about functions.

A function can be referred to in different ways: as a dynamic mechanism that performs transformation through an input and output, as the relationship between two variables, and as a rule of correspondence between two sets (Malik, 1980). Functions can appear as a graph, a verbal description, a table, or an algebraic expression. Students need to connect fragments of function representations in order to build a comprehensive understanding of the function concept (Best & Bikner-Ahsbahs, 2017). This is

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Figure 1. The function task (reformulated from Olsson, 2018).

particularly important in upper secondary school, as the function concept has a central role in organizing and connecting many important mathematical ideas (Michelsen, 2006). An important aspect of understanding the function concept is students' underlying algebraic knowledge of variables (Leinhardt et al., 1990; Lepak et al., 2018). For instance, students are more often familiar with letters as unknowns than with variables. When working with linear functions, such as $y=mx+b$, it is important to interpret the algebraic expression as a "kind of relationship between the letters, as their value changes in a systematic manner" (Küchemann, 1978, p. 26).

Exploring linear functions' parameters m and b through focusing on translations between different representations (e.g., algebraic, graphical, tables) (Akkoç & Tall, 2005; Leinhardt et al., 1990) can be fruitful for in-depth learning. In this article, the function problem in **Figure 1** was discussed by the student pairs and contains a focus on translation between algebraic and graphical representation as well as the importance of variables for understanding the function concept. Attempting to formulate a rule for a pair of perpendicular lines supports a generalization process where students make use of their findings via patterns for making a general relationship between two linear functions—hence, the rule.

Students worked in dyads on one laptop and were encouraged to use the dynamic software program GeoGebra as a tool in their problem-solving. GeoGebra provides tools to create, manipulate, and control mathematical content for students to investigate mathematical relations (Granberg & Olsson, 2015; Hall & Chamblee, 2013). Thus, the given linear function problem presents an opportunity to investigate varying parameters of the slope number and the constant. Changing an algebraic expression may cause GeoGebra to dynamically change the related graphical representation (Preiner, 2008). Thus, students get rapid feedback on performed actions, inputs, and changes in GeoGebra. However, GeoGebra does not interpret the generated information. Therefore, students would have to make sense of dynamic changes between different linear representations. Olsson (2018) found that students successfully solving a task with GeoGebra used given feedback extensively and engaged in reasoning.

Data Analysis

A deductive analytical strategy was used (Yin, 2014), which involved "identifying or creating a suitable video corpus and systematically sampling from it to examine specific research questions" (Derry et al., 2010, p. 10). The videos were watched several times before critical events (Powell et al., 2003) were identified. Events were seen as critical if (1) the dyads had some form of interaction, characterized by the three emphasized aspects and (2) there were teacher interactions tied to these sequences. Four dyads fulfilled these criteria, and each pair's work with the given problem, as well as the interactions they had with the teacher in the process, was transcribed and coded. The teachers' interactions with the dyads were coded using the coding scheme presented in **Table 1**. These codes were based on Drageset's (2014, 2019) framework for redirecting, progressing, and focusing teacher actions, but were slightly revised through an iterative coding procedure between the data set and the theoretically based framework. **Table 1** shows which codes are based on the original framework and which ones are added or revised. The following paragraphs outlines Drageset's (2014, 2019) framework starting with the funneling actions, *redirecting* and *progressing*, which is followed by focusing actions, called *focusing*, and ending with **Table 1**–an overview of the teacher action framework.

A typical redirecting teacher comment is to discard a student's suggestion or comment, called *put aside* in the coding. With such an action, a teacher is not providing help with a presumably pressing question or challenge. The second redirecting action, *advising a new strategy*, means that a teacher's comment is suggesting an alternative approach or way of thinking to solve a problem. The last redirecting action is *correcting questions*, where a teacher's question aims to move a student's focus over to another approach. In summary, redirecting actions are a teacher's strategy for shifting attention to something else.

In line with the funneling manner of actions, a teacher may aim to move a problem-solving process forward. Drageset (2014) explains four actions for attempting to guide students' *progress*. *Open progress details* are teachers' open questions with possibilities for several answers concerning the progress for solving a problem. This action includes questions on how to do, how to think, how to solve, and how to generalize patterns. Thus, an open progress action is aiming at "moving the process forward, but without pointing out the direction" (Drageset, 2014, p. 16). *Closed progress details*, on the other hand, concern how (many, large, much, big, how to do it) and what (it becomes, shall we write, is, to do). Questions typically request details needed to move the process forward, connected to steps in a procedure (Lithner, 2008). These details can be process answers (one step at a time) or details about how the process should go on to reach the answer. Another aspect of teacher action attempting to move the process forward is a *simplification* of the task at hand. To simplify a task, a teacher may change or add information, tell students how to solve it, or give hints to make a task easier (Wood, 1998). A teacher would typically pull a student toward the solution (the Topaze effect, cited in Brousseau, 2006): "It often seems that this involvement is meant to ensure the progress of the class and sometimes these comments appear to come as a consequence of a halted progress. Many of the simplification comments could also be characterized as hints" (Drageset, 2014, p. 15). The last progress action is *demonstration*. A teacher typically demonstrates how to solve the problem or shows students every step in a procedure. It is primarily a monologue given by a teacher that is occasionally broken if students ask questions or if the teacher asks students whether they understand or agree.

Table 1. Codes for teachers' actions in conversations with student-pairs

Teachers' actions	
Focusing–*Giving attention to students' thoughts and input*	
Requests for students' input	
Enlighten detail	• Student explanation • Focus on details • *Gathering information***
Justification*	• Explaining why • Justifying method and/or answer
Apply to similar problems*	• Asking to use knowledge on similar problems
Request assessment from other students*	• Asking other students to evaluate answer/solution
Pointing out	
Recap*	• Repeating an answer • Adding to an answer
Notice	• Highlighting details in a dialogue • Reminding students of new or previous information • *Explanation focus** on a student's question
Progressing–*Taking action to move the process forward*	
Open progress details	• Open questions with several answers • *Encourage testing of strategy***
Closed progress details	• Closed questions with one answer • Request for details • Request about procedure/steps
Simplification	• Adding information • Hints • Telling students what to do • *Explanation progress***
Demonstration	• Showing a procedure • Doing several steps of a procedure • *Evaluation of solution/strategy* **
Redirecting–*Bringing attention to something else*	
Put aside	• Discarding a student's suggestion/comment • *Interrupting***
Advising a new strategy	• Suggesting another approach
Correcting questions	• Question changing approach

Note. *Not used in the analysis; **Added actions as a result of the analysis process

Further, Drageset (2014) divides focusing teacher actions (Wood, 1998) in two categories: *request for student's input* and *pointing out*. A teacher may ask for a student's input through *enlighten details*, *justification*, *applying to similar problems*, or *requesting assessment from other students*. These concepts are related to what Franke et al. (2007) express as access to student thinking. If teachers *enlighten details*, it is a request for students to explain what something means or how something happens. Typically, details are brought into focus. If a teacher asks for a student's *justification* (Cengiz et al., 2011), it is a request for a more thorough explanation, often to validate why the answer found, or the method used, is right.

Another approach focusing on students' thinking is when a teacher is *pointing out* something. Sub-categories of pointing out are *recap* and *notice*. The purpose of the *recap* is to merge information to clarify important elements in a student's explanations. Also, a teacher can repeat a student's answer with the purpose of confirming or ending dialogue, or sometimes adding information to an answer. A *noticing* action (Cengiz et al., 2011) is when a teacher highlights particular aspects, concepts, or details he wants to make a student aware of. Other aspects of noticing are reminding students of new or previous information and adding information.

All teacher utterances in the interactions between the four student-pairs and their teacher were coded, and from every interaction, narratives were written about the students' situation before a teacher interacted with the students, as well as a characterization of the teacher-student interaction. After the teacher left the conversation, written narratives described how the students interacted in the moments that followed. Excerpts presented in the results section represent typical teacher actions for the given teacher-student interactions, combined with the typical way students interacted with each other and their teacher.

Students' Interactions Prior to Teacher Interaction

As outlined in the introduction, four student-pairs gave insights into collaborative interactions concerning their reasoning and processes for creating and maintaining collaboration connected to their agency (Hansen, 2021). Two pairs demonstrated *bi-directional interactions*, whereas the other two pairs interacted in a *one-directional manner* when solving the function problem (**Figure 1**). This section characterizes the interactions found in the four student-pairs without the teacher present in their conversation, based on Hansen (2021). Results section follows the student-pairs' conversations with their teacher presented with a brief summary of the course of events before and after the conversations with their teacher.

In the bi-directional interactions, there were mutual attempts to solve the linear function problem in both student-pairs. The student-pairs, Emma and Hannah, Philip and Noah, engaged in CMR, which was observed particularly through the *plausibility* and the *anchoring* of their arguments (Lithner, 2017). The students made arguments about the linear function that were acceptable and plausible not only to themselves, but to each other. Therefore, the arguments used in creating a shared understanding were an interactional accomplishment (Yackel, 2001). The students who were in a bi-directional interaction mutually justified choices (e.g., choosing different slope numbers or constant) and conclusions (e.g., connecting their general algebraic expression of the rule and the graphical explanation of the rule) by using mathematical arguments (Boesen et al., 2014). Through their turn-taking conversations (Sidnell, 2010), the four students were prompted by their interaction to anchor explanations in mathematical properties when ideas or thoughts were not consistent with a peer or feedback from GeoGebra. These situations were seen in the collaborative processes of monitoring and repairing, which are central to maintain a shared conception of the problem.

Like the two pairs engaged in bi-directional interaction, there were many mutual attempts to solve the linear function problem also among the student-pairs interacting in a one-directional manner. Olivia and Oscar, and Leah and Isaac, were also engaged in CMR (Lithner, 2017), although in a different way than Emma and Hanna, and Philip and Noah. In contrast to collectively co-constructing reasoning sequences, the students interacting one-directionally had a primary agent expressing the final outcome of a reasoning sequence. As seen in Hansen (2021), Isaac and Oscar were the primary agents, whereas Leah and Olivia were the secondary agents. One similarity between the two student-pairs was that both primary agents, Isaac and Oscar, made suggestions for solving the function problem, whereas Olivia and Leah, secondary agents, asked their peer about details concerning ideas and inputs. Also, both pairs focused on the slope numbers of linear functions related to the perpendicularity, focusing on how the parameter m was changing in a systematic manner (Küchemann, 1978), which is important underlying algebraic knowledge of variables that is important for building a comprehensive function concept (Lepak et al., 2018). Both pairs used GeoGebra, where they easily manipulated and adjusted different function representations, and they extensively used and attempted to make sense of its feedback.

RESULTS

This section provides the results from each of the four student pairs' interaction with their teacher. The excerpts presented within each subsection are typical teacher actions for the given teacher-student interactions, combined with the typical way students interacted with each other.

Jacob in Interaction with Hannah and Emma–Progressing Actions

Prior to the interaction with the teacher, Hannah and Emma had made a shared attempt to understand the meaning of perpendicular lines by exploring the connection between the algebraic and the graphical expression of the linear functions. Right after their turn-taking conversation, Jacob initiated a teacher-student interaction by getting in contact with the student-pair and asking the student-pair what they had tried out so far in the problem-solving process.

Excerpt 1

(14) Emma	When you say perpendicular, right, should it be 90 degrees?
(15) Jacob	Yes. Because when you measure an angle, any angle, then you should've... 90 degrees, there, yes (pointing to the screen).
(16) Hannah	90 degrees.
(17) Emma	Umm? For any?
(18) Hannah	If you measure here, or here, or here, or here (using the laptop).
(19) Emma	Oh, like that.
(20) Jacob	So, you tried some different things.
(21) Hannah	Yes.
(22) Jacob	What've you done so far?
(23) Emma	Well, I misunderstood at first. I thought it was a type of task like this (using her pencil to illustrate two crossing lines and pointing to a previous task on the piece of paper in front of her).
(24) Jacob	Like, this line here, and this one, aren't those perpendicular? Because here you see... (pointing at the drawn lines on the paper).
(25) Emma	Is it just to take... since we got +4... is it just to take -4x (both students look down at their own papers with graphs in a coordinate system)?
(26) Jacob	Yes, can you see what'll happen if you take -4x?
(27) Emma	Didn't we do that earlier?

(28) Hannah Yes, that might be right.

During the conversation, Jacob seemingly tried to make sense of the students' questions, suggestions, and input. With his actions, our interpretation is that he aimed to move the process forward; thus, it was a *progressing* conversation. For instance, he began an explanation of perpendicularity (15), which was interrupted by another student asking for bathroom permission. Consequentially, Hannah and Emma engaged in a dialogue where Hannah attempted to explain the meaning of perpendicularity (16-19). Furthermore, Emma suggested testing two linear functions with slope numbers 4 and -4 (25). Jacob encouraged testing of the suggested strategy (26). After his question, a short student dialogue followed where they concluded that they had already tested the suggested lines. Jacob stated that their previous attempt did not work and asked an *open progress detail* question, "What's wrong with the line in a way?" Emma replied that it was not 90 degrees. Jacob's response was initiating another *progressing* conversation.

Excerpt 2

(29) Jacob Yes. What would you've to do with the formula?

(30) Hannah It can't be 4*x*, or it must be less of each.

(31) Jacob Yes. What is the number doing? The number in front of x.

(32) Emma Slope number.

(33) Jacob Yes, so here it's the... So, if we try something, what would you change it to, to come closer to the solution?

(34) Emma 2 or something? -2.

(35) Jacob Yes (Hannah writes on the laptop).

(36) Emma Closer at least.

(37) Jacob Yes. You should try to make it even smaller.

(38) Emma -2*x*, then? Because now we're on minus... it's not 90 degrees now either.

(39) Hannah No. Wasn't it what we... Now it goes back. Maybe it's a fraction, then?

Jacob responded with *closed progress details*, hence moving the progress forward by pointing out that *something* should be changed; more specifically, the algebraic representation (29). When Hannah replied that it should be less than 4*x* (30), Jacob guided the conversation's focus to the slope number (31-39). Thus, Jacob used the students' responses to channel a focus on the slope number in the algebraic expression connected to the graphical representation.

When Jacob encouraged them to test a smaller slope number (37), Hannah and Emma continued their dialogue, which Jacob quickly *interrupted* after Hannah's question (39). Such instances happened several times in the conversation above (e.g., 20, 33). In those situations, Jacob guided the student-pair on what seemed to be his chosen solution path. When Jacob interfered, he often made *progressing* actions, such as *open progress details*, *closed progress details*, *evaluation of solution/strategy*, and *encouraging of testing strategy.* In **Excerpt 1** and **Excerpt 2**, Jacob is moving the process forward by channeling the focus to the slope number, which initiate progress for guessing and checking different linear functions in GeoGebra. This consequentially results in less student reasoning and justification for their actions, less collaborative repairing, but a continued shared agency.

After the teacher-student interaction, Hannah and Emma tested for different slope numbers: 1/2, 1/3, and then 1/4. Their inputs resulted in the perpendicular pair of lines: y=4x+2 and y=-1/4x+2. Hannah and Emma progressed to the third task of formulating a rule for when two linear functions were perpendicular. They tested their rule, but it gave them the wrong result, and they concluded that they should ask for guidance. However, Hannah and Emma did not finish their work, as they did not get any further support in their problem-solving.

Lucas in Interaction with Philip and Noah–Redirecting Actions

Philip and Noah had prior to the conversation with the teacher jointly reflected and agreed upon a shared understanding of an expression of the rule for making linear functions perpendicular to one another. Their teacher, Lucas, was passing by Noah and Philip when Noah signaled a need for teacher support by raising his hand.

Excerpt 3

(13) Noah Umm... This...

(14) Lucas You don't have to test the rule on me (interrupting).

(15) Noah I just...

(16) Lucas You must test the rule (interrupting). Can you use it for different instances?

(17) Noah	But, does the rule look okay? For the second one to be perpendicular with the first...
(18) Lucas	To be picky, this does not look like a rule (interrupting).
(19)	(Noah laughs quietly, but Philip looks serious).
(20) Lucas	Could you read out loud what you have written down here, Philip, inside the frame?
(21) Philip	-1 divided by a is the slope number to the perpendicular line (after reading their rule, both students looked at Lucas).

Noah was barely able to say one word (13) before being *interrupted* by their teacher, Lucas: "You don't have to test the rule on me" (14). His response is characterized as *redirecting*, and being of an unsupportive character. Noah further attempted to explain their rule, but was cut off a second time (15). Noah tried to ask for teacher feedback on their formula, so he asked, "But, does the rule look okay?" Then he hurried on to continue his explanation of the rule (17), but was *interrupted* a third time (18). Lucas asked whether they could use the rule (16) and for more precisely formulated mathematics (18). Thus, his teacher actions were categorized as *redirecting* and *progressing*.

Both students seemed to think they had found a valid expression for the formula, but they were not given the opportunity to share their findings and questions with their teacher, from whom they seemed to want confirmation and/or guidance. In this situation, Lucas did not show interest in the students' thoughts, reasoning, or collaboration. Lucas's tone was harsh, and his interaction cannot be described as welcoming. The student-pair did not seem to be intimidated. However, Philip looked a little disappointed (19).

After the teacher-student interaction, Noah and Philip made an explanation for their rule for perpendicular lines. Noah uttered that the slope number had to be -1 divided by the slope number from line *a*. Philip challenged Noah to write the rule like a formula. They discussed different options and agreed to call the two lines for 1 and 2 in order to make it a "bit more professional mathematics language," as they put it. They developed their formula to $a_2 = \frac{-1}{a_1}$, where a_2 was the slope number of the second line and a_1 was the slope number of the original line. Philip and Noah had just found this way of writing it when Lucas came by a second time and observed for 10 seconds before they noticed him. When they saw him, Noah began explaining the formula by writing it down on the paper. Lucas responded by saying, "I leave this to you," then he left the conversation.

In **Excerpt 3**, Lucas noticed students' findings, which they had written down on a paper. With redirecting teacher action Lucas did not ask for students' explanations for their findings, but prompted a better mathematical expression of the rule. In the continuing student-student interaction Philip and Noah responded by adjusting their findings into a more precise expressed formula. They continued to anchor their suggestions in mathematical properties through collaborative processes and with shared agency.

Sophie in Interaction with Oscar and Olivia–Focusing Actions

Oscar had prior to the conversation with the teacher suggested a rule for making a pair of perpendicular lines, justified in linear functions with slope numbers with opposite signs. Olivia supported his input and explanation. Olivia and Oscar tested, in line with their formulated rule, another pair of linear functions with slope numbers 2 and -2. Oscar observed the input in GeoGebra and said that "this is absolutely not perpendicular. Then we need a new rule." Olivia gently asked, while crossing two fingers, indicating perpendicular lines, "Perpendicular... It's 90 degrees?" Oscar did not notice her question and suggested another line, and that they should try -0.5*x*. Olivia asked again if perpendicular meant 90 degrees. He said yes and evaluated Olivia's input of the new linear function with the slope number -0.5, which he thought looked "very perpendicular." They used GeoGebra to check if it was perpendicular, which the tool confirmed.

Their teacher, Sophie, approached the student-pair and asked if they had any success in making perpendicular functions. Both students answered, and Olivia said that they had made two pairs.

Excerpt 4

(8) Sophie	You tried something here (pointing at the laptop screen). When you tried that, what did you think?
(9) Oscar	No. But, first, at least I thought that 2x and therefore -2x would become perpendicular. That didn't work. It became too steep... or too gentle (makes a hand movement).
(10) Sophie	Why did you think... How did you find that 2x, and therefore -2x, would make a perpendicular line?
(11) Oscar	Umm, because we tried with only x, and then it worked. But it didn't with 2x. So... Hmm, I don't know.
(12) Sophie	Yes. You might consider making another pair of lines perpendicular to one another... to look for any relationship. Because now, you have two, right, these two are perpendicular to one another, and these two are perpendicular to one another. So, try to make another pair being perpendicular to one another.

Sophie initiated a conversation with Olivia and Oscar by *gathering information* on what progress they had made and how they had started their problem-solving process. Sophie continued to request the students' thoughts and input when pointing out performed actions in GeoGebra (8). Oscar replied with what *he* had thought would make a perpendicular pair of linear functions and why. He evaluated the graphical representation and concluded it was either too steep or too gentle (9). Again, Sophie *requested* the students' thoughts and brought details into focus (10). Thus, Sophie *enlightened details* (8, 10) emphasizing the students' reasoning. Oscar explained why they tested 2 and -2 and justified his answer by connecting it to the first linear pair with slope numbers 1 and -1. Sophie ended the conversation by encouraging the student-pair to find several pairs of perpendicular lines (12). Her teacher actions addressed details already highlighted by Oscar. Thus, Sophie brought the students' ideas to the center of attention, which is characterized as *noticing*, a *focusing* action (12). In **Excerpt 4**, Sophie continued the conversation with attention to the students' thoughts and attempted to promote students' reasoning by focusing on their findings when addressing details already highlighted by Oscar. After the teacher-student interaction, Olivia and Oscar continued their same interactional pattern: Oscar continued making suggestions, which Olivia translated into actions on the laptop.

Lucas in Interaction with Leah and Isaac–Progressing Actions

Prior to the conversation with Lucas, Leah and Isaac had found a perpendicular pair of linear functions and agreed upon the relationship entailing a slope number with the same number, only with different signs: -1 and 1. In the following teacher-student conversation, Leah initiated the conversation, asking their teacher, Lucas, if -1 and 1 were opposite numbers.

Excerpt 5

(10) Lucas	I don't know if it [opposite numbers] is a concept.
(11) Isaac	But we want to formulate that the slope number for this line (pointing to the laptop screen) is x, and this is -x.
(12) Leah	Is the negative number of... umm.
(13) Lucas	Yes, rather that.
(14) Isaac	Is the same slope number, only negative.
(15) Leah	Is the negative number of the slope line... err (frustrated). Yeah, but this slope line (pointing), no, this line has the slope number. And this line (pointing) has the negative of the slope number.
(16) Lucas	Yes, that's better than saying opposite numbers. But how can you be sure that they're perpendicular to each other? Can you make sure about that in GeoGebra? Maybe do that first.

Lucas's initial response, "I don't know if it is a concept" (10), partially answered the student-pair's question, but more importantly, Isaac and Leah continued to explain what they meant by opposite numbers and what they attempted to find (11-15). Leah and Isaac engaged in a dialogue discussing the relationship between the linear functions. This presented an opportunity to address mathematical properties, such as slope number, constant, algebraic expression, graphical representation, and coordinate system. However, Lucas was *progress*-oriented and did not use the students' reasoning. Nevertheless, Lucas provided positive feedback on their use of concepts and said that their formulations were better than "opposite numbers" (13, 16). Then he asked them to use GeoGebra to prove the functions' perpendicularity (16). Lucas acted to move the process forward by asking for *open progress details*: "But how can you be sure that they're perpendicular to each other?" Such a question may have several possible answers and be aiming for progress. In the same line, Lucas also said, "Can you make sure about it in GeoGebra? Maybe do that first." Thus, he *simplified* the question by adding a step for the procedure of investigating perpendicularity between the linear functions.

Further into the unfolding situation, Isaac and Leah attempted to use GeoGebra to investigate the linear pair's perpendicularity. However, they did not succeed at first. Lucas responded and said that

> "I think we can say that [they're perpendicular] about the two lines upon each other. However, I think we have a small challenge... (pointing to the laptop screen). If you add a couple of points on every line, then you can make line segments between them. That might be hard work."

Lucas *simplified* by suggesting making another input using GeoGebra to examine the perpendicularity before he left the conversation. Lucas used Isaac and Leah's reasoning to guide them in their problem-solving process of the function problem. He particularly evaluated their input in GeoGebra. Moreover, Lucas focused on *simplifying* Isaac and Leah's reasoning path by giving hints for validating their findings, probably to pull them in the direction of investigating the connection between the slope numbers.

Isaac and Leah were engaged the entire time during the teacher-student conversation. When Lucas left the conversation, they kept their dialogue going. Isaac and Leah managed to use GeoGebra to confirm that the linear functions were perpendicular to each other. In the situation that followed, Isaac said he wanted to make a formulation for their findings, and Leah said they should test for other pairs of lines before attempting to formulate a rule. However, Isaac suggested a rule anyway, and observed that his assumption was not going to work for every number; hence, it would not work for every slope number. For a couple of minutes, Isaac was testing different linear functions in GeoGebra, and Leah observed his actions. Then, Isaac tested two linear functions with the slope numbers 2 and -0.5. However, they did not evaluate their result, and started testing several other linear functions.

In **Excerpt 5**, Lucas acted with progressing actions aiming for an evolving problem-solving process. His actions focused on (1) students' use of the concept slope number and (2) simplifying students' strategies by adding steps to the solution method. Consequentially Leah and Isaac used GeoGebra to confirm that the linear functions were perpendicular to each other. In the further unfolding events Isaac and Leah wanted to pursue different solution paths. Isaac acted with primary agency and neglected Leah's suggestion, which seemed to exclude Leah from making her thoughts visible. Therefore, Leah is placed in a position as a secondary agent where she observed, evaluated, and seemingly attempted to understand Isaac.

DISCUSSION

Initially we asked: What are the opportunities and limitations of teacher actions for the productivity of students' interactional patterns? In the study, three teachers interacted with the four student-pairs. Sophie, who interacted with Olivia and Oscar, primarily focused on the students' thoughts and reasoning for their suggested ideas and actions. Thus, her teacher actions were categorized as *focusing actions* (Wood, 1998). Sophie did *enlighten details* (Drageset, 2014) through her gathering of information and request for the students' explanations. Sophie requested a response from both students in a pair when asking about their performed actions in GeoGebra or when specifically asking about details brought into the conversation by the students. Sophie contributed to making details in the students' mathematical reasoning explicit, thus interacting with powerful teacher moves (Franke et al., 2007) for promoting a learning environment where students actively engage in problem-solving and construction of their own understandings (Stein et al., 2008). In the teacher-student interaction with Olivia and Oscar, Sophie mainly interacted with the primary agent, Oscar. Consequentially, it is likely that it was Sophie's focusing actions that facilitated reasoning from the primary agent, which was translated by the secondary agent into actions in GeoGebra.

Opportunities and limitations in Sophie's actions in teacher-student interactions are related to the category of focusing teacher actions (Drageset, 2014). Sophie's attention to the students' thoughts and input is a foundational aspect of supporting students in learning mathematics through their own attempts to make sense of mathematics and explore mathematical ideas (Norqvist et al., 2019), such as the linear function concept. Sophie provided timely support (van de Pol et al., 2018) when the student-pairs needed feedback in their problem-solving process. However, she primarily interacted with the primary agent and missed the opportunity to support both students in mathematical reasoning and the development of collaborative processes. In the interactions with Olivia and Oscar, Sophie had the opportunity to engage in other aspects of focusing actions, such as *assessment from other students*, where she could have promoted a collaborative interaction. If Olivia were presented with an opportunity to evaluate an answer or an idea, she could have refused to answer or attempted to contribute to the dialogue. For the latter outcome, it would be crucial for Sophie to both provide timely support, probably more extensive in the beginning and reducing over time, as suggested by van de Pol et al. (2018). However, Howe et al. (2007) say that a teacher should be relatively non-directive with the guidance, therefore Sophie could start with requesting Olivia to share details of what they had found, which is *closed progress details*. From initiating a student's talk, Sophie could further have progressed with *focusing* action where Olivia would have had the opportunity to *explaining why* and *justifying* an idea, thus, engaging in collaborative processes of building and monitoring and encouraged to use CMR. Moreover, it could facilitate for a change in the pair's agency dynamics, where potentially Olivia could feel encouraged to participate with shared agency.

Sophie was present and engaged in interacting with the students, but she did not encourage the students to explain to each other, which is emphasized as important for developing students' exploration and autonomy (Hufferd-Ackles et al., 2004) and crucial for constructing and maintaining a shared conception (Roschelle & Teasley, 1995). Olivia and Oscar remained in a one-directional interaction throughout the problem-solving process, which can have an impact on their learning potential. There were probably more learning opportunities for Oscar who frequently used CMR, compared to Olivia, who did not. However, that does not mean that Olivia did not learn the mathematics involved, or other skills, such as using GeoGebra as a useful tool or other social aspects of interacting with a peer. It indicates that students with secondary agency is not engaged with anchoring their arguments in mathematical properties, which potentially can lead to learning of the mathematics involved. However, Sophie's request for the students' input and thoughts could have been a fruitful start for encouraging students to interact with each other, and it would have been interesting to observe Sophie interacting with other student-pairs with different group dynamics than Oscar and Olivia's. Since a teacher's guidance is shaped by students' mathematical ideas and contributions (Staples, 2007), it is probably the case for who and how a teacher reacts and respond to a group's dynamics, such as Sophie's main interaction with the primary agent Oscar.

Lucas was acting with *progressing actions* (Drageset, 2014) when interacting with Leah and Isaac. The conversation began when Leah asked Lucas about "opposite numbers," indicating slope numbers with different signs, to which Lucas answered, but he refrained from elaborating on the answer. Consequentially it possibly made an opening for Leah and Isaac to engage in a math-talk, which became a short discussion about the relationship between the linear functions. It is likely that Lucas entered the conversation at the right time, and he had the opportunity to support the students' learning in a timely manner (van de Pol et al., 2018), but as the conversation progressed, Lucas missed the opportunity to focus his actions on the students' reasoning. He moved the problem-solving process forward with progressing actions (Drageset, 2014): *open progress details* about validating the linear function's pair perpendicularity, and *simplification* by adding steps in a procedure (Lithner, 2008) for solving a sub-question in GeoGebra. A main focus on progressing action limits the students' opportunities to produce mathematics (Schoenfeld, 2013) through building their own theory sequences of arguments (Lithner, 2017). Lucas commented on and evaluated the students' inputs in GeoGebra; therefore, he missed an opportunity to support the students' own interpretations of GeoGebra's feedback, which is important in order for students to construct shared reasoning pathways (Olsson, 2018).

Similar to Olivia and Oscar, Leah and Isaac interacted in a one-directional way. During the conversation with Lucas, both Leah and Isaac were asked questions and explained their interpretation of “opposite numbers.” Thus, both the secondary and primary agent were engaged, unlike in the interaction between Sophie, Olivia, and Oscar, where Oscar mainly responded to Sophie’s teacher actions. Despite the differences seen in the teacher-student interactions concerning the one-directional pattern, we observed that neither of the teachers’ approaches changed the students’ interactional dynamics concerning their agency in the conversations that followed. We propose that there are more opportunities for learning, in general, for the primary agents if a teacher mainly acts toward them. Consequentially, such actions strengthen the one-directional relationship between the students, and the primary agent is given more authority in the development of the collaborative processes and ownership of the reasoning process.

In the teacher-student interaction between Lucas, Philip, and Noah, we observed *redirecting* and *progressing teacher actions* (Drageset, 2014) where Lucas discarded the students’ suggestions and comments about their problem-solving process and findings. His teacher action was to *put aside* the students’ ideas and channel the focus into something else. In this case, Lucas aimed his focus on the student responses concerning more precisely formulated mathematics. However, Lucas’s way of acting could have been an attempt to *challenge* (Alrø & Skovsmose, 2004) the students’ perception of the rule for making linear functions perpendicular. Philip and Noah mutually engaged in problem-solving, making a shared understanding and having shared agency. We believe that their shared agency was a central component of the perseverance of their productive collaborative interaction.

Although a fruitful result concerning Philip and Noah’s continuation of building their shared understanding was seen, we would not expect such an outcome in general, as Lucas’s tone was harsh and his approach could be perceived as intimidating, thus probably influencing the students’ willingness to approach the teacher or collaboratively explore mathematical properties to solve the linear function problem. It seems that Lucas’s way of interacting with Philip and Noah stood out compared to his interactions with Leah and Isaac. Perhaps Lucas saw potential in Noah and Philip’s mathematical thinking, and he wanted to challenge them to make their thinking even more explicit.

Lucas’s way of interacting with Philip and Noah sheds light on how difficult it is, even for teachers as experienced as he, to promote a classroom community facilitating collaboration, shared agency, and mathematical reasoning. There is still a need for better understanding of how teachers can support students’ reasoning for building mathematical understanding (Ayalon & Hershkowitz, 2018; Maher et al., 2018; Stockero et al., 2019), but even more so, there is a need to make experienced and new teachers aware of the impacts their own actions have when interacting with student-pairs or groups.

The teacher-student interaction seen between Jacob, Hannah, and Emma is characterized as *progressing* conversations (Drageset, 2014). Jacob probably aimed at moving the process forward by asking *open progress details* questions and responding with *closed progress details*, as well as *evaluation of solution/strategy* and *encouraging of testing strategy* regarding the students’ inputs in GeoGebra. Several times, Hannah and Emma, who were in a bi-directional interaction, attempted reasoning about mathematical properties of the problem, but this was interrupted by Jacob. With his funneling actions, he seemed to pull the students toward important aspects (the Topaze effect, cited in Brousseau, 2006) of making a pair of perpendicular functions: (1) connecting the algebraic expression with the graphical representation and (2) zooming in on the slope number.

Opportunities for productive interactions with Jacob’s teacher actions were through following a given procedure for what to focus on and how to look for a path to further generalize a rule from a pattern between the linear function pairs in GeoGebra; thus, they were in line with timely support making new information easier for the students to access (van de Pol et al., 2018). Jacob’s support resulted in the students’ testing of different pairs of slope numbers in GeoGebra, moving them closer to discovering a pattern to generalize into a rule. At the same time, primarily leading students toward the solution method without asking them to justify their answers or explain why (*justification action*) limits students’ opportunities to think for themselves and discover central mathematical properties for connecting pieces of knowledge of functions into a robust function (Best & Bikner-Ahsbahs, 2017).

In summary, the teachers’ actions influenced students’ interaction, but mainly their reasoning. The specific teacher actions influenced in following ways: (1) Sophie’s *focusing actions* contributed to new suggestions for solving the problem made by the primary agent, and implemented into action by the secondary agent, (2) Lucas interacted differently with two pairs; first, with *progressing actions* channeling students’ focus to further investigating the connection between the slope numbers, which the primary agent responded to when suggesting a rule, then by guessing and checking in GeoGebra, which was observed by the secondary agent. Secondly, with *redirecting* and *progressing actions* Lucas channeled students’ focus to more precisely formulated mathematics, which resulted in collaboratively built reasoning anchored in mathematical properties of the linear function. However, as previously stated, we believe that the productivity of the interaction was a result of students’ shared agency, and not because of timely or supportive teacher guidance. In the last teacher-student interaction, (3) Jacob’s *progressing action* contributed to channel students’ focus to the slope number, which resulted in guessing and checking in GeoGebra by both students. Moreover, both students were less engaged in anchoring their guesses in intrinsic mathematics. Their conversation included less reasoning and less repairing since they did not make any claims about their different suggestions.

A teacher’s guidance is shaped by students’ contributions (Staples, 2007). The three teachers in this study responded to uttered reasoning and actions made in GeoGebra, and their guidance were consequentially shaped by that. Even though the teachers wanted to encourage collaborative work and reasoning, uttered in conversations, they primarily responded to the mathematical ideas of the conversation, not the social aspect of collaborative processes and exercised agency. Yackel and Cobb (1996) argue that students’ reasoning and sense-making cannot be separated from their participation in making a shared mathematical understanding of the problem. Moreover, Yackel and Cobb (1996) delineate normative norms from sociomathematical norms: where normative norms are “general classroom social norms that apply to any subject matter”, whereas sociomathematical norms are “what counts as an acceptable mathematical explanation and justification” (p. 460-461). We could view social norms as collaborative processes and agency for a collaborative environment, and the sociomathematical norms as students' creative mathematical reasoning. We could further argue that teachers' actions influenced students' sociomathematical norms and not their social norms. Since the students' reasoning was influenced, but their participation was not, consequentially, we highlight the importance of teachers' awareness, not only of the mathematical content in a conversation, but also students' roles in an interaction, which emphasize the importance of both normative norms and sociomathematical norms for students' productive interactions.

When assessing students' collaborative processes, a central focus is the quality of students' interactions (Child & Shaw, 2018; Francisco, 2013), and we propose evaluating students' way of participating through different agencies, as important for understanding opportunities for students' interactions. As teachers or as teacher-educators an indication of productive interaction is to look for (1) turn-takings, where students listen to one another and build arguments depending on actions and ideas of each other. We argue that this is an important aspect for recognizing a shared agency (Mueller et al., 2012). Turn-takings in students' dialogue are critical to establish a shared understanding from an interplay of ideas in a conversation (Barron, 2000; Martin & Towers, 2015; Sidnell, 2010). Another indication to notice is (2) plausible arguments, where both students make their thinking visible through speech and actions, and base their ideas, suggestions, and explanations on mathematical concepts and relations. Thus, it is an indication of coordination of language and actions (Sarmiento & Stahl, 2008), as well as CMR (Lithner, 2017) important for learning through their own, and as mutual, attempts to construct mathematical knowledge (Maher et al., 2018; Norqvist et al., 2019). As observed when a student pair was mutually engaged in plausible arguments, their reasoning was often by both students, anchored in mathematics in all instances of collaborative process. This indicate a particularly important aspect for driving the problem-solving process forward and was often seen in situations of repairing.

CONCLUDING THOUGHTS

Following three teachers' interactions with four dyads, and building on a previous analysis of the dyads' interactional patterns (Hansen, 2021), we have discussed the importance of teacher actions for students' opportunities of productive interactions through mathematical reasoning, collaboration, and agency. Jacob and Lucas acted with *progressing actions* which resulted in less reasoning anchored in mathematical properties and more guessing and checking in GeoGebra (Hannah and Emma, and Leah and Isaac), where both student pairs maintained their interactional pattern as respectively bi-directional and one-directional, after the teacher interaction. However, another outcome from Lucas's progressing actions was seen when he acted with *redirecting* and *progressing actions* which could have had an impact on the continued creative reasoning where students maintained their bi-directional interaction after the teacher interaction (Philip and Noah). Sophie's teacher action was *focusing actions* which also could have impacted creative reasoning by the primary agent observed and acted on by the secondary agent (Olivia and Oscar), where students continued their one-directional interaction after the teacher interaction. Moreover, the students who were initially engaged in a bi-directional or a one-directional interaction did not change their ways of interacting after interacting with their teacher. Neither the nature of the collaborative process nor the students' agency seemed to be particularly influenced by the teachers' actions. One explanation for this pattern might be that, to a large extent, the teachers approached the primary agent in their interactions with the one-directional dyads, thus mainly initiating a conversation between the teacher and the primary agent. On the other hand, we observed that the dyads with a shared agency and productive collaboration maintained this interactional pattern in spite of the teachers' funneling actions.

Although the teachers wanted to encourage collaborative work and reasoning, their actions were shaped by students' mathematical ideas, not the social aspects of collaborative processes and exercised agency. Furthermore, concerning these aspects it is important that teachers notice students' roles, so that teacher actions potentially can promote reasoning in collaborative processes such as monitoring and repairing and co-construction of arguments in shared agencies. These findings can probably apply to different collaborative instances with different mathematical problems, or possibly in other subjects. If applicable to other contexts, a teacher's action for more productive student interactions should evaluate a student's role and aim to promote students' shared agency through turn-takings and making plausible arguments.

It is important for both students in a pair to engage in the following actions: reasoning where they justify their arguments and attempt to pose different claims, where both should be encouraged to make counterarguments or justifications by anchoring the suggestions in mathematical properties. Teacher actions facilitating the mentioned students' actions are important for their engagement in the collaborative processes of monitoring and potentially repairing, promoting students to become authors of mathematical ideas (Langer-Osuna et al., 2020). With such outcomes of teacher actions, we suggest that teacher authority (Langer-Osuna, 2018) could potentially afford students' shared agency through important collaborative processes for progress in problem solving processes and construction of their own mathematical knowledge. However, while we propose the importance of what could have been done to facilitate such outcome, it is yet to be studied what kind of teacher actions, as suggested by Drageset (2014), that would influence a productive interaction seen in students' exercised agency and collaborative processes. This is an issue for further research. We acknowledge a teacher's challenge in observing students' interactions. However, when situations allow for observation, we think the two mentioned aspects, properties for turn-taking and plausible arguments, as often observed in bi-directional interactions (Hansen, 2021), may be an important indication for evaluating opportunities for productive student interactions. Which is a step towards a better understanding of how to facilitate mathematical reasoning and argumentation in classrooms (Maher et al., 2018; van de Pol et al., 2018).

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