Pythagoras - Journal of the Association for Mathematics Education of South Africa
ISSN: (Online) 2223-7895, (Print) 1012-2346

Exploring the relationship between commognition and the Van Hiele theory for studying problem-solving discourse in Euclidean geometry education

Authors:
Sfiso C. Mahlaba[1]
Vimolan Mudaly[2]

Affiliations:
[1]Department of Mathematics Education, Faculty of Education, North West University, Mafikeng, South Africa

[2]Department of Mathematics Education, Faculty of Education, University of KwaZulu-Natal, Pinetown, South Africa

Corresponding author:
Sfiso Mahlaba,
sfiso.mahlaba@nwu.ac.za

Dates:
Received: 15 Oct. 2021
Accepted: 11 May 2022
Published: 29 July 2022

How to cite this article:
Mahlaba, S.C., & Mudaly, V. (2022). Exploring the relationship between commognition and the Van Hiele theory for studying problem-solving discourse in Euclidean geometry education. *Pythagoras, 43*(1), a659. https://doi.org/10.4102/pythagoras.v43i1.659

This article is an advanced theoretical study as a result of a chapter from the first author's PhD study. The aim of the article is to discuss the relationship between commognition and the Van Hiele theory for studying discourse during Euclidean geometry problem-solving. Commognition is a theoretical framework that can be used in mathematics education to explain mathematical thinking through one's discourse during problem-solving. Commognition uses four elements that characterise mathematical discourse and the difference between ritualistic and explorative discourses to explain how one displays mastery of mathematical problem-solving. On the other hand, the Van Hiele theory characterises five levels of geometrical thinking during one's geometry learning and development. These five levels are fixed and mastery of one level leads to the next, and there is no success in the next level without mastering the previous level. However, for the purpose of the Curriculum and Assessment Policy Statement (CAPS) we only focused on the first four Van Hiele levels. Findings from this theoretical review revealed that progress in the Van Hiele levels of geometrical thinking depends mainly on the discourse participation of the preservice teachers when solving geometry problems. In particular, an explorative discourse is required for the development in these four levels of geometrical thinking as compared to a ritualistic discourse participation.

Keywords: commognition; Van Hiele theory; Euclidean geometry; geometrical thinking; visual mediators.

Introduction and problem statement

A review of the Grade 12 National Senior Certificate examination diagnostic analysis from 2016 to 2020 reveals that the average pass percentage of Grade 12 mathematics learners in South Africa is below 60% (Department of Basic Education, 2016, 2017, 2018, 2019). This comes after Van Putten, Howie and Stols (2010) felt that South African teachers are not well prepared to teach Euclidean geometry. In alleviating the situation, Machisi (2021) suggests the use of unconventional teaching approaches such as the Van Hiele theory-based teaching and learning approach which as Machisi (2021, p. 1) concluded 'meets learners' needs better than conventional approaches in learning Euclidean geometry'. Furthermore, in alleviating the difficulties faced by mathematics learners, Sfard (2008) proposed that teachers should aim to transform learners' discourse participation from ritualistic to explorative discourse participation. In particular, Sfard proposes this transformation of discourse participation because she believed that learners' mathematical thinking can be encoded from the way they communicate about mathematics. We have seen Sfard's theory being applied in other mathematical domains in South African research such as functions (Mpofu & Mudaly, 2020), numeracy (Heyd-Metzuyanim & Graven, 2016) and equations (Roberts & Le Roux, 2019).

While commognition has a potential for alleviating difficulties in all domains of mathematics, the Van Hiele theory is specifically dedicated to guide teachers on how to alleviate learning difficulties related to Euclidean geometry. The current theoretical article locates the problem in the fact that these theories are currently operating in isolation yet they have a similar purpose of improving learning in mathematics. Wang (2016) combined these two theories (commognition and Van Hiele) with a focus on the elements of mathematical discourses, but this article takes a different approach by focusing on the type of discourse participation. The aim here is to harness the power of commognition in improving geometrical knowledge through the Van Hiele levels when solving geometry problems. Thus, we discuss the tenets of each theory, then discuss how the study viewed the amalgamation of commognition and the Van Hiele theory as means of enhancing geometry understanding during problem-solving. The findings reported in this theoretical article are from a larger PhD study but this article focuses only on the relationship between commognition and the Van Hiele theory when studying problem-solving discourse in Euclidean geometry. Thus, no empirical data will be cited to in this article.

The Van Hiele theory of geometrical thinking

The Van Hiele theory of geometrical thinking was developed by Van Hiele-Geldof (1957) and Van Hiele (1957) towards the completion of their Doctor of Philosophy degrees at the University of Utrecht. The Van Hieles posited that children go through five levels of thought development in their geometrical learning. These levels include recognition (level 1), analysis (level 2), order (level 3), deduction (level 4) and rigour (level 5). This theory was developed within the contexts of learners who are still in their secondary school education but in this review article it is viewed from the lens of preservice teachers (PSTs). Thus, any reference to PSTs is equivalent to secondary school learners in the context of this study. These four levels of geometrical thinking also have their own descriptions that determines the type of learning that occurs in that level. These characteristics per level are summarised below:

- Level 1: PSTs recognise names and recognise figures as a whole (i.e. a square and a rectangle are different).
- Level 2: PSTs begin not only to recognise objects by their global appearance but also to identify their properties with appropriate technical language (e.g. a triangle is a closed figure with three sides).
- Level 3: PSTs begin to logically order these properties through short chains of deduction and understand the interrelationship between figures through their properties.
- Level 4: PSTs begin to develop longer chains of deduction and understand the significance and roles of postulates, theorems and proofs.
- Level 5: PSTs understand the role of rigour and can make abstract deductions that allow them to understand even non-Euclidean geometries.

The Van Hiele theory is characterised by the existence of four characteristics summarised by Usiskin (1982, p. 4) and De Villiers (2012) as follows:

- Fixed sequence: PSTs progress through the levels invariantly which means that a PST cannot be at Van Hiele level *n* without having passed level *n*–1.
- Adjacency: at each level of thought, the intrinsic knowledge from the previous knowledge is extrinsic in the current level.
- Distinction: the linguistic symbols and network of relationships connecting these symbols are distinct in each level.
- Separation: PSTs who are reasoning at different levels cannot understand each other.

These characteristics describe the manner in which PSTs are to proceed through the levels and what is important to consider in each level. Within the five levels of geometrical thinking, the most pertinent ones in the Curriculum and Assessment Policy Statement (CAPS) are the first four levels (levels 1–4) which we focused on in this study because they also apply to PSTs' education. At level 1, geometrical figures are recognised by their visual appearance (form) only, without any reference to their properties and any relationship that might exist between them. At this level, PSTs are able to relate geometrical figures with objects they see in their everyday lives, for example a rectangle looks like a door. These activities are critical at this level as the foundation for the next level (Yi, Flores, & Wang, 2020). In level 2, geometrical figures are identified based on their properties, without considering the relationship that exists between these properties. Thus, secondary school PSTs see a geometrical figure in isolation, not related to other figures. A square can be recognised as having four equal sides and four right angles without relating the property of right angles to a triangle. As PSTs develop to level 3, they begin to see the relationships between the properties of geometry figures. They now can relate a square and a rectangle by ordering their properties and deducing one from the other. At level 4, PSTs' thinking and reasoning are concerned with understanding the meaning of deduction and proof. They can understand the role of theorems, postulates and properties of geometrical figures when doing proofs.

There are critical issues about these levels that apply to the development of thought in geometry, especially for PSTs to use in their instruction. The language and signs used at each level are distinct, such that a relationship that is true at one level might not be true at another (Van Hiele, 1959). The second issue to be aware of is that people who reason at different levels cannot understand each other. Hence, teachers need to attempt to reason at the level of learners, understand their routines and narratives to scaffold them to the next level. Teachers must continuously support learners to construct their deductive relational system in geometry (Van Hiele, 1959), without imposing the relational system of the teacher onto the learners. These levels are critical in the analysis of thinking and reasoning in geometry, because they reveal the characteristics of thinking for both learners and teachers. Since this study had its main focus on PSTs' thinking when solving geometry problems, it seemed useful to incorporate these levels, as they are indicators of geometrical thinking. Even though these levels were not assessed directly in this study, they are pertinent in geometry problem-solving. The PSTs' behaviour when participating in geometry problem-solving can be related to each Van Hiele level and teachers' discourse participation when solving Euclidean geometry problems (see Table 3).

Commognition

This section provides the details of Sfard's (2008) theory of commognition, with a particular focus on the aspects of the theory that relate to this study. We begin with a brief explanation of the theory in general and a few key tenets of the theory. Thereafter, we explain how the theory relates to learning and thinking, and how mathematics learning is a discourse, as we locate the current study within the theory of commognition. This section aims to describe the keywords and the language of Sfard's commognition and how they are key in describing PSTs' mathematical discourse when solving geometry problems. Furthermore, this section aims to explain how commognition has enabled the study to explain PSTs' discourses and what improvements can be made to the theory in the future.

Commognition in a nutshell

In 2008, Sfard published a book titled *Thinking as communicating: Human development, the growth of discourses, and mathematizing*, which explains a theory that can guide and be used to understand mathematical learning. Sfard (2008) uses the amalgamation of 'communication' and 'cognition' to coin the term 'commognition', which she describes as communicating about thinking. She defines commognition by putting into perspective this amalgamation, stating that commognition 'stresses that interpersonal communication and individual thinking are two facets of the same phenomenon' (Sfard, 2008, p. xvii). Here, Sfard asserts that thinking is correlated to communicating, stating that 'thinking is defined as the *individualized version of interpersonal* communication' (Sfard, 2008, p. xvii), closely relating thinking to Bakhtin's (1986, p. 126) idea of the 'superaddressee'. Sfard emphasises that communication and thinking are inseparable. She describes the underpinnings of the theory from the significance of communication, objectification, and elements of mathematical discourse that are significant in mathematics classrooms. In her elucidation of commognition, Sfard differentiates between colloquial and mathematical discourse, where the former is considered to be everyday, spontaneous discourses and the latter is specifically related to mathematics. She posits four characteristics of the latter kind of discourse. A discourse becomes mathematical because of its word usage, visual mediators, narratives and routines.

Commognition is driven by the processes of objectification which is characterised by a double elimination of using metaphors to generate new discourse. This double elimination is characterised by the processes of alienation and reification. According to Sfard (2008, p. 44), reification 'consist[s] of substituting talk about actions with talk about objects, [while alienation] consists in presenting phenomena in an interpersonal way'. A reified talk includes utterances like "*Thabo has developed the concept of geometrical proof and problem-solving*" while an alienated talk includes utterances like "*the sum of all interior angles of a triangle is* 180°". The results of objectification are mainly abstraction, which helps in differentiating discourses of mathematicians and that of, for example, street vendors about similar issues. It is this power of objectification that differentiates between colloquial and mathematical discourse. Sfard asserts that mathematical discourse is characterised by word use, visual mediators, narratives and routines. These characteristics are used to differentiate between three routines: rituals, deeds and exploration; here we only focus on routines and explorations. Thus, in this study we use the elements of mathematical discourse to differentiate between a ritualistic and an objectified discourse using each of their characteristics (see Table 1). All these will be discussed in the following sections in more detail.

Why commognition?

Commognition is a discursive theory that is utilised here for its theoretical potential to explain PSTs' thinking during geometrical problem-solving. It is a theory that acknowledges that everyone thinks on a daily basis but that others do not have direct access to this thinking. Hence, commognition regards thinking as 'an individualised version of *interpersonal communication*' (Sfard, 2008, p. 81). According to this view, thinking cannot just be an isolated activity, but becomes 'the act of communication in itself' (Sfard, 2008, p. 82). This means that whatever utterances are made through discourse by an individual are a consequence of that individual's thinking, and the best way to study that individual's thinking is to analyse their communication in discourse. Furthermore, school learning, as it is for teacher training, should present an opportunity to extend the discourses of learners and PSTs (Ben-Zvi & Sfard, 2007).

Commognition further recognises that, just like learning, thinking develops from a patterned collective activity.

TABLE 1: The comparison between rituals and explorations in commognition.

Elements of discourse participation	Ritual	Exploration
Closing condition or goal	Relationship with others (improving one's position concerning others)	Description of the world (production of endorsed narrative about the world)
By whom the routine is performed	With (scaffolded by) others	No need for scaffolding; can be performed individually
For whom the routine is performed	Others (authoritative discourse)	Others and oneself (internally persuasive discourse)
Applicability (changing the when, keeping the how constant)	Restricted: the procedure is highly situated	Broad: the procedure is applicable in a wide range of situations
Flexibility (changing the how, keeping the when constant)	Almost no degree of freedom in the course of action	The procedure is a whole class equivalence of different courses of action
Correctability	Cannot be locally corrected; has to be reiterated in its entirety	Parts can be locally replaced with equivalent subroutines
Acceptability condition	The activity has to be shown to adhere strictly to the rules defining the routine procedure; the acceptance depends on other people	The narrative produced through the performance must be sustainable in such a way that the acceptance is independent of other people
Words and mediators use	Phrase-driven use of keywords as descriptions of extra-discursive mediators	Objectified use of keywords as signifying objects in their own right

Source: Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press

Commognition recognises that thinking can be objectified or disobjectified, but rests mainly on the significance of explaining mathematical thinking through disobjectified discourses. Therefore, thinking can be explained by analysing the discourses of PSTs. Thinking, as a patterned collective activity, happens through communicating with others and ourselves. Thinking is therefore dialogical (Sfard, 2008), and thinking is modified and changed as we communicate with others.

Mathematics is considered a difficult subject in South African schooling. Furthermore, geometry is seen as the topic where learners perform the poorest and where even teachers struggle to teach geometry effectively (Naidoo & Kapofu, 2020; Tachie, 2020; Van Putten et al., 2010). Most teachers who do manage to get learners to pass geometry use the drilling of theorems and how to prove them. Some teachers rely mainly on the possibility of questions being repeated in the standardised tests (Machisi, 2021). To improve the dire situation of teachers with insufficient geometry knowledge, which leads to learners performing poorly in geometry, we need to approach this problem as a collective, ensuring that teachers are properly trained to teach geometry in secondary schools. We need to tap into PSTs' thinking when they solve geometry problems to see how they think and then design proper tasks and teaching strategies to enhance their level of geometry thinking to a suitable one, where they would be able to teach geometry effectively to ensure meaning making within learners. In this way, commognition offered a window to tap into PSTs' thinking when they solve geometry problems, to understand their thinking. In commognition, thinking is voluntary, individuals engage in thinking through their continued participation in the mathematical discourse. Hence, in this view, geometry learning for PSTs is a consequence of their continued participation in the community of mathematics, which mainly originates from the participationist theories of learning (Lave & Wenger, 1991). As a human activity, participation in the activity of communication has emotional implications, which need to be understood properly, especially if the communication is among competing peers (Heyd-Metzuyanim, 2013). Preservice teachers need to move from being ritualistic participants in the discourse of geometry to being individual geometry problem-solvers who can teach geometry effectively. Since Vygotsky (1978), the activity of being a peripheral participant in geometry discourse for PSTs begins with the help of a more knowledgeable other. Hence, the role of the lecturer (trainer) is important as a knowledgeable other. Hence, if PSTs are learning, they become more and more independent of the lecturer and their thinking as they learn does not require the aid of the lecturer, as they become independent thinkers.

Objectification in discourse

The use of metaphors is common in all discourses, where words are partitioned into an unfamiliar discourse because of their familiarity and their readiness to be used in that discourse (Sfard, 2008). The building of mathematical knowledge from concrete objects has long been recognised (Dienes, 1960). In the current article, it seemed significant to distinguish between PSTs' discourse about objects and how they communicated about mathematical objects. The 'process in which a noun begins to be used as if it signified an extra-discursive, self-sustained entity (object), independent of human agency' (Sfard, 2008, p. 300) is known as objectification. An example, in Euclidean geometry there can be a statement such as "*angle ABC is equal to angle BCA because of angles opposite equal sides*" instead of "*this angle is equal to that angle because this is an isosceles triangle*". In commognition, objectification is considered to encapsulate two inseparable discursive moves: reification and alienation. According to Sfard (2008, p. 44), 'reification is the act of replacing sentences about processes and actions with propositions about states and objects'. Hence, in this article reification describes PSTs transforming their talk about the process of problem-solving into talk about objects. Reification allows PSTs to be concise about what they are communicating, which makes it more flexible and applicable in mathematical discourse. A reified version of the statement "*in the majority of the tests and tasks dealing with Euclidean geometry proof in school, he regularly did well and achieved very good marks*" is "*he has acquired the concept of Euclidean geometry proof*".

Alienation on the other hand, involves the removal of the reified discourse from the actor. Alienation refers to 'using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings' (Sfard, 2008, p. 295). Alienation includes the use of passive voice in a particular mathematical sentence, for example "*the angle between the tangent and the chord is equal to the angle subtended by the chord in the alternate segment*", which removes any personal attachments to the statement (Sfard, 2021). Alienation allows PSTs to engage in the discourse of geometry problem-solving in an impersonal way. These alienated geometry discourses can be thought of as theorems, axioms or postulates, etc., since they are monological[1] (Bakhtin, 1986). In this view, a geometrical proof is the final stage of the process of objectification, where the human experiences and constructions are removed from the discourse; it is the stage of alienation itself. This is a hint that geometry teaching and learning should not begin with the process of proving, but that of investigation and exploration (De Villiers & Heideman, 2014). Alienation is seen as contributing to the genesis of mathematical knowledge and understanding (Morgan & Tang, 2016). Hence, once PSTs can alienate a certain mathematical discourse, they begin to understand and construct mathematical knowledge, specifically in mathematical discourse and not just colloquial discourse.

Objectification has been shown to have several advantages in the process of mathematical learning. Ben-Yehuda, Lavy, Linchevski and Sfard (2005) show that objectification may lead to mathematical discourses that contribute to increased levels of mathematical performance. Objectification further

1. A monologically understood world is objectified and corresponds to a single and unified authorial consciousness.

makes the way we communicate about mathematics more effective and provides a method of attaching objects into our mathematical discourses. Once we objectify, we create an ‘object’ or a ‘thing’ that has permanence in our mathematical discourse, which can also be an abstract entity. Through this objectified discourse, PSTs accumulate knowledge through participating in successive mathematical discourses that increase in complexity and applicability. The reification process relates directly to the mathematical objects objectified in discourse and it allows PSTs to endorse the discourse as a mathematical one. Hence, objectification, in this case, underlies the patterned ways in which we think. However, objectification removes personal experiences of learning and thinking in the discourse. As Sfard (2008, p. 56) puts it, objectified ‘descriptions deprive a person of the sense of agency, restrict her sense of responsibility, and, in effect, exclude and disable just as much as they enable and create’. This is possibly a consequence of objectification removing the PSTs’ thinking and learning experiences and the way in which they might communicate in their everyday lives in the discourse. The objectification of mathematical discourse means that colloquial discourse is reduced into a more specific mathematical discourse, consisting of specific word usage, routines, narratives, and some visual mediators. In geometry, this can be articulated by the colloquial utterance “*this angle is equal to that angle*” compared to the more objectified utterance “*<ABC = <BCA*”. These, as explained in commognition, are the main characteristics of a mathematical discourse (Sfard, 2008). Hence, mathematics from this point of view, can be identified as a discourse.

Mathematics as a discourse

Sfard (2008, p. 161) describes mathematical discourse as ‘an autopoietic system’, whose end products are exactly the constituents or objects of its present discourse (cf. Sfard, 2021). With its salient features of Lave and Wegner’s (1991) idea of ‘legitimate peripheral participation’, commognition further describes mathematical learning as participation in the mathematical discourse (Sfard, 2008). Vygotsky (1978) highlights the significance of communication in learning, and Lave and Wenger (1991) highlight the significance of participation in the process of learning. Furthermore, the idea of mathematics as a discourse has been strengthened by researchers such as Nardi, Ryve, Stadler and Viirman (2014) and Sfard (2014), saying that to not consider mathematics as a discourse would be ludicrous. The objects produced in a mathematical discourse are defined as abstract discursive objects containing mathematical signifiers and they are the products of objectification (Sfard, 2008). A simple geometrical example would be constructing an accurate diagram from a given statement.

The notion of signifiers is important, also, in considering mathematics as a discourse. In commognition, ‘signifier’ refers to any primary object that encapsulates its realisation procedures (Sfard, 2008). An example of a signifier can be ‘words or symbols that function as nouns in utterances of discourse participants’ (Sfard, 2008, p. 154). Specifically, mathematical objects are defined as abstract discursive objects with distinct mathematical signifiers (Sfard, 2008). A geometrical diagram that is used to solve particular geometrical problems can be recognised as a signifier. Human beings are able to take part in different types of discourse, and in some discourses humans fall short. For example, it might be difficult for a visual artist to participate in a specialised mathematical discourse, in the same way that it would be difficult for a mathematician to participate in a discourse about project management or geography.

Furthermore, Sfard (2008) differentiates between two types of discourses. Colloquial discourses are everyday-life discourses, visually mediated by pre-existing objects (Sfard, 2008). Colloquial discourses of mathematics use everyday language that is reified and colloquial narratives can be endorsed by PSTs through their engagement in discourse with the knowledgeable other or through repetition. Mathematical discourses are, however, distinct from colloquial discourses. What makes mathematical discourses distinct is the fact that mathematics is characterised as a domain-specific discourse that can be identified by its word usage, visual mediators, routines and narratives (Sfard, 2008). In South African mathematics education, the use of words and symbolic information is common, and it seems like an attempt to strengthen a formal mathematical discourse. This is also evident in the strong way in which symbolic language is evident in geometry proofs with very limited word usage. For example, most geometrical proofs are completed with statements in symbols such as “*<ABC = <DAC*” with a few justification statements in words such as “*tan-chord theorem*”. Mathematics is presented in an objectified way in South African high schools. The discourse does not go through the process of reification, but is alienated. This form of mathematical learning is what Freudenthal (1973) refers to as the anti-didactical view. Teachers and learners do not engage in discourse for the objective of reification, but simply aim to memorise alienated mathematical discourses. Hence, mathematics, and especially geometry, is challenging for many South African learners, as described in this article. Let us take a closer look at the elements of any mathematical discourse as delineated in commognition.

Word usage

The key to identifying the realm in which a discourse belongs is the keywords used by interlocutors in the dialogue. If you hear people dialogising and the keywords ‘tangent’, ‘chord’ or even ‘straight line’ occur, we know this dialogue belongs within the constraints of geometrical discourse in the high school curriculum. Mathematical discourses in schools are similar. There is no way one can talk of ‘quadratic equations’ or ‘the tan-chord theorem’ in any subject other than mathematics. Even though some word usages in mathematics appear in colloquial discourse, they form part of the formal mathematical discourse that helps in understanding mathematical concepts. The significance is highlighted by Sfard (2008) clearly when she states ‘word use is an all-important matter because being tantamount to what others call "word meaning," it is responsible for what the user can say about (and thus see in) the world' (p. 133). An 'integral' in colloquial discourse is different from an 'integral' in mathematical discourse and to converse with a mathematician about an integral will not be the same as that with any other citizen who has no knowledge of mathematics. Hence, in this study, the focus here was placed on PSTs' use of mathematical keywords. Using this commognition, one can observe PSTs regarding whether in their utterances they substituted a word from their colloquial discourse with a key mathematical word, and how this affected their understanding of their thinking in problem-solving. This can allow the researcher to make comments on the PSTs' use of mathematical keywords in their practice and the use of words in mathematics learning in South African schooling.

Visual mediators

These are the visible objects that can be used by interlocutors in communicating (Sfard, 2008). These visual objects are significant, because they can act as identifiers of colloquial and mathematical discourse. In mathematical discourse, visual mediators are only developed to mediate a specific discourse that has developed or will develop at a specific time and in a specific space. These visual mediators can be used as thinking aids in mathematical discourse and they can enhance the discourse. Shapes, notations and geometry diagrams included in this review provide clinical examples of visual mediators. Preservice teachers engage individually with these visual mediators. Especially in geometry, diagrams are important and, in most cases, they are key to solving geometry problems. If a geometry problem is posed in words, PSTs need to come up with their own diagram to solve the problem. This visual diagram can provide PSTs with discursive prompts, which can allow them to recall specific knowledge and ways of mathematical problem-solving. Furthermore, visual mediators are cues for visual thinking and many prominent mathematicians have relied to a large degree on their visual thinking for success (Clements, 1981; Shepard, 1978). Commognition holds that this visual thinking and imagery is enhanced by the ability of visual mediators to induce multiple ways of thinking about a certain problem.

Narratives

In Sfard's (2008) words, narratives are:

> any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures. (p. 134)

Narratives in discourse can be thought of as ideas that need to be discussed and endorsed mathematically, and once a certain narrative is endorsed, it is considered a theory (Sfard, 2008). The goal of mathematics is to produce endorsable narratives through the process of alienation in objectification. This means that PSTs need to produce narratives that can be derived through the use of mathematical rules and methods. In geometry, postulates, axioms and theorems can be considered as narratives, since they can be derived using mathematical laws. These different narratives can be used in isolation or in conjunction with others to solve different mathematical problems. A geometrical example of a narrative used in this way is a theorem such as "*the sum of angles of a triangle is equal to* 180°" to support a particular geometrical statement.

Routines

Routines are significant and special in mathematical practice. Routines, according to Sfard (2008), 'are repetitive patterns characteristic of the given discourse' (p. 134). Routines are repetitive and well-articulated discursive patterns, which may include the process of mathematical generalisation or completion of certain procedures (Berger, 2013). Routines are regulated by mathematical rules, such as what counts as a definition of a square, what constitutes a mathematical proof or even how to calculate the area of a triangle. Once a narrative has been endorsed mathematically, it can be used as a routine or a reason to endorse other mathematical narratives. Different patterned ways of communicating about geometry exist and PSTs can use any method. These can be observed in the way PSTs use words and visual mediators to derive new or substantiate existing mathematical narratives. For example, a geometry proof can be approached from different perspectives and if a proof is elucidated in an objectified manner, it stands to be true unless it is refuted by others. Furthermore, the context, the teacher, the classroom environment, the learners and other factors may influence PSTs' discourse about their thinking.

Differentiating between two types of discourse participation in commognition

Mathematics education has been characterised by different beliefs about teaching and learning. A recent development was commognition, a belief that equates learning to communicating about thinking, which becomes a 'legitimate peripheral participant' in mathematical discourse (Lave & Wenger, 1991; Sfard, 2008). Rooted within sociocultural theory, commognition provides an appropriate analytic framework for analysing PSTs' discourses. Participating in the mathematical discourse during learning can be described as based on the complex dyadic relationship between discursive routines in rituals and explorations. To describe PSTs' thinking, their utterances were examined for the objectification of the object of geometry problem-solving, and the substantiation of their endorsed narratives and considered explorations. In this subsection, we explain the distinction between ritualistic and explorative discourse participation in commognition.

Rituals are socially oriented actions performed to conform to society. One engages in rituals to avoid punishment, please someone or for gain certain rewards (Lavie, Steiner, & Sfard, 2019). For example, if Grade 12 learners memorise and reproduce proofs for certain Euclidean geometry theorems because these proofs are required in the examination, the participation of the learners in classroom discourse becomes purely ritualistic.

The main driver of rituals is societal expectations. When one feels obliged to perform in a certain way to please someone, then one engages in rituals. If learners in Mrs X's classroom were directing all answers to her without giving reasons and the teacher was the one endorsing the answers as correct or incorrect through giving applause for positive evaluation, these learners' participation in this discourse was solely motivated by getting applause or positive evaluation from the teacher (Heyd-Metzuyanim & Graven, 2016). In a way similar to ritualistic participation in the discourse, PSTs continually sought affirmation and approval from the interviewer as they solved geometry problems. In ritualistic discourse participation, interlocutors obtain discursive cues by imitating their knowledgeable other peers or teachers or rely on previous experiences. These discursive actions are scaffolded by others; hence their level of applicability is highly restricted. Here, ritualistic discussants aim to act in harmony with no degree of flexibility in their actions, merely to imitate what others are doing. As a consequence of their high reliance on performance instead of knowing, rituals tend to follow strict rules to ensure that they can be produced by others. In ritualistic discourse participation, all rituals that do not succeed are simply repeated instead of being corrected or modified, which is a further indication of the rigidity in the applicability of rituals. This form of discourse participation does not require special substantiation of the produced narrative, because it focuses mainly on the *how* of a routine (as explained earlier). Hence, the steps for the process are listed clearly and the performer needs to follow these steps to reproduce the ritual.

As opposed to rituals, explorations are not aimed at pleasing or conforming to societal expectations, but at advancing theory. Particularly, routines can be characterised as explorations if they produce narratives contributing to a mathematical theory instead of tangible objects (Sfard, 2008). Explorative participation in mathematical discourse is characterised by objectification, where words denote realistic mathematical objects, and the participant views different realisations of the same thing (such as $x^2–4$ and $(x–2)(x+4)$) to be interchangeable (Heyd-Metzuyanim & Graven, 2016). Rituals are mainly 'process-oriented', while explorations are 'outcome-oriented' (Lavie et al., 2019). Explorative activities aim at inventing, producing or discovering some truths or facts about mathematical objects. These routines are applicable in a variety of contexts because the resulting narratives are endorsed and form part of a mathematical theory. PSTs who exhibit this meta-level type of thinking can use these endorsed narratives in a wide range of contexts while providing multiple, but acceptable, means of substantiations for their routines.

Too often learners are considered 'good' at mathematics because they conform very well to society's expectations, and those who do not conform are usually labelled as 'outcasts' and 'weak at mathematics' (Heyd-Metzuyanim & Graven, 2016). This view may affect learners' formations of their identities in the classroom community. Participation in explorative discourse requires a good understanding of some *already endorsed* mathematical knowledge; hence rituals can be thought of as significant building blocks towards explorations (Lavie et al., 2019; Nachlieli & Michal, 2019). Sfard (2008, p. 223) claims that rituals and deeds are 'developmental predecessors of explorations'. Hence, meta-level learning of mathematics moves from ritualistic discourse participation towards explorative participation, and the ritual develops into exploration through the process of rationalisation. At the beginning of learning, when a routine constitutes objects or metarules unfamiliar to the learner, learning becomes highly unlikely. Hence, at this stage, for beginners to be conversant with a certain mathematical discourse, they rely on imitating a knowledgeable other in that specific discourse. The beginner imitates the rules and procedures of the knowledgeable other, adapting them until they individualise the mathematical discourse, but at this early stage, their discourse participation is ritualised. At this stage, beginners know *how* to perform a certain routine, but have not struck the balance as to *when* to perform it. For learners to be able to transform their rituals into explorations, they need to reflect continually on their performance of the ritual, while examining the rationale for the expert performance as they participate in mathematical discourse. In Vygotsky's idea, ritualistic discourse participation occurs in the Zone of Proximal Development (ZPD), where the learner can participate in the collective patterned thinking or the performance of a routine, but is incapable of individual performance. By contrast, explorations look for the connectedness of different routines so that once learners have acquired explorative discourse participation, they hold a network of interconnected routines, instead of disconnected ones. The explorative form of objectifying particular routines allows learners to compress mathematical knowledge to allow them to solve a multitude of mathematical problems with few routines. Furthermore, commognition recognises that metalevel learning occurs 'when the learner is exposed to commognitive conflict' (Sfard, 2008, p. 260). Sfard (2008, p. 296) describes the commognitive conflict as the conflict that 'arises when communication occurs across incommensurable discourses'. Interlocutors who are in commognitive conflict participate in discourses that differ in their utilisation of words, mediators and routines, which might allow them to endorse narratives that seem contradictory. Commognition warns the commognitive researcher not to confuse commognitive conflict with cognitive conflict. The difference is tabulated in Table 2.

TABLE 2: Comparing cognitive conflict and commognitive conflict.

Elements of the conflict	Cognitive conflict	Commognitive conflict
The conflict is between	The interlocutor and the world	Incommensurable discourses
Role in learning	Is an optional way for removing misconceptions	Practically indispensable for meta-level learning
How is it resolved?	By student's rational effort	By student's acceptance and rationalisation (individualisation) of the discursive ways of the expert interlocutor

Source: Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press

The link between commognition and the Van Hiele theory

A brief explanation of the relationship between the two theories as conceived in the study suffices. In particular, this relationship is described based on the type of discourse participation evident from PSTs and how this type of discourse participation was related to each level of the Van Hiele theory. Firstly, the relationship is explained based on teachers' failed behaviours during geometry problem-solving instead of their successes, because this will raise awareness of the severity of the need to develop competent geometry teachers in South Africa. Teachers' behaviours, described in Table 3, were constructed based on the properties of each Van Hiele level of geometry thinking. These behaviours can range from any example that characterises behaviour at a certain Van Hiele level of geometry thinking through the processes described by the Van Hieles. The relationship between teachers' behaviours and the Van Hiele levels is a critical one but elementary, because the teachers' behaviours were formulated based on the properties of the Van Hiele levels. Nevertheless, what is not evident here is that, since competency at each level is dependent on competency at the preceding levels, teachers' struggles at a particular level might not be because of struggling with geometry reasoning related to that level, but the preceding levels. For example, a failure to prove a geometry theorem (level 4) might be hindered by not only a deficient competency to order objects according to the relationship that exists between their properties (level 3), but also PSTs' visualisation skills (level 1).

A PMT who is operating at a ritualistic Van Hiele level 1 not only struggles with the visual identification of geometrical objects using their appearances but also fails to link descriptions with their visual appearances. This means that this PMT is not able to identify a theorem or property using the appearance of the diagram in a particular problem. This PMT is then not ready to proceed to the other levels of geometrical thinking and must stay in that level until they have mastered the explorative Van Hiele level 1. Wang (2016) found that PMTs uses phrases related to visuals such as 'looks like' even when substantiating their arguments. This explanation goes for all the levels and, as a critical finding of this study, this progression is better explained in Figure 1. As seen in Figure 1, progression to the next Van Hiele level requires that PSTs operate fully in the explorative discourse participation where they rely on their own thinking and experiences and use objectified discourse when talking about solving geometrical problems. Table 3 shows how PMTs' discourse changes as their geometrical thinking moves to a higher Van Hiele level (Wang, 2016) but this does not indicate that their discourse participation changes. Changes in discourse participation from ritualistic to explorative (discourse of experts) requires that PMTs apply some reification to their discourse allowing them to switch from colloquial to mathematical use of Euclidean geometry discourse (Sfard, 2021). These changes are observed in how PMTs use the elements of mathematical discourse in their discourse participation during geometry problem-solving. That is why PMTs who cannot differentiate between sufficient and necessary properties, conditions or even geometrical steps for a particular proof to be true are indeed still lacking not only Van Hiele level 2 but also level 1 (Sfard, 2007). In peer discussions, such changes in discourse (and perhaps discourse participation) can be a consequence of the coalescence among the discourses of the peers (Ben-Dor & Heyd-Metzuyanim, 2021; Kaur, 2020).

The transformation in discourse is a critical and necessary condition for mathematical learning (Heyd-Metzuyanim, 2018) and, in many instances, transforming from a ritualistic to an explorative discourse may be dependent on the scaffolding from knowledgeable others. As such, transforming to being able to do deductive geometrical proofs individually (from level 3 to level 4) requires that one participates in the discourse of proof ritualistically first and, through support and scaffolding from knowledgeable others, one gradually transforms to explorative discourse participation (Ben-Dor & Heyd-Metzuyanim, 2021). This is also supported by Cooper

TABLE 3: Relationship between preservice teachers' discourse participation and the Van Hiele levels of geometrical thinking.

Level of geometrical thinking	Ritualistic discourse participation. (Characterised by colloquial discourse) (Routines scaffolded by others)	Explorative discourse participation. (Characterised by objectified discourse) (Routines performed individually)
PST recognises geometrical figures by using only their visual appearance		
1	PST uses visual cues without corroborating them with properties, theorems or definitions to identify mathematical objects. Sometimes fails to link descriptions with their visual mediators.	PST corroborates visual cues, theorems, properties and definitions to identify mathematical objects. Shows a good understanding of linking descriptions with their corresponding visual mediators.
PST begins to use properties to identify mathematical objects but without ordering these properties, so a square is not recognised as a rectangle		
2	PST uses properties to identify mathematical objects as they appear but does not connect these properties to perform routines, endorse narratives or produce endorsable visual narratives. Sometimes assumes properties and definitions of mathematical objects based on the visual appearance of the diagrams.	PST uses properties to identify mathematical objects as they appear and can connect these properties with the performance of routines and endorsing narratives. Never uses visual appearance of diagrams to conclude about properties or definitions of mathematical objects.
PST begins to logically order properties of geometrical figures and can deduce properties from others		
3	PST relates figures using their properties but still relies on remembering procedures and algorithms to proceed with performing routines and endorsing narratives nor can they substantiate their conclusions. They also fail to see the link and relationship between properties of geometrical objects. Thus, they struggle to form even short deductive chains of arguments.	PST relates figures using their properties and uses explorations to produce discursive mathematical objects (*e.g. visual mediators*) and they can substantiate their conclusions. They understand the link and relationship between the properties of geometrical objects and can use this link to form short deductive chains.
PST can now display their understanding of logical deduction through proving and using axioms and theorems to substantiate their mathematical behaviours		
4	PST attempts to follow strict procedures and previous experiences to describe abstract mathematical objects and construct narratives through logical deduction but fails to substantiate the subroutines leading to the production of the narrative.	PST explores different approaches interconnectedly to invent problem-solving strategies and construct narratives through logical deduction. They can substantiate all subroutines followed in producing the narrative and are flexible in their routine performance.

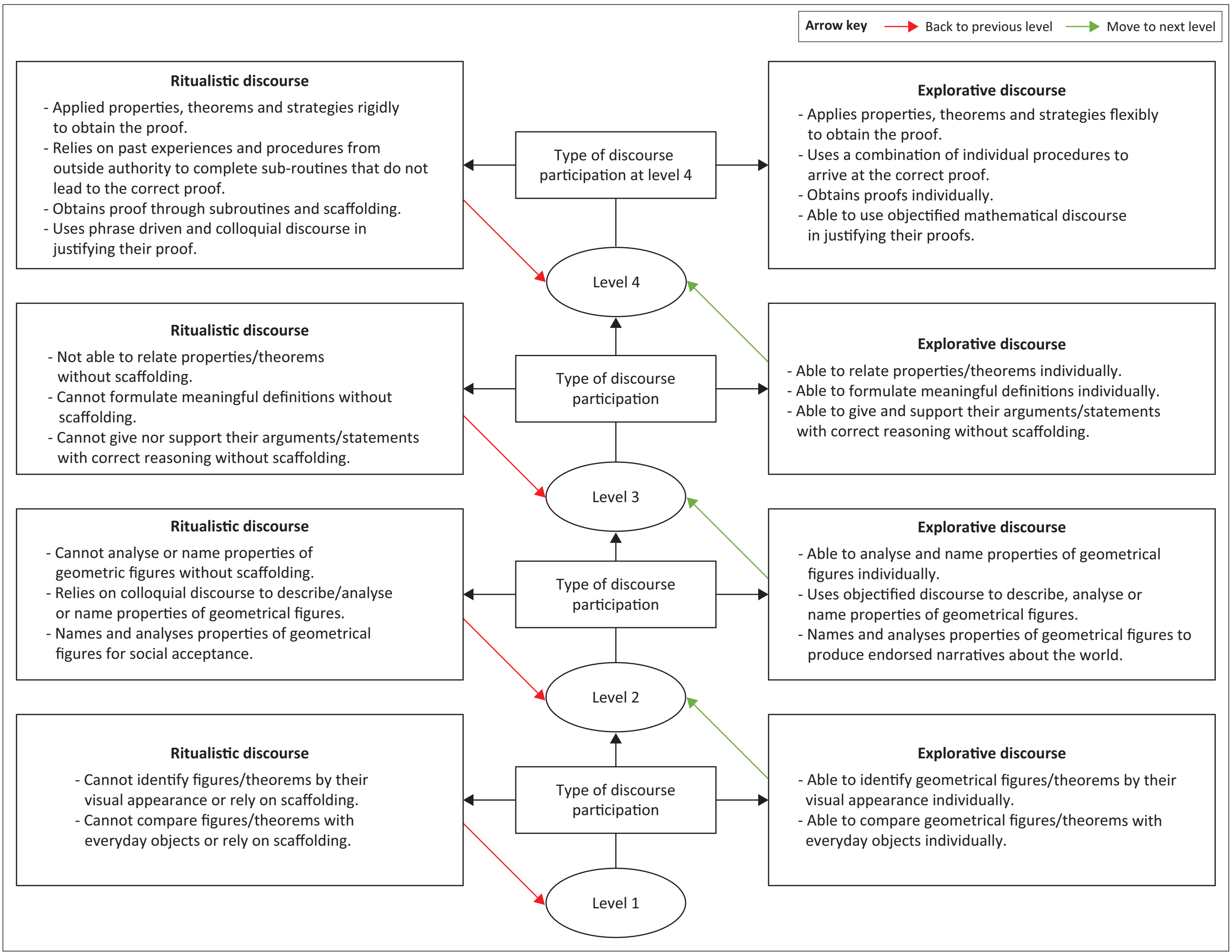

FIGURE 1: A commognitive analysis of pre-service teachers' geometrical thinking development through Van Hiele levels of geometrical thinking.

and Lavie (2021) who found that developments in mathematical thinking and specifically learning cannot avoid ritualistic discourse participation (cf. Kaur, 2020). As such, Table 3 as corroborated by Figure 1 indicates that development in geometrical thinking is reliant on PMTs transforming their discourse from ritual to exploration. Thus, PMTs participate in the Euclidean geometry discourse with the aim of producing endorsable narratives about geometrical objects. However, the transition to either a higher Van Hiele level or from ritualistic to explorative discourse participation is complex and a deeper understanding of the discourses at each Van Hiele level and how transformation occurs from ritualistic to explorative discourse participation is warranted. For example, PMTs can be operating at Van Hiele level 4 but their word usage, visual mediators, narratives and substantiation of Euclidean geometry routines may differ depending on their discourse participation (Wang & Kinzel, 2014). Even further, PMTs may be operating at a ritualistic Van Hiele level 4 and still have differing discourse in their talk about geometrical objects. Thus, the transformations in geometrical thinking represented in Figure 1 are complex and sometimes may constitute more complex discourse relationships than the ones represented in Figure 1.

Conclusion

Upon conducting this literature study, it was found that development to higher Van Hiele levels was dependent on the discourse participation of the PSTs according to Figure 1. Thus, in conclusion, we discuss the study's main contribution to the theories of commognition and the Van Hiele theory of geometrical thinking. In particular, we discuss how the findings from the study amalgamated commognition and the Van Hiele theory of geometrical thinking to contribute to new knowledge. We use Figure 1 to guide the discussion and this allows us to show how PMTs' discourse participation can be used to enhance their development of geometrical thinking through the Van Hiele levels of geometrical thinking. The below discussion is guided by the characteristics of the Van Hiele levels and how PSTs participated in discourse in this study. Thus, we discuss this contribution based on the movement from level 1 to level 2, movement from level 2 to level 3 and so on ending at level 4 which is the level at which Grade 12 PSTs should be operating. Furthermore, the discussion is guided by the type of discourse participation teachers displayed in each level, from the findings.

Development from Van Hiele level 1 to level 2

The identification of geometrical figures by their appearances only is the critical determining skill in level 0 because only the recognition of the figure is required. The PSTs who could not identify geometric figures by their appearances during discourse and those who relied on scaffolding from the interviewer lack this critical geometrical skill. Furthermore, in advanced geometry, theorems can also be identified through the appearance of the diagrams and PSTs who could not identify particular theorems that were available in the diagram lack visualisation. From the literature findings, the lack of visualising theorems can hinder PSTs' problem-solving but once a scaffold was given by the interviewer, they were able to complete subroutines that involved the theorem as exploration. Thus, PSTs who lack this visualisation skill should not be allowed to progress to level 1 because visualisation is a prerequisite in level 1. They must stay at level 0 as indicated by the red arrow in Figure 1. However, PSTs who could identify geometrical objects based on their appearances individually and could compare these objects with objects they encounter daily could be allowed to move to level 1 as they have mastered the skill of visualisation as indicated by the green arrow in Figure 1. That is because the former is still reliant on ritualistic discourse participation while the latter relies on explorative discourse participation.

Development from Van Hiele level 2 to level 3

The defining characteristic in Van Hiele level 1 is that PSTs should have moved from not only recognising objects by their appearances but also linking the objects to their properties. However, interrelating the properties of objects is still not developed at level 1. The theoretical findings indicate that PSTs who cannot name properties of geometrical objects without scaffolding and those who used colloquial discourse to name these properties must not move to Van Hiele level 3. Furthermore, PSTs who mentioned incorrect properties to gain social acceptance, relying on the usage of ritualistic discourse and past experiences should also remain in ritualistic Van Hiele level 2. These PSTs have not mastered the ability to link a geometrical figure with its property and thus they should remain in ritualistic Van Hiele level 2 until this skill is mastered. Those PSTs who are promoted to explorative Van Hiele level 3 are those who can individually link a particular property to a particular geometrical figure individually. These PSTs can use objectified discourse to mention and link geometrical properties with their geometrical figures which according to the theoretical findings relates to explorative discourse participation. This link can allow PSTs to produce endorsed narratives about the geometrical world and these PSTs can be promoted to explorative Van Hiele level 3 because they show mastery of the skill of linking geometrical figures with their properties using explorative discourse.

Development from Van Hiele level 3 to level 4

At level 3 PSTs do not just link properties with their geometrical figures but can logically order and interrelate properties to understand the relationship between geometrical figures. Mastery of level 3 means that PSTs are getting ready for logical deduction required in proofs. At this level, PSTs should not be reliant on scaffolding to link properties and still be promoted to level 4 because that shows that they have not mastered Van Hiele level 3. Furthermore, the formulation of meaningful definitions is critical in level 3 and if PSTs still rely on scaffolding they are not ready for level 4. The understanding of properties and theorems is critical in proofs and if one has not mastered the link between properties and the relationship between figures, then one is not ready to move to level 4. This is because PSTs will produce a whole proof without giving proper reasoning for their arguments or statements with correct reasons. Only PSTs who show logical understanding of how properties of different figures link and are able support their narrative individually using objectified discourse should be allowed to proceed to level 4 which is the last Van Hiele level required in the CAPS.

Discourse participation at level 4

Level 4 requires that PSTs be able to use experiences from the previous levels to understand the role of properties, theorems and the links therein when doing geometry proofs. At this level, PSTs now begin to develop longer arguments to perform geometrical proofs and can successfully substantiate each argument with an endorsed mathematical narrative. Those PSTs who continually rely on the same procedure applied in exactly the same way instead of being flexible in their strategies are not ready for level 4 and they should be demoted to level 3. At this level, relying on procedures, subroutines and scaffolding from others to obtain a proof does not guarantee that one will be able to prove similar problems in the future; independence is required to master level 4. Justifying statements during proofs using phrase-driven and colloquial discourse instead of objectified discourse also shows that one has not mastered level 3; thus, they must be demoted to level 3. A PST who performs geometrical proofs independently and uses explorative discourse when talking about geometrical proofs can be thought of as ready to teach geometry at Grade 12 level in the CAPS.

Acknowledgements

Competing interests

The authors declare that no competing interest exists.

Authors' contributions

S.C.M. completed the article individually and V.M. was the supervisor of the PhD study and contributed by checking the accuracy of the information in the article.

Ethical considerations

This article followed all ethical standards for research without direct contact with human or animal subjects.

Funding information

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data availability

Data sharing is not applicable to this article, as no new data were created or analysed in this study.

Disclaimer

The views expressed in this article are of the authors only and do not represent the official position of the two institutions.

References

Bakhtin, M.M. (1986). *Speech genres and other late essays* (V. W. McGee, Trans.). Austin, TX: University of Texas Press.

Ben-Dor, N., & Heyd-Metzuyanim, E. (2021). Standing on each other's shoulders: A case of coalescence between geometric discourses in peer interaction. *The Journal of Mathematical Behavior, 64*, 100900. https://doi.org/10.1016/j.jmathb.2021.100900

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students` access to arithmetical discourses. *Journal of Research in Mathematics Education, 36*(3), 176–247.

Ben-Zvi, D., & Sfard, A. (2007). Ariadne`s thread, deadalus` wings and the learners autonomy. *Education & Didactique, 1*(3), 123–142. https://doi.org/10.4000/educationdidactique.241

Berger, M. (2013). Examining mathematical discourse to understand in-service teachers' mathematical activites. *Pythagoras, 34*(1), 1–10. https://doi.org/10.4102/pythagoras.v34i1.197

Clements, K. (1981). Visual imagery and school mathematics. *For the Learning of Mathematics, 2*(2), 2–9.

Cooper, J., & Lavie, I. (2021). Bridging incommensurable discourses – A commognitive look at instructional design in the zone of proximal development. *The Journal of Mathematical Behavior, 61*, 100822. https://doi.org/10.1016/j.jmathb.2020.100822

Department of Basic Education. (2016). *National senior certificate diagnostic report*. Pretoria: Government Printers.

Department of Basic Education. (2017). *National senior certificate diagnostic report*. Pretoria: Government Printers.

Department of Basic Education. (2018). *National senior certificate diagnostic report*. Pretoria: Government Printers.

Department of Basic Education. (2019). *National senior certificate diagnostic report part 1*. Pretoria: Governement Printers.

De Villiers, M. (2012). *Some reflections on the Van Hiele theory*. Paper presented at the National Mathematics Congress, Swakopmund, 21 May.

De Villiers, M., & Heideman, N. (2014). Conjecturing, refuting and proving within the context of dynamic geometry. *Learning and Teaching Mathematics, 2014*(17), 20–26. Retrieved from https://www.ingentaconnect.com/content/sabinet/amesal/2014/00002014/00000017/art00006

Dienes, Z.P. (1960). *Building up mathematics*. London: Hutchinson.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordercht-Holland: D Reidel Publishing Company.

Heyd-Metzuyanim, E. (2013). The co-construction of learning difficulties in mathematics – Teacher–student interactions and their role in the development of a disabled mathematical identity. *Educational Studies in Mathematics, 83*(3), 341–368. https://doi.org/10.1007/s10649-012-9457-z

Heyd-Metzuyanim, E. (2018). A discursive approach to identity and critical transitions in mathematics learning. In T. Amin & O. Levrini (Eds.), *Converging perspectives on conceptual change: Mapping an emerging paradigm in the learning sciences* (pp. 289–304). London: Routledge.

Heyd-Metzuyanim, E., & Graven, M. (2016). Between people-pleasing and mathematizing: South African learners' struggle for numeracy. *Educational Studies in Mathematics, 91*, 349–373. https://doi.org/10.1007/s10649-015-9637-8

Kaur, H. (2020). Introducing the concept of angle to young children in a dynamic geometry environment. *International Journal of Mathematical Education in Science and Technology, 51*(2), 161–182. https://doi.org/10.1080/0020739X.2020.1717657

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York, NY: Cambridge University Press.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*(2), 153–176. https://doi.org/10.1007/s10649-018-9817-4

Machisi, E. (2021). Grade 11 students' reflections on their Euclidean geometry learning experiences. *Eurasia Journal of Mathematics, Science and Technology Education, 17*(2), em1938. https://doi.org/10.29333/ejmste/9672

Morgan, C., & Tang, S. (2016). To what extent are students expected to participate in specialised mathematical discourse? Change over time in school mathematics in England. *Research in Mathematics Education, 18*(2), 142–164. https://doi.org/10.1080/14794802.2016.1174145

Mpofu, S., & Mudaly, V. (2020). Grade 11 rural learners understanding of functions: A commognition perspective. *African Journal of Research in Mathematics, Science and Technology Education, 24*(2), 156–168. https://doi.org/10.1080/18117295.2020.1798670

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in mathematics classrooms. *Educational Studies in Mathematics, 101*(2), 253–271. https://doi.org/10.1007/s10649-018-9848-x

Naidoo, J., & Kapofu, W. (2020). Exploring female learners' perceptions of learning geometry in mathematics. *South African Journal of Education, 40*(1), 1–11. https://doi.org/10.15700/saje.v40n1a1727

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education, 16*(2), 182–198. https://doi.org/10.1080/14794802.2014.918338

Roberts, A., & Le Roux, K. (2019). A commognitive perspective on Grade 8 and Grade 9 learner thinking about linear equations. *Pythagoras, 40*(1), a519. https://doi.org/10.4102/pythagoras.v40i1.519

Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning froma commognitive standpoint. *Journal of Learning Sciences, 16*(4), 565–613. https://doi.org/10.1080/10508400701525253

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Sfard, A. (2014). University mathematics as a discourse–why, how, and what for? *Research in Mathematics Education, 16*(2), 199–203. https://doi.org/10.1080/14794802.2014.918339

Sfard, A. (2021). Taming fantastic beasts of mathematics: Struggling with incommensurability. *International Journal of Research in Undergraduate Mathematics Education*, 1–33. https://doi.org/10.1007/s40753-021-00156-7

Shepard, R.N. (1978). The mental image. *American Psychologist, 33*(2), 125–137. https://doi.org/10.1037/0003-066X.33.2.125

Tachie, S.A. (2020). The challenges of South African teachers in teaching Euclidean geometry. *International Journal of Learning, Teaching and Educational Research, 19*(8), 297–312. https://doi.org/10.26803/ijlter.19.8.16

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry. CDASSG project*. Chicago, IL: University of Chicago.

Van Hiele, P.M. (1957). *De Problematick van het Inzicht Gedemonstreed van het Inzicht van Schodkindren in Meetkundeleerstof; includes English summary.* Utrecht: University of Utrecht.

Van Hiele, P.M. (1959). The child`s thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M van Hiele* (pp. 243–252).

Van Hiele-Geldof, D. (1957). *De didaktick van de Meetkunde in de eerste klass van het VHMO Summary of unpublished doctoral dissertation with English summary* (PhD). University of Utrecht.

Van Putten, S., Howie, S., & Stols, G. (2010). Making Euclidean geometry compulsory: Are we prepared? *Perspectives in Education, 28*(4), 22–31.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Havard University Press.

Wang, S. (2016). *Discourse perspective of geometric thoughts*. Heidelberg: Springer Spektrum.

Wang, S., & Kinzel, M. (2014). How do they know it is a parallelogram? Analysing geometric discourse at van Hiele Level 3. *Research in Mathematics Education, 16*(3), 288–305. https://doi.org/10.1080/14794802.2014.933711

Yi, M., Flores, R., & Wang, J. (2020). Examining the influence of van Hiele theory-based instructional activities on elementary preservice teachers' geometry knowledge for teaching 2-D shapes. *Teaching and Teacher Education, 91*, 103038. https://doi.org/10.1016/j.tate.2020.103038