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Development of a Survey to Assess Conceptual Understanding of Quantum Mechanics among Moroccan Undergraduates

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Abstract: We developed a Quantum Mechanics Conceptual Understanding Survey (QMCUS) in this study. The survey was conducted using a quantitative methodology. A multiple-choice survey of 35 questions was administered to 338 undergraduate students. Three experienced quantum mechanics instructors examined the validity of the survey. The reliability of our survey was measured using Cronbach's alpha, the Fergusson delta index, the discrimination index, and the point biserial correlation coefficient. These indices showed that the developed survey is reliable. The statistical analysis of the students' results using SPSS shows that the scores obtained by the students have a normal distribution, around the score of 7.14. The results of the t-test show that the students' scores are below the required threshold, which means that it is still difficult for the students to understand the concepts of quantum mechanics. The obtained results allow us to draw some conclusions. The students' difficulties in understanding the quantum concepts are due to the nature of these concepts; they are abstract and counterintuitive. In addition, the learners did not have frequent contact with the subatomic world, which led them to adopt misconceptions. Moreover, students find it difficult to imagine and conceptualize quantum concepts. Therefore, subatomic phenomena are still explained with classical paradigms. Another difficulty is the lack of prerequisites and the difficulties in using the mathematical formalism and its translation into Dirac notation.

Keywords: *Conceptual understanding, learning difficulties, quantum mechanics, teaching/learning.*

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Introduction

Due to the immense applications of quantum concepts in recent times, the teaching of quantum mechanics has been integrated into several cycles and fields of university, engineering, and high school. Nevertheless, teaching/learning quantum mechanics encounters some difficulties related to many didactic and epistemological causes, such as learner misconceptions, didactic transposition, teaching methods, and epistemological obstacles.

Indeed, quantum concepts are counterintuitive, too abstract (Ayene et al., 2011; Johnston et al., 1998; Zhu & Singh, 2011), and far from learners' daily experiences. Moreover, classical paradigms have dominated in the form of sedimentation of scientific knowledge over several years of study, creating a barrier to understanding quantum concepts (Oldache & Khiari, 2016). Mathematical formalism and Dirac notation also present an obstacle to teaching/learning quantum concepts (Sadaghiani, 2005; Singh, 2001; Singh et al., 2006). All these causes lead students to develop misconceptions about quantum concepts and hinder their understanding of the phenomena of the subatomic world.

Several studies have investigated and identified various misconceptions about quantum concepts and difficulties that students face in learning quantum mechanics (Ayene et al., 2011; Baily & Finkelstein, 2015; Bao & Redish, 2002; Fischler & Lichtfeldt, 1992; Ireson, 1999; Johnston et al., 1998; Kalkanis et al., 2003; Ke et al., 2005; Krijtenburg-Lewerissa et al., 2017; Mashhadi & Woolnough, 1999; Oldache & Khiari, 2010, 2015, 2016; Olsen, 2002; Petri &

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Niedderer, 1998; Sadaghiani, 2005; Selçuk & Çalışkan, 2009; Singh, 2001; Singh et al., 2006; Styer, 1996; Tsaparlis & Papaphotis, 2009; Wuttiprom et al., 2009; Zhu & Singh, 2011, 2012a).

Students' difficulties in learning quantum mechanics and misconceptions about quantum concepts provide a database that designers of programs and teachers of quantum mechanics should consider when planning. On this basis, it is possible to propose new strategies to overcome these difficulties and make it easier for students to understand quantum concepts. Discussion of representations and interpretations of quantum concepts will inform classroom practice to help students overcome their learning difficulties.

Literature Review

In physics education research (PER), researchers examine how well learners have understood physics concepts at the end of a lecture. This research focuses on the difficulties students have in understanding concepts in physics. Researchers interested in the conceptual understanding of physics have developed several instruments to study it. In classical mechanics, we find the Force Concept Inventory (FCI) (Hestenes et al., 1992), Force and Motion Conceptual Evaluation (FMCE) (Thornton & Sokoloff, 1998), Rotational and Rolling Motion Survey, and Energy and Momentum Survey (Rimoldini & Singh, 2005). In the field of electromagnetism, we find the Concept Survey of Electricity and Magnetism (CSEM) (Maloney et al., 2001) and Brief Electricity and Magnetism Assessment (BEMA) (Ding et al., 2006).

Once these instruments are tested for reliability and validity, teachers can use them to assess their students' conceptual understanding. Researchers might also seek to examine the effects of integrating an innovation or new curriculum into the learning process on the development of understanding. For example, the FCI test in research (Hake, 1998) has made it possible to show the effects of integration interactivity on the learning process.

In quantum mechanics research, researchers often wonder how well students understand the new concepts of quantum mechanics. So, several instruments have recently appeared in quantum mechanics research to investigate conceptual understanding (Table 1).

Table 1. Different Investigation Instruments Used in Conceptual Understanding Quantum Concepts

Instruments	Authors	Themes and Concepts Covered by the Investigation
QMCA: The Quantum Mechanics Concept Assessment	Sadaghiani, 2015	Quantum measurement, the time-independent Schrödinger equation, wave functions, boundary conditions, time evolution, and probability density
QPCS : Quantum Physics Conceptual Survey	Wuttiprom et al., 2009	Photoelectric effect, waves, particles, De Broglie Wavelength, double-slit interference, the uncertainty principle
QMCS: The Quantum Mechanics Conceptual Survey	McKagan et al., 2010	Wave functions and probability, wave-particle duality, the Schrodinger equation, quantization of states, the uncertainty principle, superposition, operators and observables, and tunneling
QFMPS: The Quantum Mechanics Formalism and Postulate Survey	Marshman & Singh, 2019	Quantum states, eigenstates of operators corresponding to physical observables, time development of quantum states, measurement, expectation value of observables, the time dependence of expectation value of observables, commutators or compatibility, spin angular momentum, Dirac notation, and the dimensionality of the Hilbert space
QMS: The Quantum Mechanics Survey	Zhu & Singh, 2012b	Wave function possibilities, bound or scattering states, measurement, expectation values, time dependence of the wavefunction and expectation values, stationary and non-stationary states, the role of the Hamiltonian, the uncertainty principle, and Ehrenfest's theorem
QMVI: Quantum Mechanics Visualization Instrument	Cataloglu, 2002	Visualization of wave function, probability, expectation value, uncertainty, an indifferent barrier of potential,and moving particles through the barrier of potential
QMCI: The Quantum Mechanics Concept Inventory	Falk, 2004	Tunneling, free particle, wave packet, and energy

The Quantum Mechanics Conceptual Understanding Survey (QMCUS) was developed using the above instruments. In the context of teaching quantum mechanics in Morocco, after several revisions of the survey with three teachers that have long experience teaching quantum mechanics, the instrument was validated and limited to 35 multiple-choice questions aimed at undergraduate students. In this context, after a review of the main textbooks and programs used in Moroccan universities for the introduction to quantum mechanics, we have defined the different concepts that must be learned and understood at the end of the lecture on quantum mechanics so that a specification of the content is made taking into account the levels of content according to a taxonomy knowledge-comprehension-reasoning, with an essential weighting of the level of understanding. Then, according to this specification, the survey covers the following topics: crises of classical physics and the birth of quantum mechanics, fundamental aspect, wave-corpuscle duality, understanding of wave function, uncertainty principle and indeterminism, postulates of quantum mechanics and the process of measurement, Hamiltonian operator and Schrödinger's equation, tunneling effect, correct use of mathematical formalism and Dirac notation.

This paper demonstrates the current conceptual understanding of quantum mechanics using the QMCUS survey. The survey is addressed to students of a Moroccan university. More specifically, our research will answer the following questions:

- What learning difficulties do undergraduate students face when studying quantum mechanics?

- What misconceptions do students have about quantum concepts?

- What causes students' difficulties in understanding quantum concepts?

Methodology

Research Design: Description of the Investigation Survey QMCUS

The investigative survey QMCUS was developed with a study that consisted of multiple-choice questions. This survey was administered to undergraduate students who had attended the Quantum Mechanics lecture. The items of this survey consisted of 35 multiple-choice questions. The objectives of this survey were to collect results, perform statistical analysis to determine the difficulty level of understanding quantum concepts, and uncover students' misconceptions about their understanding of quantum concepts. One of the goals was to find out where the breakdowns in understanding quantum concepts were concentrated.

Nevertheless, two preliminaries were necessary to judge this survey as reliable and valid. Validity consisted of submitting this survey to experts in the field, particularly teachers involved in teaching quantum mechanics, to solicit their comments and recommendations, and finally, to declare it valid. The reliability was found adequate after collecting the results and statistical analysis by Cronbach's coefficient alpha, Fergusson's delta index, discrimination index, and point biserial correlation coefficient.

Consequently, it is proposed to identify the difficulties and misconceptions about quantum concepts among students in Moroccan universities, especially in two institutes of Cadi Ayyad University. To this end, the survey was developed after a preliminary analysis of several courses on quantum mechanics in the cycle of licentiate studies of the sciences of matter physics in the Faculty of Science of Moroccan universities, then a specification of the content considering the content levels according to a knowledge-comprehension-reasoning taxonomy. Under essential weighting of the comprehension level, according to this specification, the survey covers the following topics, which are compiled in Table 2.

Table 2. Themes Covered by the Investigation

Conceptual Themes	Items
Crises in classical physics and the foundation of quantum mechanics	1,2
Fundamental aspect	3,4,5,6,7,8
The wave-corpuscle duality	9,10,11,12
Understanding the wave function	13,14,15
Principle of uncertainty and indeterminism	16,17,18,19
The postulates of quantum mechanics	20,21,22,23,24,25,26,27
The Hamiltonian operator and Schrödinger's equation	28,29
Tunneling	30,31
Mathematical formalism and Dirac notation	32,33,34,35

Sample

In this work, we opted for a quantitative research method by asking a series of 35 questions to 338 students studying at two institutions of the University Cadi Ayyad Morocco, the Ecole Normale Superieure, and the Faculty of Sciences (Table 3). After completing the quantum mechanics course, students were asked to spend two hours completing the survey.

Table 3. Information of the Studied Sample

Numberof students	Option	Institution
70	Education license option Physics-Chemistry	Ecole Normale Superieure Marrakech
141	Sciences of the matter option physics	Faculty of Sciences
127	Sciences of the matter option chemistry	Faculty of Sciences

Analyzing of Data: Validity and Reliability of the Investigation Survey, and Normality Tests of Results' Distribution

To interpret the study's results, the survey used in the study must be valid and reliable. Therefore, the survey was submitted to a professor from Ecole Normale Superieure Marrakech, who teaches quantum mechanics, and two other professors from the Faculty of Science at Cadi Ayyad University. After discussions, the survey went through several versions until the final version was suitable.

To study the reliability of a survey in PER, researchers use various indices that examine the reliability of the survey, such as Cronbach's alpha, Ferguson's delta, *KR 21*, etc. To analyze the reliability of each item in the survey, we also use the difficulty index, the discrimination index, and the point biserial correlation coefficient.

The difficulty index, which indicates the percentage of students who didn’t answer an item well, must be between 0.3 and 0.9 for the item to be considered appropriate. For the QMCUS survey items, the range of the difficulty index is from 0.14 to 0.94, with most items having a difficulty index between 0.5 and 0.7. The discrimination index of an item indicates the strength of the distinction between the best and the weakest students; it varies between - 1 and 1, and the acceptable value is > 0.3. Thus, all items have a value > 0.3, except for items 10, 11, 16, 26, and 27; these items also have a biserial point < 0.2. The biserial point for an item indicates the consistency of the item with the entire survey; according to the literature, it must be greater than 0.2; for the survey, most items have a biserial point > 0.2. Thus, we conclude that most items are reliable.

The internal consistency index Cronbach's alpha was 0.77, indicating that the study survey used is reliable. In addition, Furgeson's delta was 0.89, proving the QMCUS survey's reliability.

Thus, we can say that the survey is entirely reliable, and the items that compose it are suitable for testing the understanding of the introduction to quantum concepts. Table 4 shows the survey items' difficulty, discrimination index, and biserial point index.

Table 4. Statistical Indices

Item	Difficulty Index	Discrimination Index	Biserial Point Index
1	0.45	0.48	0.45
2	0.63	0.47	0.55
3	0.23	0.41	0.28
4	0.53	0.58	0.55
5	0.44	0.41	0.32
6	0.41	0.13	0.23
7	0.55	0.55	0.55
8	0.61	0.42	0.38
9	0.69	0.43	0.50
10	0.54	0.13	0.08
11	0.70	0.19	0.14
12	0.72	0.36	0.39
13	0.14	0.70	0.34
14	0.83	0.55	0.49
15	0.57	0.37	0.37
16	0.65	0.17	0.20
17	0.62	0.35	0.34
18	0.69	0.70	0.62
19	0.76	0.34	0.34
20	0.56	0.22	0.15
21	0.94	0.57	0.36
22	0.56	0.42	0.46
23	0.20	0.21	0.20
24	0.70	0.43	0.31
25	0.69	0.38	0.33
26	0.85	0.16	0.16
27	0.74	0.18	0.14
28	0.81	0.29	0.34
29	0.89	0.36	0.32
30	0.83	0.33	0.31
31	0.56	0.43	0.36
32	0.79	0.44	0.49
33	0.41	0.25	0.28
34	0.65	0.19	0.16
35	0.54	0.63	0.55

Statistical analysis using the SPSS package is required to determine whether or not the distribution of the students' scores (Figure 1) is normal by selecting the Kolmogorov-Smirnov and Shapiro-Wilk p-value, as indicated in Table 5. The p-value of the two tests is > 0.05, which shows that the distribution of the results is normal.

Table 5. Test of Normality

	Kolmogorov-Smirnov	**Shapiro-Wilk**
p-value	0.183	0.151

Findings /Results

Student scores are distributed according to Figure 1, which shows the intervals of scores by the number of students; Figure 2 is the boxplot of student scores, showing the symmetrical distribution around a mean of 7.14; and Table 6 gives descriptive statistics of the results of the scores obtained by students.

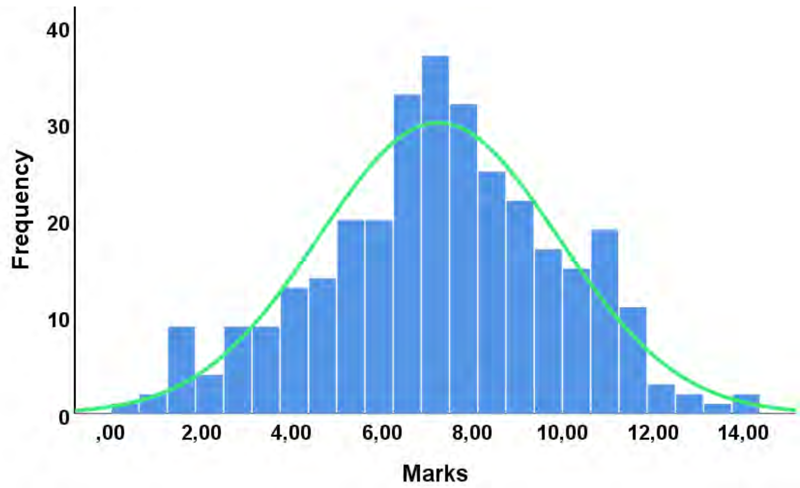

Figure 1. The Distribution of the Students' Scores.

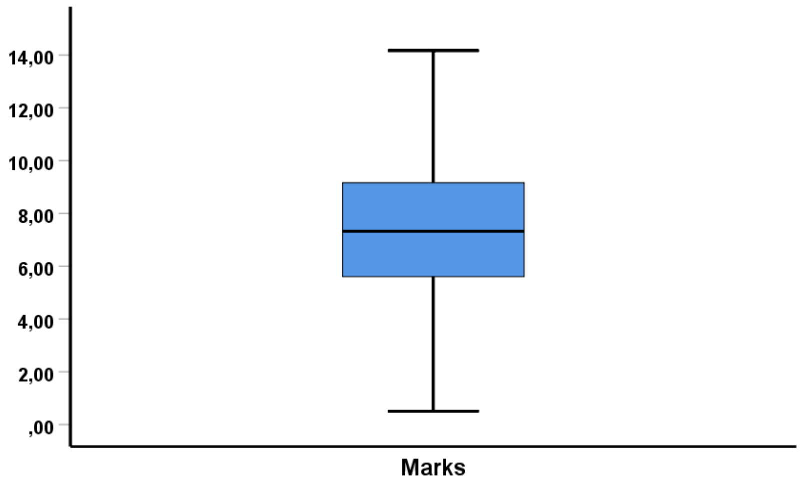

Figure 2. Box Plot of the Students' Scores.

Table 6. Descriptive Statistics of the Results

		Statistic
Marks	Mean	7,140
	5% Trimmed Mean	7,182
	Median	7,275
	Std. Deviation	2,877
	Minimum	,00
	Maximum	18,33
	Skewness	-,150
	Kurtosis	,358

The average of the obtained scores is 7.14, the three means are comparable, and the skewers and kurtosis values show that the distribution of the scores is approximately symmetrical and not flattened. First, we performed the normality test to determine that the students' scores were below the threshold. Since the distribution of scores was normal according to the tests mentioned in Table 5, we resorted to a parametric test, the t-test. The results of the t-test yielded a p-value of zero (Table 7), showing a significant difference between the grades obtained by the students and the threshold of 10.

Table 7. One-Sample Test

	Test Value = 10			
	t	**Df**	**Sig. (2-tailed)**	**Mean Difference**
Mark	-9,540	69	,000	-2,092

The results show that the students' scores are below the average of 10. Thus, these results show that students have difficulty understanding quantum mechanics concepts.

Difficulty in Understanding the Limits of Classical Physics and the Fundamentals of Quantum Mechanics

For question 1 on the Compton effect, 45% of the students did not understand the corpuscular aspect of light, considering that light is in the form of photons that enter into relativistic collisions with electrons. The most difficult question, with a difficulty index of 63% in this subject area, was Question 2, which reflects a misunderstanding of the wave aspect of electrons in the Davisson-Germer experiment.

Difficulty in Understanding the Fundamental Aspects of Quantum Mechanics and Classical Mechanics

Questions 3 and 4 were related to the characteristics of classical and quantum mechanics. Many students have no difficulty distinguishing the characteristic properties of classical mechanics. This result indicates that the classical paradigms are well learned. In contrast, 53% of students have difficulty determining the properties of quantum mechanics, such as causality and probability of measurement. Questions 5, 6, and 7 focused on understanding the corpuscle, wave, and wave-corpuscle properties. The students successfully determined the points that explain a corpuscle and a wave. However, they found it difficult to distinguish the properties of a wave-corpuscle because they misunderstood the principle of duality.

Regarding the framework of the study of a system, whether in quantum mechanics or classical physics, 61% of the students did not give the correct answers to question 8, which asks students to compare the wavelength with the object's dimensions. However, in the case of neutrons and electrons, students do not come up with the wavelength associated with the sizes of these two subatomic particles.

Difficulty in Understanding the Wave-Particle Duality

In this theme of understanding the wave-corpuscular duality, question 12 was the most difficult, with a difficulty index of 72%. This question concerns the relationship between the energy and impulses corpuscular parameters E and $\vec{p}$and pulsation and wave vector parameters ω and $\vec{k}$.

$$E = \hbar\omega \text{ and } \qquad \vec{p} = \hbar\vec{k} \qquad (1)$$

The correct answer is (a), while 72% chose either (b) or (c) (Appendix). The students could not distinguish between the corpuscular and wave parameters and did not indicate which attributes and properties declare a particle or a wave, which is found in questions 6 and 7. This result proves that the students do not acquire the prerequisites before learning quantum mechanics.

In question 11, only 47% of the students gave the correct answer. Question 11 is about the double-slit experiment, which illustrates the wave behavior of photons, electrons, etc. These particles passing through a double-slit show the phenomenon of interference with a screen.

Difficulty in Understanding the Wave Function

Question 14 was the most difficult in this theme, with a difficulty rating of 83%. Students failed to distinguish a possible wave function in the case of a particle confined in an infinite potential well. This system is well treated in class by solving Schrödinger's equation with eigenvalues. The functions found are eigenfunctions of the Hamiltonian, which correspond to the eigenvalues, which are the discrete energies of the system. These eigenstates are stationary states which check the boundary conditions. While a state can present itself in an eigenstate to another observable, such as the position observable, the state in the question is in a Dirac peak centered on the observed value of the position inside the wells, so the function presented in question is a possible function. Thus, according to this system, well studied in class, the students think that only the wave functions that satisfy this condition are the only possible functions or states expressed in sinusoidal functions, or functions represented graphically by a sinusoidal curve.

Difficulty in Understanding the Principle of Uncertainty

In our study, only 31% of students who correctly answered question 18 asked about the meaning of the uncertainty principle, which shows the difficulty in understanding the uncertainty principle.

For question 16, only 36% of the students gave the correct propositions: d, e, f, and h (Appendix). The principle of uncertainty is quoted in class in a relation that connects the position and the momentum of a particle at a given moment by the link:

$$\Delta x . \Delta p \geq \frac{\hbar}{2} \quad (2)$$

A limitation on uncertainty on two quantities does not concern the quantities' position and momentum. Nonetheless, it involves other quantities, such as time and energy, but it also links the conjugate quantities, the quantities connected by the Fourier transform. Finally, the uncertainty relation concerns the quantities their observables do not commute.

Difficulties in Understanding the Postulates of Quantum Mechanics and Measurement

We notice that this theme covered by the investigation reveals the incomprehension of the postulates of quantum mechanics. The most challenging question is question 21, with a difficulty index of 94%, and question 26, with a difficulty index of 85%.

For question 20, an observable $\hat{Q}$corresponding to a physical quantity Q, having a discrete spectrum of non-degenerate eigenvalues, the states $\{|q_n\rangle\}$ are the eigenstates of the observable $\hat{Q}$ corresponding to the eigenvalues q_n. At t=0, the state of the system is described by the ket$|\psi\rangle$ Choose the right propositions:

a. Measurement of the quantity Q gives, as a result, one of the eigenvalues of the observable$\hat{Q}$

b. The measurement of the quantity Q gives as a result$\sum q_n$

c. Immediately after the measurement, the system state reduces to any eigenstate of the observable.

d. The probability of obtaining the result q_n is$|\langle q_n|\psi\rangle|^2$

Only 44% of the students gave the correct propositions, a, c, and d.

For question 21, consider the following conversation between two students about an observable $\hat{Q}$ for a system in a quantum state $|\psi\rangle$ that is not a proper state of the observable.

Ahmed: The action of an observable $\hat{Q}$ corresponds to a physical quantity Q on a quantum state $|\psi\rangle$, corresponds to the measurement made by this observable, such that

$$\hat{Q}|\psi\rangle = q_n|\psi\rangle \quad (3)$$

With q_n is the observed value.

Said: No, the measurement reduces the state according to the following operation

$$\hat{Q}|\psi\rangle = q_n|\psi_n\rangle \quad (4)$$

With$|\psi_n\rangle$ is an eigenstate of the observable $\hat{Q}$corresponding to the eigenvalue q_n.

Which of these two students do you agree with, justifying your answer?

a. I agree with Ahmed only

b. I agree with Said only

c. I do not agree with either Ahmed or Said

d. I agree with both

e. The answer depends on the observable $\hat{Q}$

Only 6% of students could give the correct answer, which is c (neither Ahmed nor Said was right); this reveals two representations among the students carried by Said and Ahmed.

Difficulties in Understanding the Hamiltonian Observables and the Schrödinger Equation

In this topic, the two questions appeared difficult with an index of greater than 81%, which shows the incomprehension of the concept of the Hamiltonian observable and the Schrödinger equation.

In a question about the most fundamental equation in quantum mechanics, only 12% of students could choose the correct answer, b, according to equation 5.

$$H|\psi\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t} \qquad (5)$$

Which is time-dependent. The Schrödinger equation is the most fundamental concept in quantum mechanics. We can say that it is the equivalent of the basic principle of dynamics in classical physics. The answer that expresses the Schrödinger equation independent of time is:

$$H\Psi = E\Psi \qquad (6)$$

Is valid only if the wave function represents an eigenstate of the Hamiltonian, which we call the equation with eigenvalues. For equation 7:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x) \qquad (7)$$

It is a special case when the potential is null, and the system is monodimensional.

Difficulties in Using the Mathematical Formalism and Dirac Notation

Questions 32-35 examined students' difficulties with mathematical formalism and Dirac notation. For question 34, all propositions were correct. In contrast, only 35% of the students could choose the right propositions. In this question, an observable action on states is investigated using Dirac notation, focusing on understanding the position and momentum. We have the observable $\hat{P}$which corresponds to the momentum. The eigenstates of this observable are the kets $\lfloor p\rangle$ with eigenvalues p, then:

$$\hat{P}\lfloor p\rangle = p\lfloor p\rangle \qquad (8)$$

And we have $$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} \qquad (9)$$

For question 35: Assume that $\{|q_n\rangle, n = 1,2, \dots\dots, \infty\}$forms a complete orthonormal basis consisting of the eigenstates of an observable $\hat{Q}$ that corresponds to a physical quantity with q_n are the non-degenerate eigenvalues. $\hat{I}$ is the unit operator. Choose among the following propositions the correct ones.

a. $\sum\lfloor q_n\rangle\langle q_n| = \hat{I}$ (10)

b. $\langle\psi|\hat{Q}|\psi\rangle = \sum q_n|\langle q_n|\psi\rangle|^2$ (11)

c. $\langle\psi|\hat{Q}|\psi\rangle = \sum q_n\langle q_n|\psi\rangle$ (12)

54 % of students fail to give good answers, which are a and b.

Discussion

Difficulties in Understanding the Limits of Classical Physics and the Foundation of Quantum Mechanics

Generally, all quantum mechanics programs present some phenomena that classical physics could not explain, such as radiation of the black body, photoelectric effect, Compton Effect, electron diffraction, etc. However, the prerequisites have not yet been acquired. For example, the Compton Effect requires the module of special relativity. The Davisson-Germer experiment requires the wave optics course, which explains the phenomenon of diffraction manifested by the electrons. The radiation of the black body requires the module of statistical physics. Thus, students are expected to find still it challenging to understand the limits of classical physics and quantum mechanics. Furthermore, the corpuscular aspect of light was not well understood after the photoelectric effect and did not translate to explain the Compton Effect. On the other hand, at this stage of introduction to quantum mechanics, it is not easy for the students to assimilate that the electrons (matter) behave like waves in the Davisson-Germer experiment.

Moreover, studies show that students have difficulty juxtaposing the behavior of waves and particles. For example, research has shown that many high school and undergraduate students mistakenly see electrons exclusively as particles and photons as bright spherical balls with defined trajectories (Greca & Freire, 2003; Hubber, 2006; Ireson, 1999, 2000; Mashhadi & Woolnough, 1999; Müller & Wiesner, 2002; Olsen, 2002).

Students find it difficult to imagine the wave aspect for the particles of matter and the corpuscular aspect for light; this explains the persistence of classical paradigms. Moreover, students still try to explain quantum phenomena through a classical physics vision.

But suppose we avoid the teaching of the introduction of quantum mechanics, the crises of classical physics as a starting point, without sensitizing students to transition from the classical paradigm to the quantum paradigm. In that case, the likelihood of developing erroneous misconceptions on the part of students is high (Gil & Solbes, 1993).

There is another approach to teaching the introduction of quantum mechanics, which is based on the concepts of analytical mechanics, such as the Hamiltonian, the action, and the Poisson brackets. Then, from the correspondence principle, we integrated them into quantum mechanics (Oldache & Khiari, 2010; Yang, 2006).

Difficulties in Understanding the Fundamental Aspects of Quantum Mechanics and Classical Mechanics

A large majority of students find it difficult to determine properties of quantum mechanics, such as causality; they do not understand the meaning of causality in classical and quantum mechanics and even think there is no causality in quantum mechanics. This answer is wrong because the Schrödinger equation represents the causality of the evolution of a quantum particle. Moreover, learners confuse indeterminacy and non-causality as two different concepts: the first is related to the Heisenberg principle; however, causality is connected to the Schrödinger equation.

On the other hand, among the aspects of quantum mechanics, probability and statistics, students fail to make a relation between the measurement process, which is probabilistic, and the statistical distribution of measurement due to not understanding the postulate of quantum measurement. Another confusion among students is that the spectrum of observables is always quantized. However, there are observables with a continuous spectrum and others that are continuous and quantized depending on the system studied.

Most students do not understand the attributes declaring corpuscle, wave, and specially wave-corpuscle. Nevertheless, students successfully determined the details expressing a corpuscle and a wave. However, they found it difficult to distinguish the attributes of a wave-corpuscle concerning the misunderstanding of the principle of duality. Students do not understand the distinct framework of the study of the system. It is in quantum mechanics or classical physics; the first criterion is the system's action; if it is comparable to the value of Planck constant *h*, then the system will be studied in quantum mechanics. The second criterion is the De Broglie wavelength; when the particle is in motion, it will calculate the associated wavelength according to De Broglie. If it is greater or equal to the dimensions of the particle, we will study the system in quantum mechanics. If the wavelength is much less than the dimensions of the particle, then the system will be sufficiently situated in the study of classical physics.

To sum up, these difficulties could be attributed to a lack of knowledge prerequisites. The properties of a wave are not assimilated because the students have not yet learned the wave optics modulus and the wave modulus, which presents a knowledge gap hindering them from understanding this thematic part fully.

Moreover, the learners still find it strange to juxtapose two different corpuscular and wave aspects to the same particle, and they cannot build the wave-corpuscular concept (Ayene et al., 2011; Fischler & Lichtfeldt, 1992; Johnston et al., 1998; Mashhadi & Woolnough, 1999).

With the persistence of the classical paradigm, students still imagine and conceptualize quantum phenomena from a classical point of view. On the other hand, even if they accept the quantum theory, they develop some misconceptions, such as considering that all physical quantities are quantized. In contrast, some quantities have a continuous spectrum of eigenvalues. Furthermore, the students do not assimilate the meaning of specific properties such as probability, causality, and indeterminacy in quantum mechanics, which means that they do not understand the physical sense of the postulates of quantum mechanics; they only accept them. Moreover, students find it counterintuitive to juxtapose two properties: causality expressed by the Schrodinger equation and non-causality in the measurement process. These difficulties are already cited in previous works (Bao & Redish, 2002; Johnston et al., 1998; Marshman & Singh, 2015; Sadaghiani, 2005; Singh, 2001; Singh et al., 2006; Singh & Marshman, 2015; Zhu& Singh,2012a; Styer,1996).

Another epistemological cause is the debate between academic scientists and schools of interpretation around these concepts of probability and causality (Jaynes, 1990), which leaves some confusion among learners.

Difficulty in Understanding the Wave-Particle Duality

The behavior of duality is challenging to understand among students. According to studies (Ayene et al., 2011; Ireson, 1999, 2000), the origins of these difficulties are that students carry three categories of understanding of the concept of duality. The first category of conceptualization is classic. In this description, the students consider quantum particles as waves or corpuscles. The second category is a mixed description in which the learners think of the coexistence of two entities, waves and corpuscles. The third category is a quasi-quantum description, in which students understand that quantum objects can behave both like corpuscles and waves. In another study (Didiş Körhasan & Miller, 2019), the authors detected four categories of mental models among students about duality, Quantum conceptualization, semiquantum conceptualization, wave conceptualization, and intermediate conceptualization; and the study showed that 2/3 of the students find it difficult to distinguish between quantum and classical ideas.

Understanding double-slit experience depends partly on learners' representations of wave and particle behavior. Some learners see photons as classical particles with defined trajectories, influencing their understanding of the experience. For example, some learners consider the photons to deflect at the edges of the slit and travel in a straight line towards the screen (Dutt, 2011). Another common problem is an incomplete understanding of the associated De Broglie wavelength. Students do not always understand the influence of speed and mass on wavelength and the impact of length waves on the interference figure (Dutt, 2011; Vokos et al., 2000). Studies show that students' misconceptions fall into three categories (Krijtenburg-Lewerissa et al., 2017). In the classical category, students believe that light has no impulse, and those electrons and photons pass through a slit deflect by moving on linear trajectories. Students think there is no relation between the momentum and the Broglie wavelength in the mixed category. The interferences do not appear by singular photons. In the third category, learners think there is no relation between the impulse and the interference figure.

Difficulty in Understanding the Wave Function

The wave function represents a key concept in quantum theory; it means all the information required on the quantum particle, it constitutes the first postulate of quantum mechanics, mathematically it is represented by a vector in the space of Hilbert, then its comprehension is primordial. But students found it challenging to understand the meaning of the wave function. The origins of these difficulties are diverse, epistemological, didactic, and historical. At the didactic level, the notion of wave function presents itself to students on many levels, complicating their understanding. The historical basis of the wave function has known several progressions, firstly with the contribution of De Broglie by associating any moving particle with a wave. However, this wave remains mysterious; what does it represent? According to the supporters of De Broglie, it is a guiding wave that explains this false representation among students. The contribution of Born, who proposed the probabilistic meaning of the wave function, was generalized by considering the wave function as a mathematical tool representing the information required by the state.

In a comparable study (Oldache & Khiari, 2015), the authors found that 85% of the students thought of a real wave that guides the particle in its movement (49%) and others considered that it is a wave emitted by the particle during its movement (36%) by making an analogy with electromagnetic waves.

In a previous study (Zhu & Singh, 2012b), the same question 14 was asked to 226 students in an American university. The result shows that 60% of the students did not understand that the schematized wave function presents a possible wave in a well of finite potential. The confusion that students had here is that they did not distinguish between the eigenstates of the observable position schematized in this case and the eigenstates of the Hamiltonian resulting from the resolution of the time-independent Schrödinger equation. Then, the misconception that students had is that they distinguish only the eigenstates of energy (Hamiltonian observable), although the states can be presented as eigenstates or superposition of the eigenstates of any observable.

Difficulty in Understanding the Principle of Uncertainty

Generally, the students find it challenging to understand that specific conjugate quantities cannot be determined simultaneously with precision only to a particular limit, such as the position and the momentum of a quantum particle, according to equation 2. The accuracy given on one quantity corresponds to the decrease in precision.

The uncertainty principle has a broader meaning; it expresses the non-commutation of two observables. Indeed, if two observables do not commute, they do not share the same base constituted by their eigenstates. Then, if we measure by one of the two observables, it gives us an eigenstate which includes a superposed state of the other; it is like the precision on the first quantity makes it lose the accuracy on the second quantity.

The origins of these difficulties are that only in progress the Heisenberg principle appears as a relation that concerns only the momentum and the position. Another cause is that the other quantities are not treated in class. Moreover, the students did not transpose this principle into mathematical formalism, especially using the commutator notion. Furthermore, students confuse the meaning of uncertainty in classical with its importance in quantum mechanics. In classical physics, uncertainty is an extrinsic property. The origin of uncertainty in classical physics is related to devices and experimenters, while the principle of uncertainty in quantum mechanics reflects the subatomic nature.

However, due to teaching practices that do not present the uncertainty principle, students adopt several erroneous misconceptions about this principle. Indeed, students often have false representations about the uncertainty principle; they explain the imprecision related to the position of a particle because of its high velocity (Singh, 2008a). In another study (Zhu & Singh, 2012b), students thought, according to the uncertainty principle, that the uncertainty of the position is smaller when the expected value of the momentum is immense. Other students stated that the expected value of the position is enormous when the expected value of the momentum is small. In the study (Ayene et al., 2011), the researchers using a phenomenographic investigation, found that students adopt four categories of conceptualizations of the uncertainty principle. They are (1) uncertainty is an extrinsic property of the measurement, (2) the principle of uncertainty related to errors or uncertainty in the measurement, (3) uncertainty is a consequence of perturbation of the measurement, and (4) uncertainty is a principle of quantum mechanics.

Question 18 was proposed to Algerian students (Oldache & Khiari, 2015) to know the students' misconceptions about the kinematic properties of microscopic objects. The results show that only 20% of the students questioned thought that this principle represents an intrinsic property of nature, against 68% who believed that it is due either to an imperfection of the measuring instruments (38%) or to a limitation of human knowledge (28%).

Then to help the students to build a correct conception of the uncertainty principle, it is necessary to present in class some illustrations of the principle as the phenomenon of diffraction by a slit, and the limitation of the wave packet in space which is the principle can explain. The principle must be generalized to all the conjugated quantities; later, by advancing in the course of quantum mechanics, it is necessary to explain the principle by the concept of the commutator. Also, the discussion of the students' interpretations avoids adopting misconceptions.

Difficulties in Understanding the Postulates of Quantum Mechanics and Measurement

Question 20 reflects the lack of understanding of the probabilistic measurement process and badly circumvented reduction principle. It has an epistemological similarity with the principle of reduction and the acausal jump during the measurement of an eigenstate of the observable, which we made the measurement. We can say that there is a resemblance to the misconceptions of students and schools of interpretation that have criticized this vision of measurement and reduction stemming from the Copenhagen school.

For question 21, the same result was found in the study (Zhu & Singh, 2012a), with 29% of students who chose the correct answer in an American university; this reflects that these difficulties have a universal aspect. The first misconception among students is that learners believe that an observable action on any state corresponds to a measured eigenvalue corresponding to the state (Ahmed's answer). At the same time, this is false because the state is not an eigenstate.

The second misconception among the students is that they think, like Said, that the state is reduced, but with false reasoning (equation 4), whereas if:

$$|\psi\rangle = \sum C_n |\psi_n\rangle \quad (13)$$

Then $\hat{Q}|\psi\rangle = \hat{Q}\sum C_n|\psi_n\rangle = \sum C_n q_n |\psi_n\rangle$ (14)

This result leads to the fact that $\hat{Q}|\psi\rangle \neq q_n|\psi_n\rangle$ (15)

The study asked the same question (Singh &Marshman, 2015) in the case of the Hamiltonian operator; the researchers found that only 23% chose the correct answer, which is close to our result. Responses like this indicate that even after instruction, students have a deeply held misconception that the measurement process and state reduction are represented by an equation of the type discussed by Ahmed and Said.

The other difficult question was question 26, with a difficulty index of 85%. The spectrum of possible energies is a discrete spectrum for a particle confined in a well of infinite potential. Knowing that the state of the particle is a superposed state of the energy eigenstates, if we measure the energy, we will find one of the eigenvalues of the energy corresponding to the eigenstates composing the state of the particle. Thus, for the state $\sqrt{\frac{2}{7}}\psi_1(x) + \sqrt{\frac{5}{7}}\psi_2(x)$, if we measure the energy, we will find either E_1 or E_2, and the state immediately after the measurement is either $\psi_1(x)$ or $\psi_2(x)$.

In a similar study (Zhu &Singh, 2012a), for a similar question asked to American students, they confused the expectation value $\frac{2}{7}E_1 + \frac{5}{7}E_2$ with the energy measure. Some students who thought that the measure of energy gives E_1+E_2 reflect that these students still believe in a classical way, which explains the persistence of classical paradigms among students.

Difficulties in Understanding the Hamiltonian Observable and the Schrödinger Equation

Students find it challenging to understand the meaning of the Hamiltonian observable: is it an observable that describes the evolution of a system or an observable that measures the system's energies? Another confusion among students is that they do not differentiate between the time-dependent and time-independent equations (Singh & Marshman, 2015).

In the study (Singh et al., 2006), a question was asked: Is the equation $H\psi=E\psi$ always true? Only 29% could answer that not only if there is a proper state of the Hamiltonian, while 39% considered that it is always true, 10% considered it always true if the Hamiltonian does not depend on the time $H \neq H(t)$, 11% thought that $H\psi$ measure the energy of ψ believing that the action of the Hamiltonian reduces the state to a proper state of energy according to:

$$H\psi=E_n\psi_n \tag{17}$$

Alternatively, they thought that the Hamiltonian action corresponds to the energy measure according to equation 6.

In a study by Singh (2008b), the author asked the students whether the action of the Hamiltonian observable on a system always gives the system energy. Almost all students answered yes and gave false justifications according to equations 6 or 17.

Difficulties in Understanding the Tunnel Effect

Students believe that the particle loses energy when it crosses a potential barrier, while it is the probability that decreases when a particle crosses the potential barrier. Research has shown that learners transfer their classical reasoning when interpreting the tunneling effect (Wittmann et al., 2005); the learners think that the particle can cross the potential barrier when it spends energy.

Students confuse the decay of the amplitude of the wave function with the decay of the energy. According to McKagan et al. (2008), there are three reasons for this misconception among students. First, the wave function is plotted in the graphical representation by mentioning the energy. The second reason is that the learners think the particle can exceed the potential barrier and dissipate energy. The third possible reason is that the students relate the particle's energy to the amplitude of the wave function by analogy with waves in electromagnetism.

Difficulties in Using the Mathematical Formalism and Dirac Notation

The learning of quantum mechanics requires an essential prerequisite in mathematics in terms of linear algebra, matrix calculation, vector space, differential equations, Fourier transform, probability calculation, integrals calculation, etc.

Among the aspects of the difficulty of quantum mechanics is the mathematical formalism; students often fail to make physical sense of the quantum concepts and ideas expressed by the mathematical formalism (Singh, 2008b). Moreover, they have difficulties using mathematics in physics, especially in quantum mechanics, which requires students a critical prerequisite to learn quantum mechanics, which may, in this case, be an obstacle to teaching/learning quantum mechanics. The Dirac notation has been proposed to simplify mathematical formalism, but students with this new notation find it challenging to use it correctly. In addition, they find it challenging to translate the mathematical formalism to Dirac notation; indeed, students find it difficult to calculate the expectation value, and the probability, using Dirac notation, and they do not differentiate between the meaning of ket and bra (Singh and Marshman 2015).

In a study by Marshman and Singh (2018), the researchers found that for undergraduate students, only 23% were able to give the correct proposal for the same question 34 asked American students. The students' answers are inconsistent when expressed $\langle x|\hat{P}|p'\rangle$without using Dirac notation. According to this study, students have well-chosen the first two answers because they can make the momentum operator $\hat{P}$ act on the ket $|p'\rangle$, and then the two expressions obtained are:

$$\langle p|p'\rangle = \delta(p-p') \tag{18}$$

And the projection of the ket $|p'\rangle$ in the position space is represented in Equation 18.

45% of the students considered that the choice

$$\langle x|\hat{P}|p'\rangle = p'\langle x|p'\rangle = \frac{p'}{\sqrt{2\pi\hbar}}e^{i\frac{p'x}{\hbar}} \tag{19}$$

Is correct.

Whereas
$$\langle x|\hat{P}|p'\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p'\rangle = \frac{-i\hbar}{\sqrt{2\pi\hbar}}\frac{\partial}{\partial x}e^{i\frac{p'x}{\hbar}} \tag{20}$$

Is incorrect. Thus, students have difficulty transforming Dirac's notation into a proper mathematical formalism.

On the other hand, to understand the difficulties of determining the expectation value of an observable, there were interviews with students to determine the expected value and its meaning. In a similar study (Marshman & Singh, 2015), question 35 was proposed to the students at the University of Pittsburgh, with 45% of the students giving two correct answers. 45% of the students chose the two right propositions, a and b. Only a few students reasoned that the expectation value is the average of many measurements on states prepared in the same initial state. To calculate the expectation value, first, they have to express the state on the eigenvalues basis of the observable, so:

$$|\psi\rangle = \sum C_n|q_n\rangle \quad \text{with} \quad C_n = \langle q_n|\psi\rangle \tag{21}$$

Therefore
$$\langle\psi|\hat{Q}|\psi\rangle = \sum q_n|\langle q_n|\psi\rangle|^2 \tag{22}$$

Conclusion

The study showed that Moroccan students have difficulty understanding quantum concepts. Analysis of the students' incorrect choices reveals some of the assumed misconceptions, and comparing these results with other studies showed the universality of these difficulties. The concepts of quantum mechanics are too abstract, counterintuitive, and far removed from the daily experiences of learners; which explain that the subatomic phenomena are inaccessible to the learners' senses. Quantum concepts represent a new paradigm that is different to the classical paradigm. The mathematical formalism and Dirac notation further complicate understanding, leading students to form misconceptions and attempt to explain quantum phenomena based on the laws of classical physics.

In their planning, teachers of quantum mechanics should consider this and adapt their teaching methods to the concepts carried by learners by opening up classroom discussions about interpretations of quantum concepts. New active learning strategies must allow learners to discuss their misconceptions and break away from the classical paradigm. Another aspect that can remedy the lack of understanding of quantum mechanical concepts is the use of new technologies, such as simulations, visualizations, programming, cartoons, educational games, and interactive tutorials, which have been shown in various research and studies to be effective in overcoming the problems of misunderstanding quantum mechanical concepts.

Recommendations

This study highlights the difficulties and misconceptions that learners have with quantum concepts. To support the theme of overcoming them, quantum mechanics teachers need to consider the existence of these difficulties and misconceptions among learners. The results show that the origins of these difficulties are multiple, didactic, and epistemological; a comparison with other studies shows the universality of these difficulties.

Teachers must change and innovate their teaching methods, integrate new technologies, and propose new active learning strategies to learners. All these efforts will help students overcome the difficulties they face in understanding quantum concepts. We recommend that researchers conduct a qualitative study to categorize these difficulties and representations. Further studies should be performed using other data collection, such as interviews and recordings, to analyze student responses through phenomenographic analysis.

Limitations

This study is limited to Moroccan students. The topics covered in this study are limited to Moroccan undergraduate courses only after the course content of Introduction to Quantum Mechanics has been determined. Thus, it does not cover the topics taught in quantum mechanics II, such as the harmonic oscillator, the state of the electron in the hydrogen atom, and the perturbations.

The difficulties identified in this study can be further explored through qualitative research. By conducting a phenomenographic analysis through interviews and video recordings, we can gain further insight into students' representations of quantum concepts and comprehensively categorize these representations.

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Authorship Contribution Statement

Ait Bentaleb: Admin, Conceptualization, design, analysis, writing. Dachraoui: Data analysis/interpretation. Hassouni: Editing/reviewing, supervision, final approval. Alibrahmi: Critical revision of manuscript, final approval. Chakir: Critical revision of manuscript, final approval. Belboukhari: Concept and design, data acquisition, data analysis/interpretation.

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Appendix

Quantum Mechanics Conceptual Understanding Survey

Crises of classical physics and birth of quantum mechanics

1. In the Compton Effect, there is conservation of momentum and energy before and after the light-electron shock, the wave aspect does not allow to explain the phenomenon, and according to the shock theory, the phenomenon can be explained. Choose the correct proposals:

   a. The light behaves in the phenomenon as corpuscles.

   b. Light is always in the form of waves, but there is no information to explain the phenomenon.

   c. Each grain of light comes into contact with an electron.

2. In the Davisson-Germer experiment, a diffraction pattern is observed when a beam of electrons is directed onto a crystal lattice. According to you, the electrons in this experiment behave like, choose the good answers:

   a. Waves

   b. Corpuscles

   c. Corpuscles that move on trajectories described by waves

   d. I do not know

Fundamental aspect

3. What are the distinguishing characteristics of classical mechanics?

   a. Causality

   b. uncertainty

   c. probability

   d. statistics

   e. indeterminacy

   f. superposition

   g. determination

   h. continuity of physical quantities

   i. quantification of physical quantities

4. What are the distinguishing characteristics of quantum mechanics :

   a. Causality

   b. uncertainty

   c. probability

   d. statistics

   e. indeterminacy

   f. superposition

   g. determination

   h. quantification of physical quantitiescontinuity of physical quantities

5. Among the following properties, what are those that characterize a corpuscle

   a. Mass

   b. Position

   c. Dimensions

   d. Impulse

   e. Energy

- **f.** Trajectory
- **g.** Wave vector
- **h.** Pulse
- **i.** Interference
- **j.** Diffraction
- **k.** Propagation
- **l.** Moving
- **m.** Shocks
- **n.** Speed
- **o.** Period
- **p.** intensity

6. Among the following properties, what are those that characterize a wave?

- **a.** Mass
- **b.** position
- **c.** Impulse
- **d.** Energy
- **e.** Trajectory
- **f.** Wave vector
- **g.** Pulse
- **h.** Interference
- **i.** Diffraction
- **j.** Dimensions
- **k.** Propagation
- **l.** Moving
- **m.** Shocks
- **n.** Speed
- **o.** Period
- **p.** Intensity

7. Among the following properties, what are those that characterize a wave-corpuscle?

- **a.** Mass
- **b.** position
- **c.** Impulse
- **d.** Energy
- **e.** Trajectory
- **f.** Wave vector
- **g.** Pulse
- **h.** Interference
- **i.** Diffraction
- **j.** Dimensions
- **k.** Propagation
- **l.** Moving

m. Shocks

n. Speed

o. Period

p. Intensity

8. In 1923 De Broglie had announced the wave aspect of matter, then for any corpuscle of mass m and impulse p associated a wave that have a wavelength $\lambda = \frac{h}{p}$ .Which of these corpuscles can be studied in the framework of quantum mechanics?

a. the wavelength associated with a dust grain of diameter *1 mm,* mass $m=10^{-15}$ *kg* and velocity *v=1 mm/s*, is $\lambda=6.62.10^{-16}$ m

b. the wavelength associated with a vehicle of length *6 m*, mass m=3.10 ^{3}Kg and speed v=60 Km/h, is $\lambda=1,3.10^{-38}$ m

c. the wavelength of the wave associated with a neutron of mass $m=1,67.10^{-27}Kg$ and speed v=10m/s is $\lambda=3.96.10^{-10}$ m

d. A beam of electrons is accelerated by a potential difference V = 100 Volts. The wavelength associated electrons is $\lambda= 3.88.10^{-14}$ m. $e= 1.6.\ 10^{-19}$ *C.*

The wave-corpuscle duality

9. according to the principle of wave-corpuscle duality, the electron behaves:

a. As a wave and as a particle simultaneously

b. Sometimes as a wave and sometimes as a particle

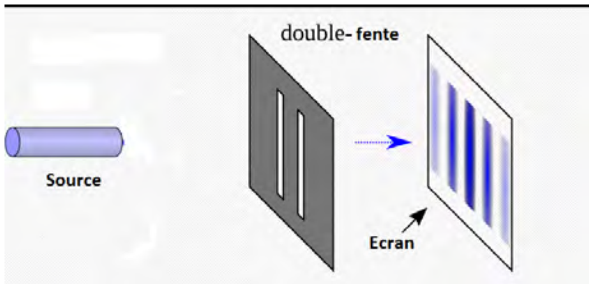
c. Neither as a wave nor as a particle

d. No clear idea

10. In an experiment, electrons are sent to a detector screen, while passing through a double slit.

And in a second experiment the light is sent to a detector screen through a double slit.

And in a third experiment dust is sent to a screen through a double opening.



Figures A, B and C can be observed from the above experiments

What figure do you think can be observed when:

___**1.** The light passes through the double slit

___**2.** The dust passes through the double opening.

___**3.** The electrons pass through the double slit.

___**4.** Light passes through only one slit when the other is covered.

___**5.** Electrons pass through only one slit when the other is covered.

If you think that none of the figures below can be observed, answer with D.

11. Choose the appropriate answer in accordance with the principle of complementarity, among the following propositions: A. being a corpuscle B. being a wave C. being a wave-corpuscle

D. we cannot know if it is wave or corpuscle.

What wave/corpuscle behavior occurs when:

___**6.** An electron sent from a source to a screen.

___**7.** The light sent from a source to a screen

___**8.** The electron when detected on the screen

___**9.** the light when will be detected on the screen.

12. The corpuscular quantities energy E and impulse $\vec{p}$ are related to the wave quantities pulsation ω and wave vector $\vec{k}$, by the following relations, choose the correct propositions.

a. $E = h\omega \quad et \quad \vec{p} = h\vec{k}$

b. $E = E_c + E_p \quad et \quad p = mv$

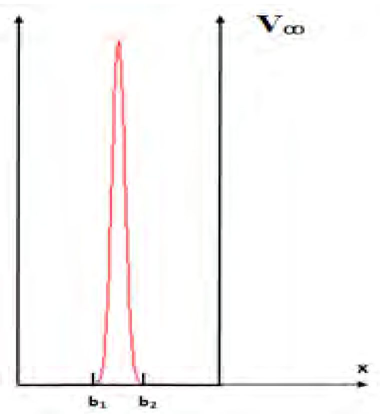
c. $E = \dfrac{p^2}{2m} \quad et \quad \Delta x.\Delta p \geq \dfrac{h}{2}$

Understanding the wave function

13. What do you think about wave function represents?

a) A real physical wave
b) A wave that guides the particle during its movement
c) A function related to the electron's charge distribution within the atom
d) A mathematical tool to describe the quantum state of the particle
e) A wave emitted by the particle
f) No clear idea

14. Consider the wave function $\psi(x,0)$ at t=0 of a particle confined in an infinite one-dimensional potential well diagrammed below, which is zero for $x<b_1$et $x>b_2$. Know that the area of the function below is 1. Choose the correct propositions.



a.This is a possible wave function.

b. It is not a wave function because it does not satisfy the boundary conditions, especially it cancels inside the well.

c.It is not a wave function because the probability outside the finite well is zero, whereas quantum mechanics predicts a non-zero probability.

15. For a free particle in a one-dimensional space, choose the correct proposals

a. Its state is represented by a plane wave $v(p,x) = \dfrac{1}{\sqrt{2\pi h}} e^{ipx/h}$

b. Its state is represented by a wave packet $\psi(x) = \dfrac{1}{\sqrt{2\pi h}} \int \psi(p) e^{ipx/h} dp$

c. Its state is represented by a superposition of plane waves having closing wave vectors

Uncertainty principle and indeterminism

16. Heisenberg's uncertainty principle states that some properties of a quantum particle cannot be measured simultaneously with precision. Which properties?

a. Position and mass

b. Mass and size.

c. Density and speed.

d. position and momentum.

e. Time and energy

f. quantities that their observables do not commute

g. all quantities

h. conjugate quantities

17. According to Heisenberg's uncertainty principle, choose the correct propositions.

a. It is possible to determine both the position and the momentum simultaneously of a quantum particle.

b. Impossible to determine them simultaneously

c. If the quantum particle's position is well determined, then its velocity is entirely undefined.

d. It is impossible to determine with precision a physical quantity

e. There is no device at the moment that can measure the position and velocity of an electron with precision.

f. the precision of a quantity depends on the precision of the conjugated quantity; the more the precision of one quantity increases, the more the precision of the other decreases.

18. Heisenberg's uncertainty (or indeterminacy) principle states that the position and momentum in the monodimensional case can only be determined within limits set by the inequality: $\Delta x.\Delta p \geq \frac{h}{2}$ .This principle is related to; (choose the correct propositions)

a. Errors carried by the measuring instruments

b. A limit of human knowledge

c. A fundamental property of subatomic nature

d. I do not know

19. According to quantum mechanics, the position of a subatomic particle

a. Can be accurately determined at this time

b. Has a precisely determined value, and the percentage of error in the measurement results from the measuring devices

c. To different forecasts, each with a specific probability of existence

d. If the particle is free, its position cannot be totally determined

e. Cannot be determined accurately because of the high velocity of the particles so that it is difficult to measure

The postulates of quantum mechanics

20. An observable Q corresponding to a physical quantity Q, having a discrete spectrum of non-degenerate eigenvalues, the states are $\{|q_n\rangle\}$ the eigenstates of the observable corresponding Q to the eigenvalues q_n . at t=0 the state of the system is described by $|\psi\rangle$, choose the correct propositions:

a. The measurement of the physical quantity Q gives as result one of the eigenvalues of the observable Q

b. The measurement of the physical quantity gives Q as a result $\sum q_n$

c. Immediately after the measurement, the system state reduces to any proper state of the observable.

d. The probability of obtaining the result q_n is $|\langle q_n|\psi\rangle|^2$

21. Consider the following conversation between two students Ahmed and Said about an observable for Q a system in a quantum $|\psi\rangle$ state that is not a proper state of the observable Q.

Ahmed: the action of an observable $\boldsymbol{Q}$ (corresponds to a physical quantity $\boldsymbol{Q}$) on a quantum $|\psi\rangle$ state, corresponds to the measurement made by this observable, such that with $Q|\psi\rangle = q_n|\psi\rangle$,with q_n is the observed value.

Said: No, the measurement reduces the state according to the following operation , $Q|\psi\rangle = q_n|\psi_n\rangle$ with is $|\psi_n\rangle$ an eigenstate of the observable corresponding Q to the eigenvalue q_n

With which of these two students do you agree, justifying your answer?

a. I agree with Ahmed only

b. I agree with Said only

c. I don't agree with Ahmed or Said

d. I agree with both

e. The answer depends on the observable Q

22. According to the principle of superposition, the particle is generally found, before the measurement, in:

a. A well-defined state

b. A superposition of states

c. No state

d. No clear idea

23. If you make measurements for a physical quantity on systems prepared in the same state, that is not a proper state of the observable Q corresponding to the physical quantity.

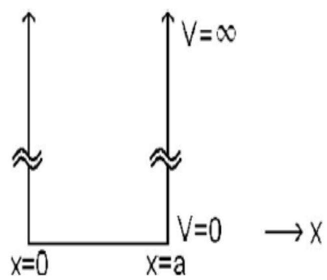
Choose the correct answers:

a. You find several possible states with different probabilities

b. The results found are equiprobable

c. You find identical results

24. You are measuring the position of a particle in an infinite one-dimensional potential well, existing in the first excited state. Choose the sentences that seem correct to you:



a. Immediately after the position measurement, the wave function will peak around a particular position value.

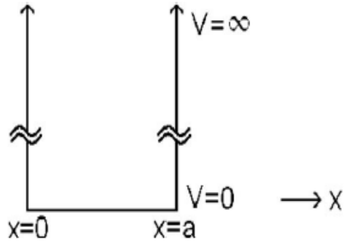
b. A long time after the position measurement, the wave function will return to the wave function corresponding to the first excited state.

c. Even if we wait a long time, the wave function will not return to the wave function corresponding to the first excited state.

d. If you make a second measurement of the position after a long time, you find any result of eigenvalues of the position.

25. The wave function of an electron in an infinite monodimensional well of width a at t=0 is $\psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$ and $\phi_1(x)$ are $\phi_2(x)$ are the fundamental and first stationary states of system. (The eigenstates with the eigenvalues of the Hamiltonian of the system are $\phi_n(x) = \sqrt{2/a}sin(n\pi x/a)$, $E_n = n^2\pi^2\hbar^2/(2ma^2)$ with n=1,2, 3.......), Choose the correct proposals:

a. the wave function at $\psi(x,t)$ an instant t is $\psi(x,t) = \left(\sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)\right)e^{\frac{-iEt}{\hbar}}$



b. the expectation value of the energy in the state $\psi(x,t)$ is $E = \frac{2}{7}E_1 + \frac{5}{7}E_2$

c. if the measurement of the energy at t=0 gives the value E_2, then the expression of the wave function after this measurement is $\phi_2(x) = \sqrt{\frac{2}{a}}\ \sin(\frac{2\pi x}{a})$

d. Immediately after the measurement of energy, the position of the electron is measured, the possible values of position are all values between 0 and a, and their probability density of measuring each position is $\left|\phi_2(x)\right|^2$

26. Consider a particle confined in an infinite monodimensional potential well, a wave function at time t=0 is $\sqrt{2/7}\psi_1(x) + \sqrt{5/7}\psi_2(x)$, that you make the energy measurement with $\psi_n(x)$ the eigenstates that correspond to the eigenvalues of the energy E_n. choose the correct proposals.

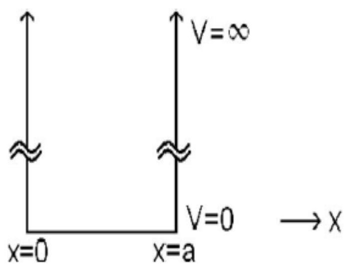
a. The energy measurement can give either E_1 or E_2.

b. The normalized wave function (excluding the time part) just after the measurement is either or $\psi_1(x)$ $\psi_2(x)$.

c. The measurement of energy gives $\frac{2}{7}E_1 + \frac{5}{7}E_2$

d. the maximum value of energy that can be measured is $E_1 + E_2$

27. A particle is in an infinite monodimensional potential well of width a. The wave function at t=0 is $\psi(x,0) = \frac{\psi_1(x) + \psi_2(x)}{\sqrt{2}}$, with $\psi_n(x)$ the eigenstates that correspond to the eigenvalues of the energy E_n, choose from the following correct ones:



a. If you measure the position of the particle at t=0 the probability density of finding it at x is $\left|\frac{\psi_1(x) + \psi_2(x)}{\sqrt{2}}\right|^2$

b. If you measure the energy of the system at t=0 the probability of finding E_1 is $\left|\int_0^a \psi^*_1(x)(\frac{\psi_1(x) + \psi_2(x)}{\sqrt{2}})dx\right|^2$

c. If you measure the position of the particle at t=0 the probability of finding the particle between 0 and a is $\int_0^a x\left|\psi(x,0)\right|^2 dx$

The Hamiltonian$\widehat{H}$ operator and Schrödinger's equation

28. The Hamiltonian operator describes H

a. The temporal evolution of a quantum state

b. The measurement of the energy of a system

c. The measure of the energy of a stationary state

29. What is the most fundamental equation in quantum mechanics?

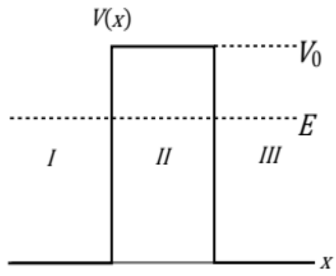
a. $H\psi = E\psi$

b. $H\left|\psi\right\rangle = i\hbar\dfrac{\partial\left|\psi\right\rangle}{\partial t}$

c. $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$

Tunnel effect

30. Consider a particle with mass m and energy E, crossing a potential barrier V(x) with the following profile:



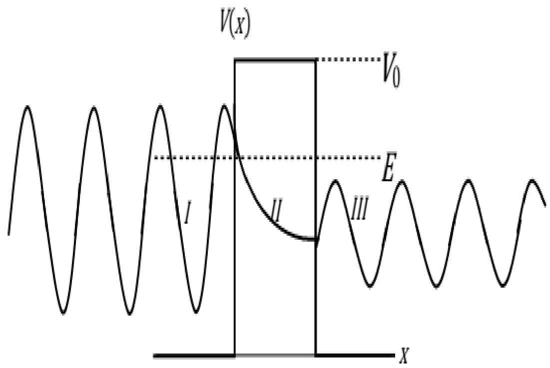
Note that in the region II E< V_0. Choose the correct proposals:

a. The general solution of the time independent Schrödinger equation in region II is $Ae^{ikx} + Be^{-ikx}$

b. The value of the constant k in a is $\sqrt{\dfrac{2m}{\hbar^2}V_0}$

c. The particle is in a scattering state in regions I and III, and in a bound state in region II.

31. Consider a particle represented by a wave function, crossing a potential barrier from zone I to zone III with E< V_0. Choose the correct proposals



a. The energy of the particle decreases exponentially.

b. The probability density decreases exponentially.

c. The energy and probability density of the particle decrease exponentially.

d. In classical physics, the particle cannot exist in zone III.

Correct use of mathematical formalism and Dirac notation

32. For a spinless particle confined in a one-dimensional space, the quantum's system state at t=0 is represented by the ket in $\left|\psi\right\rangle$ Hilbert space, and $\left|x\right\rangle$ are $\left|p\right\rangle$ the eigenstates of the position observable and X the moment P observable respectively. Choose among the following propositions, those are correct about the wave functions in the position space and the moment space for a quantum state.

a. The representation of the wave function in the position space is $\psi(x) = \left\langle x|\psi\right\rangle$

b. The representation of the wave function in the moment's space of is $\psi(p) = \left\langle p|\psi\right\rangle$

c. The representation of the wave function in the moment's space is $\psi(p) = \dfrac{\hbar}{i}\dfrac{\partial\left|\psi\right\rangle}{\partial x}$

d. $\psi(x)\ et\ \psi(p)$ are related by the following Fourier transform $\psi(x) = \dfrac{1}{\sqrt{2\pi\hbar}}\int \psi(p)e^{ipx/\hbar}dp$

33. Choose the correct expressions:

a. $\left\langle x|\psi\right\rangle = \int x\psi(x)dx$

b. $\left\langle x|\psi\right\rangle = \int \delta(x-x')\psi(x')dx'$

c. $\left\langle p|\psi\right\rangle = \int \left\langle p|x\right\rangle\left\langle x|\psi\right\rangle dx = \dfrac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(x)dx$

d. $\langle p|\psi\rangle = \int \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)dx$

e. $|\psi\rangle = \int |p\rangle\langle p|\psi\rangle\, dp$

f. $|\psi\rangle = \int \psi(x)|x\rangle dx$

34. $|p'\rangle$ is an eigenstate corresponding to the eigenvalue p' of the moment operator, for a particle confined in a monodimensional space, choose the corrects proposals.

a. $\langle p|p|p'\rangle = p'\langle p|p'\rangle = p'\delta(p-p')$

b. $\langle x|p|p'\rangle = p'\langle x|p'\rangle = \frac{p'}{\sqrt{2\pi\hbar}}e^{ip'x/\hbar}$

c. $\langle x|p|p'\rangle = -i\hbar\partial/\partial x\langle x|p'\rangle = \frac{-i\hbar}{\sqrt{2\pi\hbar}}\partial/\partial x e^{ip'x/\hbar}$

35. Assume that $\{|q_n\rangle, =1,2,3\ldots\infty\}$ forms a complete orthonormal basis consisting of the eigenstates of an observable Q that corresponds to a physical quantity with q_n are the non-degenerate eigenvalues. $\hat{I}$ is the unit operator. Choose among the following sentences the correct ones.

a. $\sum_n |q_n\rangle\langle q_n| = \hat{I}$

b. $\langle\psi|Q|\psi\rangle = \sum q_n |\langle q_n|\psi\rangle|^2$

c. $\langle\psi|Q|\psi\rangle = \sum q_n \langle q_n|\psi\rangle$