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Guiding Students' Attention Towards Multiplicative Relations Around Them: A Classroom Intervention

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Abstract

Learning fractions poses a challenge for many elementary school students, including applying fraction knowledge in novel contexts. For instance, there are substantial individual differences in students' tendency of spontaneous focusing on quantitative relations (SFOR), which is related to the development of rational number knowledge. In this study, 4th grade students (N = 129) took part in a quasi-experimental study comparing an intervention condition (n = 71) aimed at improving students' multiplicative relational reasoning and fraction knowledge with a control condition (n = 58) of business as usual fraction instruction. Five lessons of intervention activities were designed to promote students ability to recognize and describe multiplicative relations in their everyday surroundings. There was an overall positive effect on the students' mathematical knowledge. Students who participated in the intervention improved their ability to recognize and describe multiplicative relations embedded in pictures representing everyday situations. There were no significant differences in the development of fraction knowledge despite replacing five traditional fraction lessons. These findings provide further evidence that researchers and educators should continue to pay attention to issues surrounding students' spontaneous mathematical focusing tendencies.

Keywords

fraction, fraction magnitude, fraction representation, multiplicative relations, intervention, spontaneous focusing on quantitative relations

Understanding mathematics is essential for understanding the world; as stated by Pythagoras and Galilei, "the world everywhere around us is written in the language of mathematics". Yet, according to recent research on spontaneous mathematical focusing tendencies, children and adults alike differ in how much attention they focus on mathematical features in the world around them (Hannula & Lehtinen, 2005; McMullen, Chan, Mazzocco, & Hannula-Sormunen, 2019; McMullen, Verschaffel, & Hannula-Sormunen, 2020; Verschaffel et al., 2020). These individual differences are not only relevant for understanding the world around us but are also related to the development of mathematical knowledge. For instance, individual differences in spontaneous focusing on quantitative relations (SFOR) have been found to be related to the development of rational number conceptual knowledge (McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015).

Solid knowledge of fractions is essential not only for higher mathematics learning, such as algebra (Booth & Newton, 2012), but also for getting by in contemporary society (Advisory Committee on Mathematics Education, 2011; National Mathematics Advisory Panel, 2008). However, learning of fractions is considered a serious challenge in mathematics education (Siegler, Thompson, & Schneider, 2011). Both school-aged children and adults have been found to lack mathematical knowledge regarding rational numbers and especially fractions (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Siegler et al., 2012; Vamvakoussi, Van Dooren, & Verschaffel, 2012) and instructional approaches for supporting rational number knowledge are needed. Consequently, the current study examines the effectiveness of a teacher-led intervention that is based on previous findings of spontaneous mathematical focusing tendencies.

Students' Difficulties in Learning Fractions

Many primary school students struggle to understand fraction magnitudes (Jordan et al., 2013; Siegler et al., 2011). One issue is that the understanding of natural numbers sometimes interferes with the learning of fraction magnitudes (Ni & Zhou, 2005; Siegler et al., 2011; Vosniadou, Vamvakoussi, & Skopeliti, 2008; Wynn, 1995). Even adults often inappropriately use knowledge about natural numbers when reasoning about fraction magnitudes (Vamvakoussi et al., 2012).

A distinction can be made between part-whole interpretations of fractions and understanding fraction magnitudes (Fuchs, Malone, Schumacher, Namkung, & Wang, 2017). Learning only the part-whole interpretation of a fraction (for example, 1/5 is one particular slice of a whole pizza) does not capture the understanding that, with the example provided, 1/5 is one fifth of the distance from zero to one. Furthermore, the part-whole interpretation does not work with negative fractions, and it causes misunderstandings when dealing with improper fractions (Siegler et al., 2011). Understanding fractions by their magnitude captures the understanding that numbers are theoretically based somewhere on the number line, and their magnitude determines their relative position to another number (Siegler et al., 2011).

The overuse of natural number features when learning and reasoning about rational numbers is referred to as the natural number bias. The natural number bias refers to the tendency to erroneously apply knowledge relevant to natural numbers even when it is not applicable to the rational number task at hand (Ni & Zhou, 2005). This bias causes students to make errors, such as claiming that $\frac{1}{4}$ is bigger than $\frac{1}{3}$ (because 4 > 3) or that $\frac{1}{2} + \frac{1}{3}$ equals $\frac{2}{5}$ (because of reasoning with whole-number additive principles). As well, numbers are considered to have only one symbolic representation, and more digits are thought to mean a bigger number (Nunes & Bryant, 2015). Consequently, prior knowledge of natural numbers can cause many systematic mistakes when dealing with fractions (Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016).

Despite these difficulties with formal fraction instruction, studies show that even very young children appear to have a basic understanding of ratios and an ability to reason multiplicatively on tasks (Boyer, Levine, & Huttenlocher, 2008; Feigenson, 2007; McCrink & Wynn, 2007; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1999; Wing & Beal, 2004). Notably, children seem to be able to solve multiplicative reasoning problems, even before they are taught explicit operations of multiplication or division, when given the opportunity and right sort of representation (e.g., sharing sweets fairly when sweets have a different number of units) (Nunes & Bryant, 2015; Nunes, Bryant, Evans, & Barros, 2015). Given that fractions can be seen as the expression of a part-whole relation, which can be used more generally to express a multiplicative relation between two quantities of the same or different natures, this seemingly basic understanding of multiplicative relations should build a foundation for learning fraction and rational number concepts in school (Ni & Zhou, 2005). However, as presented earlier, the difficulty of learning fractions persists for many students. This disconnect between multiplicative and part-whole relations presents a ripe opportunity to build on by improving students' understanding of multiplicative relations to support their learning of fractions.

Spontaneous Focusing on Quantitative Relations

Opportunities to notice and use fractions or multiplicative relations do not happen only in explicitly mathematical situations in the classroom. Rather, the opportunities to describe mathematical relations in the world are limitless if one directs students' attention to the mathematical aspects all around (Lehtinen & Hannula, 2006). Even young children are found to spontaneously focus on numerosity and to differ in their individual tendency to pay attention to the mathematical features around them (Hannula & Lehtinen, 2005). Consider a situation in which there is one red apple and three green apples on the table. One person might not notice or describe the number of apples at all, while another could say there are four apples in total. Someone else might notice that "1/4 of the apples are red" or that "there are three times as many green apples as red apples". This kind of unguided attention towards a quantitative relation between the sets of apples is labeled spontaneous focusing on quantitative relations (SFOR) (McMullen, Hannula-Sormunen, & Lehtinen, 2014).

Evidence suggests that the amount and intensity of attention paid to quantitative or multiplicative relations in situations that are not explicitly mathematical, referred to as SFOR tendency, substantially differs between children (McMullen et al., 2014; McMullen et al., 2016). SFOR tendency is related to students' arithmetic skills in early grades (McMullen et al., 2014) and rational number conceptual development in late primary school, even after taking into account students' general mathematical achievement, non-verbal intelligence, multiplicative reasoning, and arithmetic fluency (McMullen et al., 2016; Van Hoof et al., 2016). It has been theorized that children with a higher SFOR tendency acquire more self-initiated practice with multiplicative relations and fractions in everyday life (Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). Multiplicative reasoning about real-life situations is expected to support fraction understanding (Nunes & Bryant, 2015); thus, self-initiated practice may support the development of rational number knowledge.

Enhancing Mathematical Focusing

Importantly, it is possible to enhance an individual's spontaneous mathematical focusing tendencies and get them to pay more attention to the mathematical aspects of their surroundings. For instance, Hannula, Mattinen, and Lehtinen (2005) found that, through social interaction (i.e. prompts and activities that guided children's attention on small exact numerosities in everyday surroundings), young children's tendency to spontaneously focus on numerosity could be increased, leading to positive effects on their early numeracy skills. Interventions on mathematical focusing tendencies should aim to make mathematical features of students' everyday environments explicit targets of focusing (Hannula et al., 2005; Hannula-Sormunen et al., 2020). In essence, students should be supported to become more aware of how the mathematical concepts that they learn about in class can also be found in their surrounding environment and embedded in their everyday actions. This increased awareness of the mathematical features around them is expected to lead to increased mathematical focusing outside of the intervention activities (Hannula et al., 2005; McMullen, Hannula-Sormunen, Kainulainen, Kiili, & Lehtinen, 2019).

To increase students' ability to use multiplicative relations in their everyday lives, intervention activities should make multiplicative relations explicit targets of focusing. This can be achieved by teaching students how to recognize and describe these multiplicative relations using explicit multiplicative phrasing, such as "twice as many" (McMullen, Hannula-Sormunen, et al., 2019). The intervention activities should first increase students' tendency to recognize when and where multiplicative relations exist in everyday situations. This appears feasible, as previous intervention studies have demonstrated that targeted training in noticing the mathematical features of everyday life does indeed increase participants' mathematical focusing (Braham, Libertus, & McCrink, 2018; Hannula-Sormunen et al., 2020; Hannula et al., 2005; McMullen, Hannula-Sormunen, et al., 2019). These effects have been found even after relatively short intervention periods (Braham et al., 2018; McMullen, Hannula-Sormunen, et al., 2019), and have been sustained even 20 weeks after the intervention (Hannula-Sormunen et al., 2020).

Secondly, alongside recognizing the mathematical features around them, it is necessary for students' to be able to describe them using exact mathematical language. Therefore, the present intervention aimed to improve students' mathematical language by providing explicit terms for describing everyday multiplicative relations. Knowledge of mathematical language may have a direct impact on mathematical development (Hornburg, Schmitt, & Purpura, 2018). Earlier research suggests that participation in a dialogical reading intervention focused on mathematical language can improve young children's mathematical knowledge, as well as their mathematical language (Purpura, Logan, Hassinger-Das, & Napoli, 2017). Improving students' mathematical language about multiplicative relations may support their ability to recognize these relations as explicit targets of focusing in their everyday life.

Increasing students' tendency to make multiplicative relations explicit targets of focusing in their everyday lives allows for connecting the representational and analytic meanings of multiplicative relations and fractions (Nunes & Bryant, 2015). Instead of treating multiplicative relations as solely a part of the formal, written mathematical system (e.g., $2 * 3 = 6$), our intervention explicitly connects these analytic meanings with the representational meanings of mathematical relations that exist in everyday situations (e.g., There are 2 rows of 3 windows, so there are 6 windows). By enriching the connections between the two meanings of multiplicative relations, we expect to have a positive impact on students' understanding of these meanings.

The Present Study

The considerable body of evidence regarding primary school students' difficulties in dealing with fractions encourages further research on how to support students' learning (Jordan et al., 2013; National Mathematics Advisory Panel, 2008; Siegler et al., 2011). An earlier intervention showed that it is possible to improve students' SFOR tendency through guided activities (McMullen, Hannula-Sormunen, et al., 2019). However, there is no evidence of how such an intervention affects students' knowledge of multiplicative relations or fractions.

The aim of the present study was to test how intervention activities aimed at improving students' focusing on multiplicative relations impact their mathematical knowledge, including multiplicative relations and fractions. We carried out a pre-registered quasi-experimental classroom study comparing students who participated in the intervention activities with those who had their normal fraction instruction. We examined students' mathematical knowledge at three measurement points: before, after, and three months after the intervention. We measured the students' ability to describe multiplicative relations when their attention was guided toward quantitative aspects, as well as their fraction representation and size knowledge. The overall effectiveness of the intervention on the three knowledge measures was examined based on a pre-registered analysis plan (https://osf.io/hmytv).

Method

Participants

Students (ages 10 to 11 years) from three schools and seven 4th grade classrooms (N = 129) took part in a quasi-experimental intervention study with pretest, posttest, and delayed posttest during the spring semester. The first two schools with two classrooms each were assigned to the experimental group (n = 71), whereas the third school with three classrooms was assigned to the control group (n = 58). Due to the novel nature of the classroom activities in the experimental condition, and the relatively low variance in academic skills between schools in Finland (Mattila, 2005), classes were assigned to conditions by schools in order to avoid any cross-pollination effects of the intervention because of students discussing the novel activities with each other. The schools were located in similar urban areas in Southern Finland.

Permission to take part in the study was received from teachers, parents, students, and heads of the schools. The ethical guidelines of the University of Turku on research integrity were followed.

Design

Both groups followed the same pretest–intervention–posttest–delayed posttest schedule. The tests lasted 45 minutes and were conducted during mathematics lessons. After the pretest, both intervention groups went through five consecutive 45-minute mathematics lessons, followed by the posttest on the next mathematics lesson. No other mathematics lessons took place during the intervention period. The classroom teacher taught the lessons, and the tests were run by a trained researcher. Pretest, intervention, and posttest were completed over a period of two weeks, and there were no other mathematics lessons during that time. The delayed posttest was completed three months after the intervention. Between the posttest and delayed posttest, the classrooms did not have any lessons specifically targeted at fractions.

Professional Development of the Teachers

The experimental group teachers (n = 4) were trained in professional development sessions to run all five intervention lessons. The first and last authors met the teachers two at a time in a face-to-face session, which lasted for 90 minutes and took place one week before the intervention. Teachers were instructed on how to support students in recognizing multiplicative relations from their surroundings and describe the relations verbally using precise mathematical language. The teachers were also given guidance on how to look at the multiplicative relations from multiple perspectives (e.g., describing something as one-half can also be described as twice as many). It was highlighted that the multiplicative relations found in the surroundings could be estimates and did not need to be entirely precise. The teachers were given printed copies of the lesson plans, and all five lessons were walked through together.

The control group teachers (n = 3) continued with five traditional fraction lessons during the intervention. The teachers started teaching the fraction content from the 4th grade mathematics book normally. No specific instructions or restrictions were placed on the teaching of fractions during the lessons, nor did they have any professional development sessions.

Experimental Group Program

Experimental group lessons were developed from previous interventions aimed at promoting spontaneous mathematical focusing tendencies (Hannula et al., 2005; McMullen, Hannula-Sormunen, et al., 2019). The aim of the program was to make the multiplicative relations more recognizable targets of focusing in students everyday lives through guided activities (McMullen, Verschaffel, & Hannula-Sormunen, 2020).

Introduction Lesson (Lesson 1)

The aim of the Lesson 1 was to introduce the students to the idea that multiplicative relations and fractions can be found in many different situations in their everyday lives. It provided guidance on how to recognize and basic vocabulary on how to describe these relations when they are found. The teacher was encouraged to organize the whole lesson so that it would elicit the most individual thinking and sharing of thoughts for the class (e.g., the think-pair-share method). First, the students were asked to think and share examples of daily situations in which mathematical thinking and multiplicative relations are relevant and applicable.

The teacher then presented a PowerPoint presentation, including examples of how to use multiplicative relations when describing pictures. The pictures were used to elicit discussions between the students and the teacher about how to describe multiplicative relations in everyday situations. For example, the students could describe Figure 1 using fractions “two-thirds of the apples are green’’ or multiplicative relations: “there are two times more green apples than red apples’’. Students were also encouraged to describe the multiplicative relations in multiple ways, such as using the reverse relation: “one-third of the apples is red” or “the amount of red apples is half of the amount of green apples”. The discussion was then carried out to find similar examples of multiplicative relations from the classroom. Students were encouraged to recognize and describe their findings in small groups and whole-class discussions.

Figure 1

One of the Pictures Used to Elicit Observations of Multiplicative Relations

Note. Copyright of the picture: (Saku Määttä).

Scavenger Hunts (Lessons 2 and 3)

In Lessons 2 and 3, students did two different scavenger hunt-type activities similar to those used in previous intervention on SFOR tendency (McMullen, Hannula-Sormunen, et al., 2019). The aim of the scavenger hunts' was to guide students' attention toward multiplicative relations in school. The students were first shown examples of sentences in which they needed to describe multiplicative relations (e.g., "_____ is twice as tall as the teachers' desk"). The examples were discussed in whole class and small groups. Thereafter, groups of two or three students went around the school to find answers to fill-in-the-blank statements about multiplicative relations (see Figure 2 for example prompts). For instance, an answer to the first task in Figure 2 could be: "The piano is two times as tall as the chair''.

Figure 2

Part of the First Scavenger Hunt Sheet (Translated)


a) ____________ is two times as tall as ____________

b) ____________ is halfway to ____________

c) One out of three ____________ are ____________

d) Two out of five ____________ are ____________

e) $\frac{1}{3}$ of the ____________ are ____________

For the second scavenger hunt lesson, the teacher formed another scavenger hunt by taking a selection of students' answers from the first hunt. Teachers then deleted one of the inputs that the students found on the first hunt to make a new set of statements (e.g., "____ is two times as tall as chair"). Students then went on and tried to solve these new scavenger hunt tasks.

Which One Does Not Belong? (Lessons 4 and 5)

The last two lessons used the format of "which one does not belong" tasks. During the lessons, the students rehearsed finding and describing multiplicative relations from pictures. First, the students were presented with a series of picture sets. In each set, three of the pictures contained an instance of the same multiplicative relation (e.g., one-half), and one did not. In the example in Figure 3, the cutlery does not belong to the other pictures, as the other pictures represent half (of something) and the cutlery represents one-fourth (as one-fourth of the cutlery is knife). For some sets, there can be multiple correct answers. The picture sets were discussed in whole-class and small groups.

During the last lesson, the students created picture sets in groups of two to four for other students to solve. The teacher had pedagogical autonomy on how to organize the creation process (i.e., whether to make digital sets of pictures or use materials found from the classroom). However, all teachers elected to organize the activity so that the students' formed the picture set by using physical materials found around the classroom (e.g., pencils, books, etc.). The students then went around the classroom and tried to solve other groups' tasks.

Figure 3

A "Which One Does not Belong" Picture Is Used as an Example in the Classroom

Note. Copyright of the pictures: top right picture (Hilma Halme), bottom left picture (Saku Määttä), bottom right picture (Hilma Halme).

Testing Procedure and Measurements

The 45-minute tests took place during the mathematics lessons and were taken using paper and pencil. The tasks presented were in the same order for all students. The tasks measured three different aspects of mathematical knowledge: multiplicative relations, fraction representations, and fraction size.

Knowledge of Multiplicative Relations

Picture Description Tasks

At all three time points, a transformation task was used to measure the students' ability to recognize and describe multiplicative relations. This task was based on previous measures of guided focusing on quantitative relations (McMullen, Hannula-Sormunen, et al., 2019; McMullen et al., 2016). The task in the pretest and delayed posttests involved a teleportation device that sent objects to outer space, and the posttest involved a magician transforming items. The task contained four items at each time point. On the first and third items of these tasks, students were asked to write down a description of exactly how the quantity of three sets of objects changed (e.g., "Describe how the quantity of the items changed when they teleported from earth to new planet"). Objects changed in multiple ways (e.g., color, shape), including a consistent multiplicative relation (e.g., quantities multiplied by three). On the second and fourth items, students were shown the same types of material as the previous item, but with different numbers of items, and were asked to draw the number of objects that should arrive based on what happened in the previous task. Students were given 1 minute for the writing items and 1.5 minutes for the drawing items.

In the writing task, students were given one point for each multiplicative relation they described per set of objects (e.g., three times more, half the amount, a third less). The relation did not have to be mathematically correct but needed to be a specific multiplicative relation, for example, saying that the items tripled when they actually doubled was still correct. In the drawing task, one point was given to each set of objects with the correctly drawn amount. There was a maximum of three points per item and 12 points overall.

Additional picture description tasks were included in the posttest and delayed posttest. The task measured the students' ability to recognize and describe multiplicative relations from pictures. Students were shown photos of real-life situations and asked to describe mathematical relations as precisely as possible (e.g., "What mathematical relations do you see in the picture? Describe all the mathematical relations you find from the picture as precisely as possible"). On the posttest, only the car picture description item was included (Figure 4, Item 1). In the delayed posttest, four additional pictures were included with the car picture (Figure 4, Items 2-5). Students had 2 minutes to write their answer on each item.

Figure 4

Picture Description Task Items 1-5

Note. Copyrights of the pictures: Item 1 image Andrey Khronelok/BigStockPhoto, Item 2 (Hilma Halme), Item 3 (Minna Hannula-Sormunen), Item 4 (Jake McMullen), Item 5 (Jake McMullen).

Students were given one point for each described multiplicative relation or fraction (e.g., “two out of six cars are red,” “half of the cars are on the other side”, “4/8 of the eggs are blue”). The relation did not have to be mathematically correct but should have been a specific multiplicative relation. For example, saying that one-third of the eggs on Item 2 are yellow, when the correct amount is one-fourth, would still get a point. There was no theoretical maximum number of points for the task. The highest score obtained by a student in a single item was 8 points.

The picture description tasks had an acceptable reliability (pretest α = .74; posttest α = .64, delayed posttest α = .85). As stated in the preregistration, the use of non-specific responses such as “less” and “more” were separately coded for the transformation writing tasks and picture description tasks. However, this measure was not included in the main analysis due to poor reliability (see Appendix A in the Supplementary Materials).

Fraction Representation Knowledge

Fraction representation knowledge was measured by two tasks at each measurement point.

Number Sets Task

The number sets task (Geary, Bailey, & Hoard, 2009; McMullen, Hannula-Sormunen, Lehtinen, & Siegler, 2020) was used to measure students’ fraction representation knowledge. In the task, the students chose as many symbolic and non-symbolic representations as possible that equaled the given fraction (Appendix B, Supplementary Materials). The first item was matching ½ and the second item was matching ¼. Both items had 15 different representations, nine of which were correct. The task was timed with one minute per item. A correct choice was awarded one point and an incorrect choice resulted in deducting one point, with a maximum of 18 points for the task. The task had a good reliability (pretest α = .74; posttest α = .77; delayed posttest α = .89).

Coloring Task

The coloring task was used to measure students’ fraction representation knowledge by asking them to color a specified proportion of eight figures, including four geometrical shapes and four pictorial representations (Appendix C and Appendix D, Supplementary Materials). First, students were asked to color a third of each geometrical shape. Then, students were asked to color one-third of six apples, one-half of four fish, three-fourths of four cars, and two-thirds of nine cans. The coloring was correct, if a student colored the correct number of objects (e.g., 2 out of 6 of the apples) or the correct part of every figure separately (e.g., 1/3 of each apple). One point was given for each correct answer, resulting in a maximum of 8 points. The task had a good reliability (pretest α = .82; posttest α = .85; delayed posttest α = .83).

Fraction Size Knowledge

Fraction size knowledge was measured with three tasks at each measurement point.

Missing Value Task

In the missing value task, the students were presented with two fractions, with one fraction missing either the numerator or denominator. The students were to fill in the missing numerator or denominator that would make the two fractions equal. The task contained five items: A) $\frac{1}{3} = \frac{2}{?}$, B) $\frac{6}{8} = \frac{3}{?}$, C) $\frac{5}{10} = \frac{?}{30}$, D) $\frac{2}{3} = \frac{?}{15}$, and E) $\frac{4}{12} = \frac{1}{?}$. Students received one point for each correct answer, with a maximum of five points. The task's reliability was good (pretest α = .73; posttest α = .88; delayed posttest α = .87)

Fraction Comparison Task

In the fraction comparison task (Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015) students circled the larger of two given fractions or both when their size was equal (e.g., "Circle the larger fraction. If the fractions are equal, circle both"). The task contained six items: A) $\frac{5}{9}$; $\frac{5}{7}$, B) $\frac{2}{3}$; $\frac{3}{5}$, C) $\frac{7}{15}$; $\frac{3}{4}$, D) $\frac{2}{3}$; $\frac{6}{9}$, E) $\frac{2}{5}$; $\frac{2}{7}$, and F) $\frac{5}{4}$; $\frac{6}{8}$. Students received one point for each correct answer, with a maximum of six points. The task had a good reliability (pretest α = .90; posttest α = .91; delayed posttest α = .93).

Number Line Estimation Task

In the number line estimation task, students estimated the location of a given fraction on a 14.5 cm long number line with endpoints of 0 and 1 (e.g., "Below is a number line between 0 and 1. Place a fraction $\frac{1}{4}$ on the number line"). The task contained six items: A) $\frac{1}{4}$, B) $\frac{3}{5}$, C) $\frac{1}{2}$, D) $\frac{1}{3}$, E) $\frac{3}{4}$, and F) $\frac{2}{4}$. Each item was on a separate page, and the page contained only the task instructions and the number line. Students' accuracy on the estimated and true values was measured by percent absolute error (Siegler et al., 2011). First, the distance between the student's answer and the correct spot was measured for each item (in cm). The distance was then divided by the total length of the number line (14.5 cm), resulting in a percent absolute error score.

The number line estimation task had a good reliability (pretest α = .91; posttest α = .91; delayed posttest α = .87).

Data Analysis

The data were analyzed based on pre-registered analysis using IBM SPSS 25. The graphical figures were created using JASP version 0.14. To create separate composite measures of fraction representations and fraction size knowledge, the students' scores were standardized for each task separately at each time point. The average of the standardized component task scores—in the case of fraction representations: number sets and coloring tasks, and, in the case of fraction size: missing value, comparison, and number line estimation—were then calculated. This process assigned equal weight to the different tasks used in the measure, no matter the scales. Lastly, an overall measure of mathematical knowledge was created by averaging the standardized knowledge aspect scores: multiplicative relations, fraction representations and fraction size (for Pearson correlation coefficients of the knowledge aspects see Appendix E, Supplementary Materials). Again, this assigned equal weight to the overall measure of each aspect of mathematical knowledge. Multiple-tests were corrected by using Bonferroni-Holm corrections. One student from both of the groups was deleted from the sample due to missing two out of the three tests. This criterion was determined in the pre-registration. Students who were not present at all three testing points were excluded from the confirmatory analysis results, due to the repeated measures ANOVA analysis list-wise deletion of the missing data.

Results

Confirmatory Results of Intervention Effects

Descriptive statistics of all tasks are reported in Appendix F, Supplementary Materials. Table 1 presents the combined overall score of all three measures of mathematical knowledge by experimental and control group and the descriptive statistics for each knowledge measure separately across the three testing points for those students included in the main analysis. Reliability was good for the overall scores at each time point and at least sufficient for all knowledge measures at each time point. Intra-class correlations (ICC) of each knowledge measure between classes at pre-tests varied between 0.03 and 0.11. This indicates that there was relatively little variance explained by classroom level effects at the pretest. On the pretest, separate ANOVAs revealed there were no statistically significant differences between the experimental and control groups in overall knowledge or in any specific aspect of mathematical knowledge (Fs < 0.41, ps > .201). Because of the relatively low differences between the classrooms the confirmatory analysis follows the pre-registered analysis plan without taking into account the clustering of students within classrooms.

Table 1

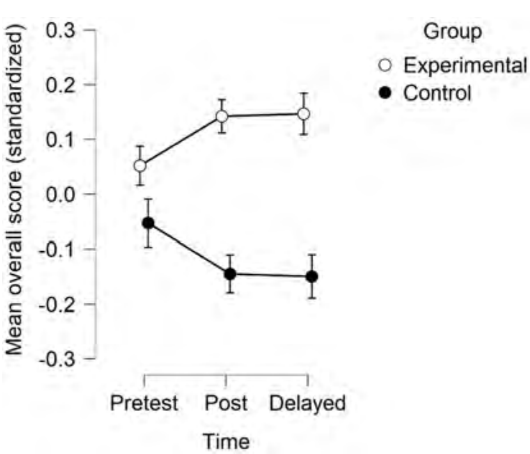
Standardized Means, Standard Deviations (in Parentheses), Cronbachs's Alphas, and ICC's of the Overall Scores and the Three Knowledge Measures Across Testing Points for Experimental and Control Groups

	Overall score			Multiplicative relations			Fraction representations			Fraction Size		
	Pre	**Post**	**Delayed**	**Pre**	**Post**	**Delayed**	**Pre**	**Post**	**Delayed**	**Pre**	**Post**	**Delayed**
Experimental (*n* = 54)	0.05 (0.80)	0.14 (0.82)	0.16 (0.85)	.056 (1.03)	.281 (1.02)	.271 (1.21)	.009 (0.95)	.066 (0.90)	.080 (0.88)	.091 (0.79)	.082 (0.81)	.131 (0.75)
Control (*n* = 53)	-0.05 (0.65)	-0.15 (0.74)	-0.16 (0.66)	-.057 (0.98)	-.286 (0.90)	-.276 (0.63)	-.009 (0.79)	-.067 (0.87)	-.081 (0.89)	-.093 (.0.68)	-.083 (0.79)	-.133 (0.81)
Cronbach's alpha	.81	.82	.87	.74	.68	.87	.83	.86	.85	.71	.68	.66
ICC	0.07	0.16	0.22	0.03	0.16	0.15	0.11	0.17	0.21	0.07	0.07	0.19

To examine the effects of the intervention on students' knowledge, a repeated measures ANOVA was run with overall scores as the dependent variable, measurement point (pre-, post-, and delayed post-test) as a within-subject variable, and group membership (experimental, control) as a between-subject variable. There was a statistically significant interaction effect of measurement point by group, $F(1, 107) = 4.09$, $p = .022$, $\eta_p^2 = .038$, on overall scores. As shown in Figure 5, the experimental group improved their overall mathematical knowledge compared to the control group.

Figure 5

Means for the Standardized Overall Scores on the Pretest, Posttest, and Delayed Posttest for the Experimental and Control Groups

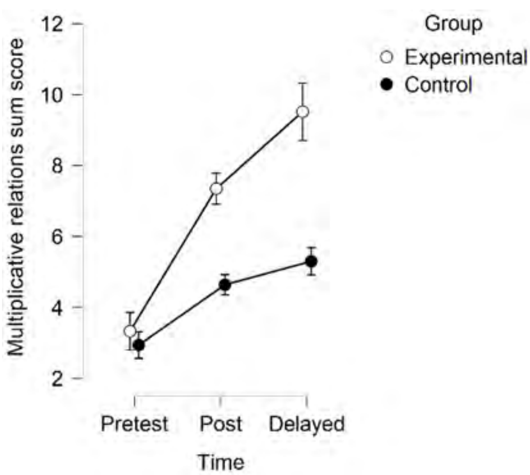


Note. Error bars represent the standard errors for the means.

Subsequent repeated measures ANOVAs were run with measurement points (pretest, posttest, and delayed posttest) as a within-subject variable and group membership (experimental, control) as a between-subject variable on each knowledge measure separately to investigate the effects of the intervention on different aspects of mathematical knowledge. There was a statistically significant interaction of measurement point by group for multiplicative relation knowledge, $F(1, 107) = 7.90$, $p = .002$, $\eta_p^2 = .07$, indicating a moderate positive effect. Figure 6 indicates that the experimental group had greater improvements in their multiplicative relations knowledge than the control group.

Figure 6

Means of the Multiplicative Relation Knowledge Scores on Pretest, Posttest, and Delayed Posttest for the Experimental and Control Groups



Note. Error bars represent the standard errors for the means.

There were no statistically significant interaction effects of measurement point by group for fraction representation knowledge, $F(1, 107) = 0.89$, $p = .41$, $\eta_p^2 = .008$) or fraction size knowledge, $F(1, 107) = 0.48$, $p = .61$, $\eta_p^2 = .005$). Hence, the experimental and control groups' fraction knowledge developed in similar ways over the three measurement points (see Table 1 for the mean values).

Exploratory Results

In our pre-registered analysis, we planned to examine specific effects of the intervention on students' natural number bias if the overall intervention effects on their size knowledge was statistically significant. Despite there being no statistically significant intervention effect on fraction size, there was some differentiation in students size knowledge from pretest to delayed posttest (see Table 1), and such delayed effects would be in line with previous delayed effects found for interventions on spontaneous mathematical focusing tendencies (e.g. Hannula et al., 2005). Thus, we examined the effect of the intervention on students' natural number bias from pretest to delayed posttest. Since all fraction comparison items were designed to capture potential natural number biases, whereas the other fraction tasks were not clearly delineated as capturing a natural number bias, we ran ANCOVA analyses on all fraction tasks separately (Table 2). Students' present at both pre- and delayed posttest were included in the analysis (Experimental group $n = 58$, Control group $n = 54$). Pretest scores were used as covariates, intervention group as the independent variable, and delayed posttest scores as the dependent variable. As shown in Table 2, there were no differences between the experimental and control groups on the representation, missing value, or fraction comparison tasks. However, the groups differed in their accuracy on the number line estimation task. The experimental group's percent absolute error margin decreased compared to the control group from pretest to delayed posttest, $F(1, 112) = 7.63$, $p = .007$, $\eta_p^2 = .06$. These results suggest that while the intervention may not have diminished students' natural number bias, as shown by the lack of difference in the fraction comparison tasks across the two conditions, it may have had some positive effects on their number line estimation skill.

Table 2

Adjusted Means, Standard Errors (SE), and ANCOVA Statistics of All Tasks at Delayed Posttest

Task / Group	Adjusted Mean (*SE*)	95% CI Lower	95% CI Upper	*F*	Sig	η_p^2
Multiplicative relations						
Experimental	9.50 (.83)	7.86	11.13	10.44	.002	.09
Control	5.65 (.86)	3.96	7.35			
Coloring						
Experimental	4.48 (.21)	4.06	4.90	0.005	.942	.00
Control	4.46 (.22)	4.03	4.89			
Number sets						
Experimental	9.14 (.48)	8.18	10.10	2.79	.089	.02
Control	7.98 (.50)	6.98	8.97			
Missing Value						
Experimental	1.68 (.24)	1.21	2.16	0.04	.847	.00
Control	1.62 (.25)	1.13	2.11			
Fraction comparison						
Experimental	1.71 (0.23)	1.26	2.16	0.07	.798	.001
Control	1.63 (0.24)	1.16	2.10			
Number Line Estimation PAE[a]						
Experimental	19.0 (2.00)	15.10	22.90	7.63	.007	.06
Control	26.8 (2.00)	22.80	30.90			

Note. Experimental group n = 58. Control group n = 54.

[a]Number line estimation average PAE (%) scores (less is more accurate answer). PAE mean represents the average distance from correct mark per item.

In order to confirm these exploratory results were not dependent on the nested nature of the data within classrooms and differences in instruction after the intervention, two competing linear mixed models were run with delayed post-test estimation as the dependent variable, experimental condition as the grouping variable, and pre-test estimation as a covariate. In the first model, classroom effects were not included in the model. In the second, random intercepts and slopes for each classroom were estimated. Results show that the model fit decreased with the inclusion of random classroom effect (BIC = 307) when compared with the model without random classroom effects (BIC = 302) and we therefore did not follow-up these results with further analysis of classroom effects.

Discussion

The present study aimed to explore how an intervention designed to promote the ability to recognize and describe multiplicative relations embedded in everyday life affects primary school students' mathematical knowledge. The effects of the intervention on student's ability to recognize and describe multiplicative relations, fraction representation knowledge, and fraction size knowledge were measured. The intervention was effective in promoting the students' overall mathematical knowledge. This appeared to be concentrated on the students' ability to recognize and describe multiplicative relations. There were no statistically significant effects of the intervention on students' fraction knowledge when compared with traditional fraction instruction. However exploratory analysis revealed a potential positive effect on the experimental group students' knowledge on the number line estimation task in the delayed posttest. The intervention replaced, rather than supplemented, five mathematics lessons on fractions, yet there was no difference in fraction knowledge between the two groups. Thus, the added value of the intervention on students' multiplicative relational knowledge suggests an overall positive outcome.

Implications for Teaching and Learning About Fractions

The present study is a second iteration of an intervention program with classroom activities based on enhancing students' focusing on multiplicative relations. Akin to the previous study by McMullen, Hannula-Sormunen, et al. (2019), two of the intervention lessons were scavenger hunt-based activities, where students found examples of multiplicative relations from around their schools. In addition to these activities, students participated in an introductory lesson and "which one does not belong" activities in the classroom. In all lessons, the students were engaged in recognizing and describing multiplicative relations embedded in everyday objects and situations using phrases such as "half" and "twice as many". While results from the previous iteration of the intervention suggested that these activities improved students' SFOR tendency (McMullen, Hannula-Sormunen, et al., 2019), the results of the present study are the first to indicate that these activities also lead to improvements in students' performance on other aspects of mathematical knowledge, in this case, their ability to recognize and describe multiplicative relations when explicitly guided to do so. This augments previous evidence suggesting that focusing on multiplicative relations may play a valuable role in promoting mathematical knowledge (McMullen et al., 2016).

The teaching and learning of fractions is an integral part of the primary school curriculum, yet students struggle with learning fractions (Jordan et al., 2013; National Mathematics Advisory Panel, 2008; Siegler et al., 2011). Importantly, students should come away from mathematics instruction with an understanding of numbers and their relations that includes both the analytic meaning, rooted in the formal mathematical system emphasized in typical classroom instruction, and the representational meaning, which is explicitly connected to real-world quantities and relations (Nunes & Bryant, 2015). The current study shows that it is possible to enhance students' knowledge of multiplicative relations tied to their representational meaning without hindering their more formal analytic fraction knowledge. This might be due to providing students with explicit instruction and extensive opportunities to practice bridging the gap between the formal, analytic meaning and the real-world representational meaning of fractions and multiplicative relations. Bridging this gap might have led the students' to recognize more mathematical features around them in and out of school, hence yielding more self-initiated practice with multiplicative relations and fractions.

Integrating guided focusing activities with normal fraction instruction could expand the ways in which fractions are taught and help students gain a deeper understanding of fractions and their representations. According to Feltovich, Spiro, and Coulson (1997), the development of more adaptive expertise is supported by environments that require meaning making and simultaneously activate multiple concepts. Our intervention allowed students to practice reasoning about formal mathematical concepts in multiple situations. Thus, the intervention may support the development of a more richly connected mental representation of these concepts, akin to the type of knowledge needed for adaptive expertise (Hatano, 2003). Such richly connected knowledge is more likely to be transferable to application in novel contexts, including in other real-world situations and in the mathematics classroom (Lehtinen et al., 2017).

One major factor contributing to the results may be that the lessons succeeded in supporting the students' explicit mathematical language (Hornburg et al., 2018). For instance, the first lesson gave direct instruction on when and how to use precise mathematical language to describe the various representations of fractions and multiplicative relations. Alongside this, the scavenger hunt and "which one does not belong" activities also provided students with the opportunity to practice using the mathematical language. These results are in line with previous studies suggesting that mathematical language and vocabulary support mathematical development in younger children (Purpura, Baroody, & Lonigan, 2013; Purpura, Napoli, Wehrspann, & Gold, 2017). The present study suggests that the relationship between mathematical language and knowledge also holds in older students and with more complex mathematical concepts, such as multiplicative relations.

Limitations and Further Directions

The present study has limitations that have to be acknowledged. The quasi-experimental design limits the conclusions that can be drawn about the learning outcomes. A larger randomized sample would lead to more clarity regarding the effects of the intervention; for example, examining whether there are differing effects of the intervention depending on students' prior knowledge. As well, the sample size of the study does not allow for reliable analysis of the multilevel nature of the data. Thus, the presented results might be attributable to differences between schools or class level teaching between posttest and delayed posttest, especially within the exploratory analysis in which the posttest was not regarded. However, the reported intra-class correlations are within acceptable range and the pre-registration of the analysis suggest that the presented results should be fairly robust.

The multifaceted approach of the intervention activities makes it hard to determine which of the activities contributed towards the results. Hence, it is not possible to determine the causal mechanism that led to the development of mathematical knowledge based on the intervention. It may be that increased skills in how to recognize and describe multiplicative relations around them led to the students in the experimental group to gain more self-initiated practice with multiplicative relations (Lehtinen et al., 2017), which could have led to sustained growth with their formal fraction knowledge despite missing five lessons on the topic. It is particularly difficult to determine how the development of multiplicative relations and fractions were entangled since the intervention contained a mix of these concepts (e.g. both twice as many and half were used, often in connection with each other). It may be for this reason that the effects of the intervention on students' fraction knowledge did not exceed those who completed their business-as-usual instruction. However, results of the exploratory analysis revealed that there might have been a delayed effect on students' ability to estimate fractions on a number line. Future studies are needed to determine whether this is truly the case.

The study places a great deal of emphasis on the teachers involved in the study. The study was relatively naturalistic, as the teachers carried out the lessons in both groups. This brings real-world limitations that would not apply in laboratory settings where teaching could be standardized. On the other hand, having the teachers carry out the intervention activities has the potential to show whether carrying out the activities and understanding the pedagogical idea of the intervention are feasible for teachers. Importantly, the results imply that teachers were able to adopt new pedagogical approaches and tools into their teaching, even with minimal introduction. The teachers voluntarily agreed to participate in the study. Therefore, the teachers may differ from the average teacher in their motivation and pedagogical approach toward mathematics instruction. Expanding this intervention to a larger group of teachers may not be straightforward, and the influences of teacher characteristics on the outcomes of the interventions should be examined in future studies.

Conclusion

The results of the present study support the existing empirical evidence of the effectiveness of interventions aimed at enhancing spontaneous mathematical focusing tendencies on students' learning (Hannula et al., 2005; McMullen, Hannula-Sormunen, et al., 2019). The interrelated relationship between SFOR tendency and traditional mathematical knowledge found in previous studies encourages further studies about the longitudinal effects of such an intervention (McMullen, Hannula-Sormunen, et al., 2019). It is crucial for students to learn that mathematics does not only exist in the school books but also everywhere around us. Being able to understand, observe, and reason about the mathematical features of the environment helps students to apply their knowledge in complex and novel situations, hence contributing to adaptive expertise and a stronger bridge between analytical and representational meaning on numbers. Directing attention to mathematical features in the world might therefore not only help students in their school path but also to understand the world more holistically.

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Data Availability: We cannot publish the original data of the study due to the restrictions negotiated with the project associated with funding of the study. We do not have permission from participants and their guardians to publish data due to requirements of their school administration in the city that the study took place in.

Supplementary Materials

The Supplementary Materials contain the following items (for access see Index of Supplementary Materials below):

- The preregistration protocol for the study
- Six Online Appendices with additional images and tables

Index of Supplementary Materials

Määttä, S., Hannula-Sormunen, M., Halme, H., & McMullen, J. (2019). *Supplementary materials to "Guiding students' attention towards multiplicative relations around them: A classroom intervention"* [Preregistration protocol]. OSF. https://osf.io/hmytv/

Määttä, S., Hannula-Sormunen, M., Halme, H., & McMullen, J. (2022). *Supplementary materials to "Guiding students' attention towards multiplicative relations around them: A classroom intervention"* [Appendice images and tables]. PsychOpen GOLD. https://doi.org/10.23668/psycharchives.5569

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