European Journal of Educational Research
Volume 11, Issue 3, 1393 - 1411.

ISSN: 2165-8714
https://www.eu-jer.com/

The Role of Visual Representations in Geometry Learning

Amalija Žakelj
University of Primorska, SLOVENIA

Andreja Klančar*
University of Primorska, SLOVENIA

Received: December 28, 2021 ▪ Revised: March 10, 2022 ▪ Accepted: April 22, 2022

Abstract: Visual representations and the process of visualisation have an important role in geometry learning. The optimal use of visual representations in complex multimedia environments has been an important research topic since the end of the last century. For the purpose of the study presented in this paper, we designed a model of learning geometry with the use of digital learning resources like dynamic geometry programmes and applets, which foster visualisation. Students explore geometric concepts through the manipulation of interactive virtual representations. This study aims to explore whether learning of geometry with digital resources is reflected in higher student achievements in solving geometric problems. This study also aims to explore the role of graphical representations (GRs) in solving geometric problems. The results of the survey show a positive impact of the model of teaching on student achievement. In the post-test, students in the experimental group (EG) performed significantly better than students in the control group (CG) in the overall number of points, in solving tasks without GR, in calculating the area and the perimeter of triangles and quadrilaterals than the CG students, in all cases with small size effect. The authors therefore argue for the use of digital technologies and resources in geometry learning, because interactive manipulatives support the transition between representations at the concrete, pictorial and symbolic (abstract) levels and are therefore important for understanding mathematical concepts, as well as for exploring relationships, making precise graphical representations (GRs), formulating and proving assumptions, and applying different problem-solving strategies.

Keywords: *Area and perimeter, digital resources, geometry, learning mathematics, visualisation.*

To cite this article: Žakelj, A., & Klančar, A. (2022). The role of visual representations in geometry learning. *European Journal of Educational Research*, *11*(3), 1393-1411. https://doi.org/10.12973/eu-jer.11.3.1393

Introduction

The use of digital technologies and resources in mathematics classes changes the learning process. Its meaningful and purposeful use can enhance learning. According to Clark-Wilson et al. (2020), the use of digital technologies in math teaching has two main functions: '(1) a support for the organisation of the teacher's work (e.g., producing learning materials, class analytics, such as grades, lessons attendance, formative and summative assessment of students, and so on) and (2) a support for new ways of doing and representing mathematics' (p. 1225).

The use of digital technologies and resources in the learning process allows the production of rich learning environments by employing various digital materials and digital support tools, applets, animations and simulations; it supports various approaches to teaching, such as modelling, simulation, experimentation and researching, as well as solving mathematical problems and authentic problems (Klančar et al., 2019). Furthermore, Zbiek (2003) highlights the importance of well-deliberated design and didactically appropriate use of digital technologies in the process of teaching and learning mathematics, especially in terms of developing mathematical intuition, understanding mathematical concepts, researching relations, making precise graphical representations (GRs), making and proving assumptions, using different problem-solving strategies and so on.

The importance of integrating digital technologies into mathematics teaching is also highlighted by Borwein and Bailey (2003), Cuban et al. (2001), Kokol-Voljč (2006), Lee and Hollebrands (2008), Thurm and Barzel (2022) and Viberg et al. (2020). They indicate that teaching with technology can enhance the learning of mathematics by facilitating realistic, problem-solving and collaborative approaches to teaching and learning. Technology can amplify students' abilities to solve problems or reorganise how students think about problems and their solutions. Multiple representations of concepts help students recognise and change their conceptions.

* **Corresponding author:**
Andreja Klančar, University of Primorska, Koper, Slovenia. ✉ andreja.klancar@pef.upr.si

However, several authors have stated that although the use of technology in the math classroom has been increasing, the results of its use do not correspond to their perceived potential to enhance the learning experience (Bray & Tangney, 2017; Lameras & Moumoutzis, 2015; Oates, 2011; Reed et al., 2010; Selwyn, 2011; Wright, 2010). Cencič et al. (2010) also point to this aspect, stating that in teaching mathematics, the use of digital technologies can contribute to a more understandable and interesting way of presenting learning content—however, provided that the integration of technology as a teaching aid is carefully deliberated. In this context, the student must always be at the forefront and not the technology, which should always serve as a didactic aid (Pustavrh, 2014).

Literature Review

Visualisation of Concepts

In mathematics, teaching the introduction of mathematical concepts is designed following a concrete-pictorial-symbolic sequencing of instruction, starting with the concrete, then passing through the pictorial to the symbolic (abstract) representations and transiting between them (Volk et al., 2017), where the child on the first (enactive) stage solves problems based on action—through manipulation of concrete materials or objects in different activities. In the enactive stage, students acquire procedural knowledge as they learn to perform many activities effectively through imitation and exercise. The next stage is the iconic stage or pictorial way of representing the world. It is determined by the senses, for example, the visible perceptions and rules of their organisation, which are the basis for a symbolic way of presenting the world. The highest level is the symbolic (abstract) level or symbolic way of presenting the world, where words, numbers and other agreed symbolic systems and rules are used to represent ideas, objects and relationships.

In the context of acquiring and constructing conceptual representations, one of the evolving research areas in mathematics is the visualisation of mathematical concepts (Presmeg, 2014). Researchers in the areas of didactics and pedagogy indeed differ slightly in their definitions of visualisation; nevertheless, the essential emphases remain quite similar. Kosslyn (1996) defines visualisation as creating the mental image of a given concept. Lipovec and Podgoršek (2016) define visualisation as a spontaneous identification of mathematical relationships in graphical presentation. Atanasova-Pachemska et al. (2016) provide a definition of visualisation in mathematics, that is, visualisation means the process of shaping images (e.g., sketches/pictures drawn on paper, mental images or virtual pictures) and their successful application in mathematical research and understanding of mathematical problems (e.g., concepts, identification of mathematical relationships, and so on).

In summary, visualisation is the creation, application and reflection of diverse visual representations. In the process of visualisation, the student is expected to create, identify and shape visual representations and then apply them meaningfully in solving problems and also reflect on them (Figure 1). Lipovec and Podgoršek (2016) distinguish between static GRs (e.g., images, schemes, diagrams and so on) and dynamic GRs (e.g., video, applet and so on).

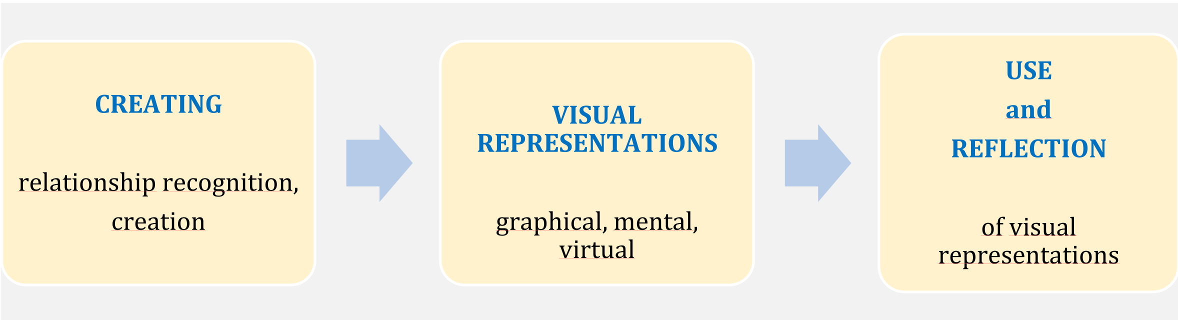

Figure 1. Visualisation Process

Visualisation and Development of Geometric Representations

Geometric concepts can be visualised in different ways as follows:

- with concrete physical models (Figure 2A),
- with static GRs (e.g., images, schemes, displays and so on) (Figure 2B),
- with dynamic GRs (e.g., video, applet and so on) (Figure 2C),
- with constructed or drawn representations or with representations made with a computer program (Kmetič et al., 2014) (Figure 2D).

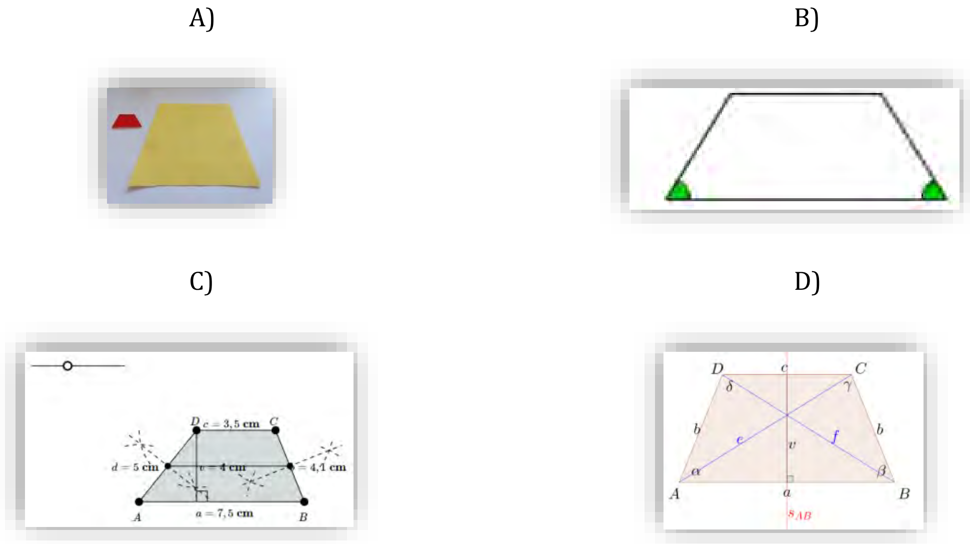

Figure 2. Visualization of Geometric Concepts

Using adequate didactic aids (e.g., concrete models, static and dynamic images, the use of the dynamic geometry programme and so on) enables students to visualise geometric concepts adapted to their competences and thus acquire them more thoroughly and in-depth (Raphael & Wahlstrom, 1989). For example, Fuys et al. (1988) point out that students who have difficulties in verbally presenting explanations represent their "explanations" with concrete objects.

Using well-organised learning situations, for example, with the use of dynamic geometry programmes, students investigate geometric concepts and relations among them; they independently construct geometric shapes, which they have recognised on various examples, the meaning of the term constructed figure, learn to distinguish between a picture or a sketch and a construction and relate Euclidian geometry with transformational and analytical geometry (Kmetič, 2008). The use of technology assists in the process of solving complex problems and encourages the development of visualisation capacities in students (Atanasova-Pachemska et al., 2016).

Computer software also allows the good visualisation of three-dimensional space and, thus, modelling and simulation of real-life phenomena and problems (Kmetič, 2008), which have a considerable effect on the development of spatial representation. Learning with programmes of dynamic geometry serves as a supplementary phase in the development of a concept. Kokol-Voljč (2006) labels the process dynamic schematisation. Programmes of dynamic geometry enables students researching by selecting the options of pulling and/or measuring, thus enabling students to explain and verify their hypotheses (Arzarello et al., 2002). The dynamic image can develop relations among the elements, thus revealing what remains equal and what can change after performing an activity. Notably, the use of the applet can be particularly effective for students who find it difficult to understand a static image, as the use of the applet allows them to determine a solution to the problem. Thus, the option of adopting dynamic schematisation represents a great educational value, as it also allows weak learners to participate in new 'discoveries' (Kmetič et al., 2014).

Multimedia-based Representations

In mathematics and mathematics education multimedia-based representations play an important role (Ollesch et al., 2017). The use of the medium affects brain function by stimulating the function of some of its parts. The processes involved in the learning of textual material differ from those involved when the material is delivered in a multimedia manner. With verbal tasks, the processes of memorisation are involved to an increased degree, whereas with multimedia-based tasks pictorial representations and visualisation play a greater role, which is extremely relevant in developing creativity and in solving problems (Gerlič & Jaušovec, 1998).

With multimedia-based representations we can represent the mathematical content in different ways (Ollesch et al., 2017). Representations are necessary for the basic understanding of mathematical concepts (Duval, 2002).

Working with multiple representations of the mathematical content enables students to benefit from complementary expressions and viewpoints of the subject matter, and they can improve and deepen their understanding (Ainsworth, 1999). Information delivered through images is easy to understand and memorise. Research shows that audio-mediated information is approximately as effective as an image. After a longe lapse of time, however, pictorial information is remembered better than aural information. Considering that the integration of sound into digital materials is rather sensitive, information conveyed simultaneously through text and sound or through image and sound is more pervasive and likelier to be anchored in long-term memory (Rebolj, 2008).

Simply presenting multiple representations to students is insufficient. Students have to understand the connections between different representations and build an appropriate mental model (Seufert, 2003).

In addition to images and sounds, animations are also an important multimedia building block. Animations are moving two- and three-dimensional images or graphs that in digital materials are intended for learning and displaying working instructions or aesthetic and psychological outcomes. The main educational meaning of animation is the creation of representations. With animation, the student can trigger a process and observe it afterwards. With highly sophisticated animations, changing the parameters is also possible. Animations and interactive computer graphics methods provide new insight into the world of mathematics.

Example 1: Determining an Area of Trapezoid

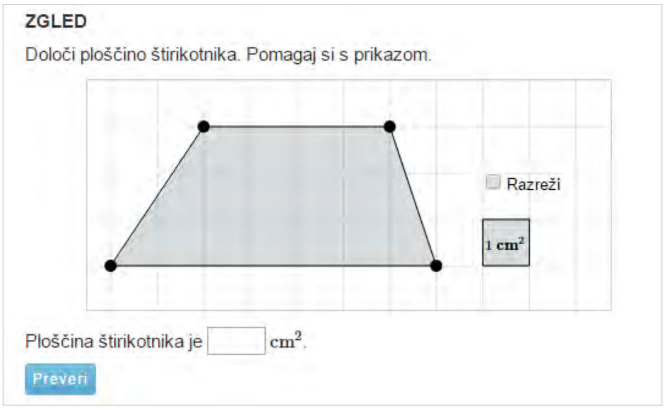

Figure 3. Determining an Area of Trapezoid (summarised from Tratar et al., 2014, p. 465)

Students develop conceptual understanding of area of chosen geometric shape (e.g., trapezoid) by using the applet (Figure 3). The applet enables students splitting and transforming trapezoid into known quadrilateral (e.g., square or rectangle) and then covering it with given standard unit to determine its area.

A learning simulation is a special type of learning animation. This type is an abstracted snapshot of reality in which everything irrelevant to the learning theme has been removed (Rebolj, 2008). Simulations are substitutes for real-world experiences. Simulations allow us to act virtually similar how we act in the real world (Shank & Cleary, as cited in Jancheski, 2011, p. 177). The student can use the simulation to manage the situation by manipulating objects, observing the results and analysing the impact of each manipulated object on the simulation.

Example 2: An Area of Triangle

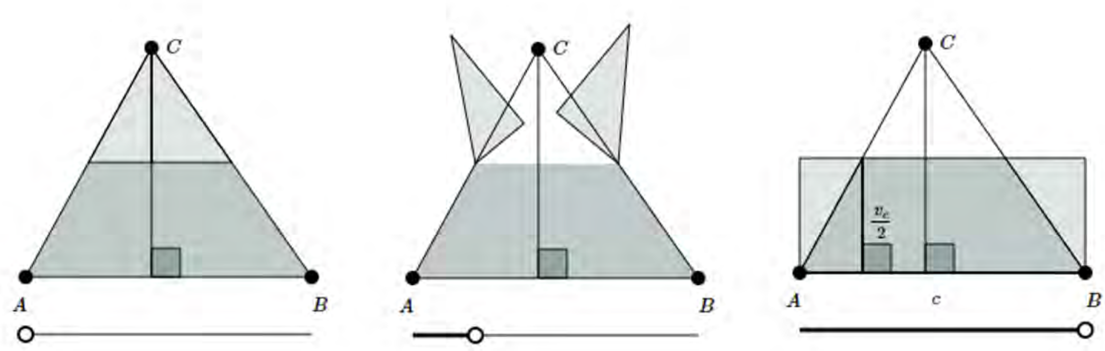

Figure 4. Area of Triangle (summarised from Tratar et al., 2014, p. 376)

Students observe a simulation through which they generalise the findings about transforming chosen shape (e.g., triangle) into known quadrilateral for the purpose of determining its area (Figure 4). Students write the findings in symbolic form as a formula. Using simulations students are developing conceptual knowledge and form strategies to solve the problem of calculating the area of a triangle or quadrilateral and other quadrilaterals and shapes. Students are researching about the area of parallelograms with the same base and the same height (Figure 5) by using the applet, where they can move vertex C, observe different parallelograms, determine or calculate their area and create a finding that the areas of parallelograms with the same base and the same height are equal.

Example 3: An Area of Parallelogram With Same Base and Same Height

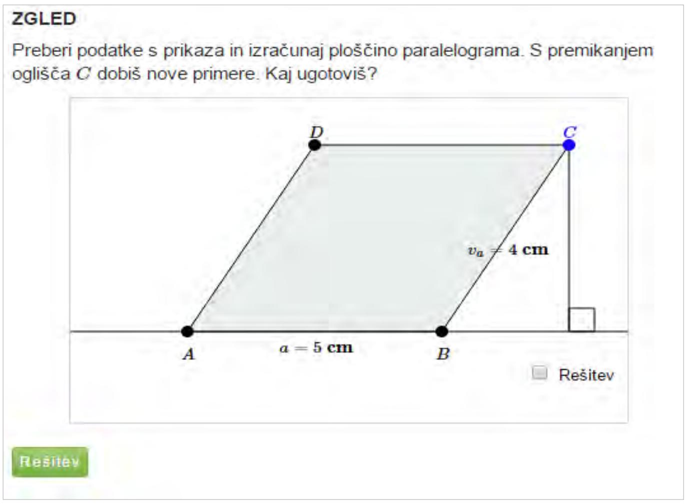

Figure 5. Calculating an Area of Parallelograms with Same Base and Same Height (summarised from Tratar et al., 2014, p. 496)

Computer simulations enables students to explore, experiment, question and hypothesise about real-life situations (Jancheski, 2011). Theirs meaningful and purposeful integration in learning process can enhance learning and learning outcomes.

Purpose and Objectives of the Study

The study aims to explore (1) whether learning geometry with the use of digital learning resources is reflected in higher student achievements in solving geometric problems and (2) to explore the role of GRs in solving geometric problems about the area and the perimeter of triangles and quadrilaterals in the context of students' achievements.

Research Questions

R1: Is geometry learning with the use of digital learning resources reflected in higher student achievements in solving geometric problems?

R2: Are there differences in the achievements of solving geometric tasks with or without visual GR between the experimental group (EG) and the control group (CG) students?

R 2.1: What is the role of visual representations in solving geometric tasks from the point of view of student's achievements?

Methodology

We used descriptive and causal experimental methods of pedagogical research (Hartas, 2010; Sagadin, 1993). The effectiveness of the model of geometry learning with the use of digital learning resources was tested using a one-factor pedagogical experiment with classes as comparison groups. The EG was formed of students who received the experimental factor (who were learning geometry using digital learning resources). The group of students who were taught in the traditional way (i.e., without the use of technology, using paper and pencil) formed the CG. The learning process in EG and CG focused on the transformation of triangles and quadrilaterals into shapes with which students know how to calculate the area and to find and use a suitable strategy to calculate the area of chosen triangle or quadrilateral.

Geometry learning in CG

The learning objectives and the content were the same for the CG and the EG students. An approach, used in CG was teacher-centred. In geometry lessons the CG students used textbooks, geometric tools (e.g., ruler, geometric triangle, compasses) and notebooks. During the lessons, the teachers called attention to correct the use of geometric tools and the consistency and accuracy in drawing visual representations (e.g., sketches and so on) and to the systematic writing of the task procedures.

Geometry Learning in EG

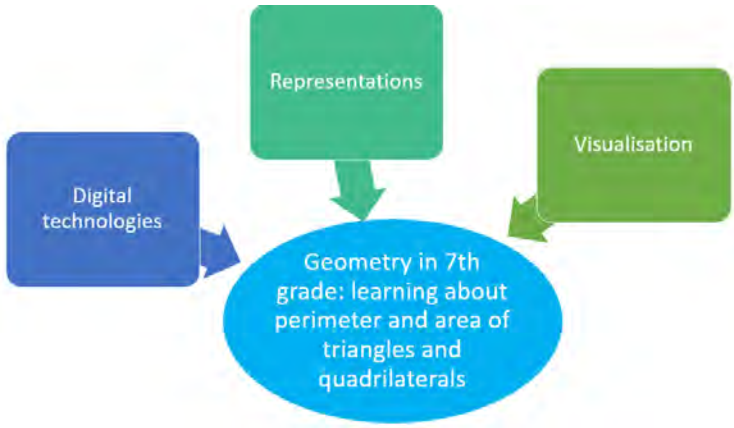

Figure 6. Model of Geometry Learning with Digital Learning Resources

For the purpose of the study—in accordance with the theory—we designed a model of geometry learning with the use of digital learning resources (Figure 6) like dynamic geometry programmes and applets (Figure 3, Figure 4 and Figure 5). In the learning process, visualisation and the exploration of geometric concepts through the manipulation of interactive virtual representations was emphasized. We used the instructional design which incorporated Bruner's (1966) three-stage learning model, in which learning activities are designed following a concrete-pictorial-symbolic sequencing of instruction (Volk et al., 2017), starting with the concrete, then passing through pictorial to symbolic (abstract) representations and transiting between them. Students learned about basic geometric concepts and built conceptual representations, with special emphasis on the visualisation of geometric concepts through various activities using digital technologies—by learning through researching and problem-solving using different digital learning materials, which were prepared for the purpose of the research and which guided the student from concrete to pictorial and then to symbolic (abstract) level, using i-textbooks, various didactic games and simulations (Figure 6). The model of geometry learning with the use of digital learning resources facilitates individualisation and the co-creation of learning paths in the context of selected learning objectives. Learning in the virtual learning environment using digital resources, besides encouraging the development of a conceptual understanding of geometry, allows students to train in the independent selection of activities with the aim of maximising the efficiency of achieving the set objective.

Experimental Sample

The EG consisted of 63 seventh-grade students (31 boys and 32 girls), whereas the CG consisted of 62 seventh-grade students (38 boys and 24 girls) from three randomly selected Slovenian schools. Six math teachers with the same level of education (university degree) and at least 10 years of work experience participated in the study.

Data Collection

We conducted pre-participation testing before starting the experiment (the first empirical recording). The post-test (the second empirical recording) was conducted one week after the experiment under the same conditions and with the same tester. The data collected with pre- and post-tests represent the dependent variables in the statistical context. For each measuring instrument, we analysed the objectivity, reliability, validity and difficulty, as well as discrimination, of tasks.

We designed the pre-participation test to examine the equivalence between the EG and CG and the post-tests for the purpose of this experiment after discussing the content of both tests. The objectivity of tests was considered regarding testing, evaluation of results and interpretation of test results. We calculated the reliability of both tests using Cronbach's alpha coefficient which shows acceptable levels of reliability (for both tests it was higher than 0.8). To verify the validity of the pre- and post- test we used factor analysis. Data collection was performed according to the same procedure for both groups. The results were evaluated uniformly in accordance with predetermined criteria for all the tested students in both groups, thus ensuring the objectivity of both tests. We conducted and discussed data analysis of pre- and post-participation tests in the research group and with peers.

The difficulty index is defined as the percentage of students who responded correctly for a particular task (Sočan, 2011). The difficulty index of most tasks in pre- and post- test was within an acceptable range, from 31.6% to 84.4%. The most tasks also had acceptable values of the index of discrimination (ID > 20%, the average value was 54%).

Statistical Processing

For statistical processing of the quantitative data we used the Statistical Package for Social Sciences (SPSS). The results of pre-participation test showed the initial equivalence between EG and CG students according to the results of t-test in overall score (t = 1.060; p = 0.291). With the use of t-test we also determined the significance of the differences between the EG and the CG in the post-test in overall score. As measurement of effect size the Cohen's d was used. Values for Cohen's d between 0.20 and 0.50 can be considered small effects, values between 0.50 and 0.80 medium, and values over 0.80 large effects (Cohen, 1988). As assuming the homogeneity of the variances was unjustified, due to the abnormal distribution, we used the Mann-Whitney test, a nonparametric equivalent of the parametric t-test, to determine significant differences between the average values of the EG and CG on individual tasks of the post-test and in the tasks with or without GR. We used rank values to calculate the test statistics. We calculated the effect size using Fritz et al.'s (2012) formula: $r = \frac{z}{\sqrt{N}}$, for which a small effect is 0.1, a medium effect is 0.3 and a large effect is 0.5. We analysed student's performance in overall score and in tasks with and without GR, which we present below.

Tasks With and Without GR

Tasks with GR were employed to deduce data, relations between geometric objects and so on from a GR and to solve a problem. Rectangle (Example 1) is a case of a task delivered with text and with GR added. To solve the task, the student reads the data from the image and uses them to solve the problem.

Example 1: Rectangle (Figure 7)

The sketch shows a shaded rectangle in a parallelogram. Consider the data written in the sketch and calculate the area of the rectangle.

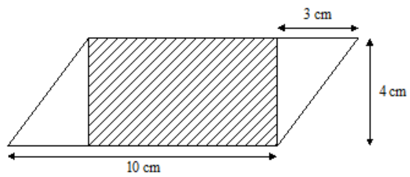

Figure 7. Example 2: Rectangle

Tasks without GR: Data are given in text only; through the process of visualisation, the student independently created a GR (e.g., a sketch) as a solution, or the creation of a GR was a task in itself. An isosceles triangle (Example 2) is a case of a task delivered with text, without GR. To solve the task, the student analyses the text, draws a sketch, reads the data from the text and uses them in solving the problem.

Example 2: An Isosceles Triangle

Calculate the perimeter and the area of an isosceles triangle with a leg of 5 cm, a base of 6 cm, and a height on the base of 4 cm. Draw a sketch.

Results

R1: Is geometry learning with the use of digital learning resources reflected in higher student achievements in solving geometric problems?

The results have proven (Table 1) that the students in the experimental group (EG), who received the model of geometry teaching with the use of digital learning resources performed better on the post-test in the overall score than their peers in the control group (CG) (EG 52.9%, CG 44.8%). The differences were statistically significant (p = 0.046) (

Table 2), effect size is low (r = 0.36).

Table 1. Descriptive Statistic for Performance in Post-Test Overall Score (%)

Group	Number of students	Mean	Std. Deviation (SD)	Std. Error of a mean (SE)	Min	Max
EG	63	52.9	22.6	2.8	3.3	100.0
CG	62	44.8	22.2	2.8	3.3	100.0

Table 2. t-Test for Independent Samples for Performance in Post-Test Overall Score (%)

Value of the t-test for independent samples	Degree of freedom	Level of statistical significance (2-tailed)	Mean Difference	SE Difference	Cohen d
2.015	123	0.046	0.081	0.040	0.36

We analysed the differences in geometry knowledge between the EG and CG students in individual tasks of the post-test (Table 3). On the post-test, six tasks (1, 3, 4, 7, 8 and 11) without visual GR and six tasks (2, 5, 6, 10 and 12) with visual GR were given.

Table 3. The Descriptive Statistic for Performance (in %) on Individual Tasks of the Post-Test According to the Group

Variable	Group	Number of students	Arithmetic mean	SD	$\bar{R}$	U	2p
Task 1 (%)	EG	63	75.0	34.5	69.76	1527	0.024
	CG	62	62.5	34.4	56.13		
Task 2 (%)	EG	63	58.7	42.6	59.60	1739	0.246
	CG	62	66.9	42.4	66.45		
Task 3 (%)	EG	63	61.0	34.3	71.13	1441	0.010
	CG	62	43.5	37.0	54.74		
Task 4 (%)	EG	63	68.3	42.4	65.71	1782	0.343
	CG	62	60.5	45.4	60.24		
Task 5 (%)	EG	63	63.5	46.0	65.19	1815	0.443
	CG	62	57.3	46.9	60.77		
Task 6 (%)	EG	63	69.0	37.5	67.56	1666	0.124
	CG	62	58.1	40.7	58.37		
Task 7 (%)	EG	63	40.5	43.9	68.05	1635	0.084
	CG	62	25.8	32.3	57.87		
Task 8 (%)	EG	63	22.2	36.8	66.45	1735.5	0.155
	CG	62	13.7	30.3	59.49		
Task 9 (%)	EG	63	36.5	47.7	65.71	1782	0.305
	CG	62	28.2	44.9	60.24		
Task 10 (%)	EG	63	24.6	35.8	66.56	1782.5	0.167
	CG	62	18.5	36.5	59.38		
Task 11 (%)	EG	63	61.9	41.8	63.86	1899	0.780
	CG	62	63.4	33.4	62.13		
Task 12 (%)	EG	63	15.1	26.4	64.05	1887	0.665
	CG	62	14.5	29.2	61.94		

According to the descriptive statistics, the table (Table 3) shows that on the post-test, the performance of the EG students was better than the CG students in 10 of the 12 tasks. At two post-test tasks, the differences between the groups according to the success in solving the tasks were statistically significant in favour of the EG (task 1: $U = 1527, 2P = 0.024$; $task$ 3: $U = 1441, 2P = 0.010$). Effect size of both tasks was low (task 1: $r = 0.20$; Task 3: $r = 0.23$). Both tasks (task 1 and 3) are tasks without a supplementary visual GR, at which the students either created the GR (as a sketch or as a support to solving) independently or the creation of the GR was a task in itself.

The analysis of the achievements of the students in the EG and CG showed that the students in the EG performed better using an adequate strategy for the computation of the area of a triangle or quadrilateral than the students in the CG ($\bar{R}$(EG) = 70.83, $\bar{R}$(CG) = 55.04). The differences were statistically significant (U = 1459.5; 2p = 0.014), the size effect was low (r = 0.22).

The students in the EG were also more successful in using an adequate strategy for the computation of the perimeter of a triangle or quadrilateral than the students in the CG ($\bar{R}$(EG) = 69.48, $\bar{R}$(CG) = 56.42). The differences were statistically significant (U = 1545; 2p = 0.038), the effect size is low (r = 0.19).

R2: Are there differences in the achievements of solving geometric tasks with or without visual GR between the experimental group (EG) and the control group (CG) students?

The results of the descriptive statistics (Table 4) show that on the post-test, compared with the CG, the EG performed better both in the tasks with GR and in the tasks without GR. In the tasks with GR students achieved an average rank of 66.30 in the EG, and 59.65 in the CG. The differences were not statistically significant (U = 1745, 2p = 0.301). In tasks without visual GR students achieved an average rank of 70.86 in the EG, and 55.02 in the CG. The differences were statistically significant (U = 1458, $2p$ = 0.014), the size effect was small (r = 0.22).

Table 4. Descriptive Statistics for the Performance (in %) in the Tasks of Post-Test With or Without Visual GR

Variable	Group	Number of Students	Arithmetic Mean	SD	$\overline{R}$	U	2p
Tasks with GR (%)	EG	63	44.6	23.2	66.30	1745	0.301
	CG	62	40.6	24.5	59.65		
Tasks without GR (%)	EG	63	58.5	25.2	70.86	1458	0.014
	CG	62	47.7	24.9	55.02		

R 2.1: What is the role of visual representations in solving geometric tasks from the point of view of students' achievements?

GRs have different roles in the post-tests.

(1) Geometric tasks with visual GR

The GR is already created (the student is a user):

- GR includes data (direct or indirect); the student must read the data from a GR and then calculate the area or/and perimeter (task 10)

(2) Geometric tasks without visual graphical representation

The GR is created by the student (student is the creator, GR is the solution):

2.1) drawing a sketch + given direct data in the text (task 3)

2.2) drawing the GR, which contains data for the procedural part (task 11)

As shown, the neuralgic points of students of the EG and CG with tasks with or without GR are highlighted in the background of examples, and the success rate of solving these tasks is analysed.

Geometric Tasks with Visual GR

a) Example: Task 10:

*The square ABCD is divided into two squares and two congruent rectangles. The area of the smaller square and the area of the rectangle are written in the figure (*Figure 8*). Calculate the perimeter of the square ABCD. Calculate the area of a shaded square.*

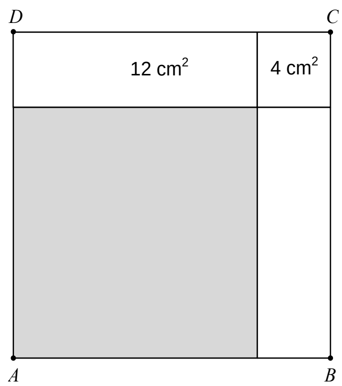

Figure 8. Task 10 (Republiški izpitni center, 2014)

The area of the smaller square and the area of the rectangle are shown in the figure.

a) *The perimeter of the square ABCD is* ___________ *cm.*

b) *The area of the shaded square is* ___________ cm^2.

Task 10 is an example of a task where the GR is already created (the student is a user). The student must read the data from the GR and then calculate the area and the perimeter. The task is a more demanding problem-solving task, the objective of which is to read the sizes of the areas of the marked shapes. Based on these parameters, the lengths of the sides are determined/identified, and these data are used to calculate the perimeter and the area of the shaded square. With this task, the EG students achieved slightly higher scores, although the achievements of the students in both groups were somewhat low. Students achieved an average rank of 66.56 in the EG, and 59.38 in the CG. The differences were not statistically significant (U = 1728.5, 2p = 0.167).

The results showed that only 22% of the students in the EG and 14% in the CG read the data in the picture correctly. EG students were more successful in calculating the perimeter; they achieved an average rank of 64.88 in the EG, and 61.09 in the CG. The differences were not statistically significant (U = 1834.5, 2p = 0.405).

EG students were also more successful in calculating the area; they achieved an average rank of 64.87 in the EG, and 61.10 in the CG. The differences were not statistically significant (U = 1835, 2p = 0.420).

The use of formulas is based on the data obtained from the GR in the tasks, and in the case of failure to obtain the data, the student subsequently could not successfully solve the task. The analysis of the results of the solution shows that 21% and 17% of the students in the EG and CG correctly used the strategy for calculating the perimeter in case the data in the GR was incorrectly understood, respectively (Figure 10) (image analysis and calculation were required), and 14% and 16% of the students in the EG and CG who correctly used the strategy for calculating the area in case the data in the GR were incorrectly understood, respectively. Some of them obtained data by measurement (Figure 9).

In the following, we will present examples of students' solutions:

a) Critical points in reading data—data obtained by measurement

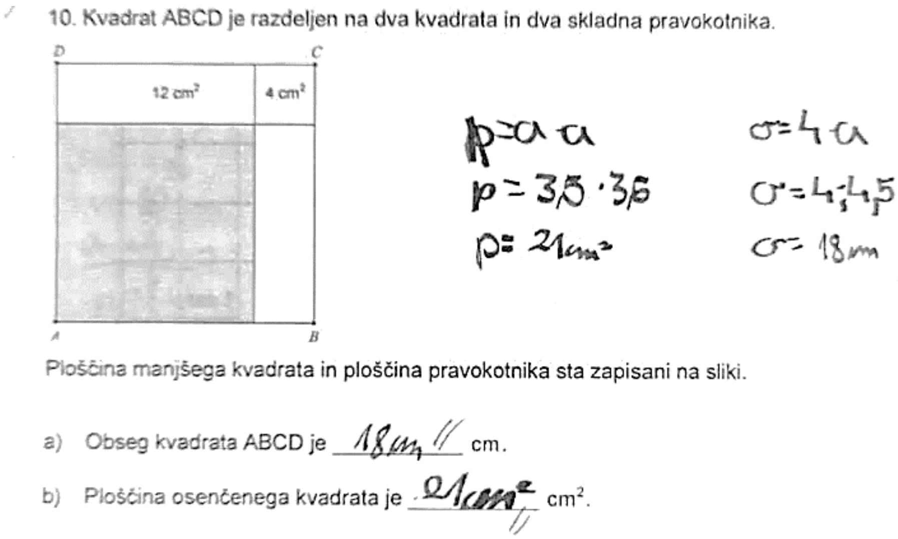

Figure 9. Example of Student Solving: Critical Points in Reading Data—Data Obtained by Measurement

The example of solving presented in Figure 9 shows that the student obtained the data from the GR by measurement and then used an appropriate strategy to calculate the perimeter and the area. The task shows that the student either did not read the instructions of the task properly or did not know how to obtain the data from the GR.

b) Critical points in reading data—task solved graphically

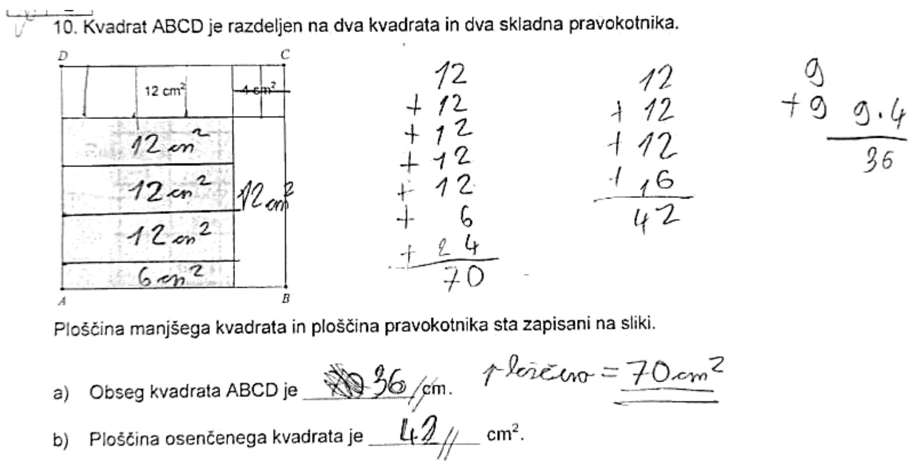

Figure 10. Example of Student Solving: Critical Points in Reading Data—Task Solved Graphically

The example of solving presented in Figure 10 shows that the student did not read carefully but inferred from the GR, transferred, covered the shaded part and solved the task graphically. Given its inaccuracy and inaccurate dimensions in the GR, the task was not solved properly. The task shows that the relationships in the GR are entirely not recognised.

c) Critical points in reading data—problem analysing indirect data

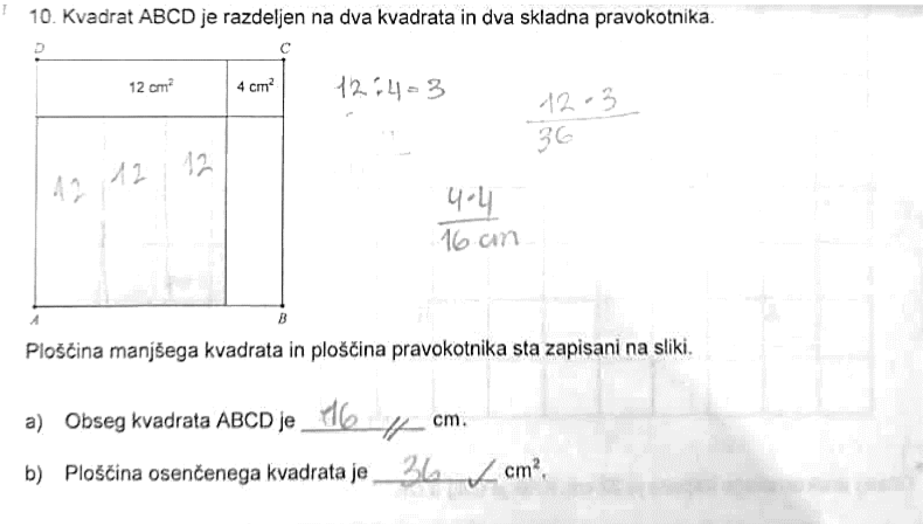

Figure 11. Example of Student Solving: Critical Points in Reading Data

As an example of solving (Figure 11), where students know how to read the data from the GR and observe relationships on the GR and—based on the read relationships—to make conclusions. The relationships between geometric objects in the GR are demanding—the student made the correct conclusions about the area but incorrectly determined the length of the sides to calculate the perimeter.

Critical points in solving geometric tasks with visual GR were the (in)correct reading of task instruction and understanding the instruction, obtaining data from GR and use of appropriate strategy for calculating the perimeter or the area of the shape. Some students obtained data by measurement and then calculated the perimeter and the area of the shape. Some students solved the task graphically.

Geometric Tasks Without Visual GR

Example: Task 3

Calculate the perimeter *and the area of an isosceles triangle with a leg of 5 cm, a base of 6 cm, and a height on the base of 4 cm. Draw a sketch.*

Task 3 is an example *of a task without visual GR*, where the GR is created by the student (the student is the creator, GR is the solution). The student must read the data provided directly in the text and, according to those data, create a GR (sketch).

The statistical analysis of the data shows that with the students of both groups, there is a weak yet important correlation between drawing the sketch and the selection of the appropriate strategy for the calculation of the perimeter (sketch-perimeter: $r = 0.249$, $p < 0.01$) or the area (sketch-area: $r = 0.339$; $p < 0.01$).

EG students were more successful than CG students in entirely solving task 3. Students achieved an average rank of 71.13 in the EG, and 54.74 in the CG. The differences were statistically significant (U = 1441, 2p = 0.010), with a small size effect (r = 0.23).

EG students were more successful than CG students in using an appropriate triangle perimeter calculation strategy. Students achieved an average rank of 68.61 in the EG, and 57.30 in the CG. The differences were statistically significant (U = 1599.5, 2p = 0.030), with a small size effect (r = 0.19).

EG students also achieved higher scores in drawing a sketch than CG students. Students achieved an average rank of 68.72 in the EG, and 57.19 in the CG. The differences were statistically significant (U = 1592.5, 2p = 0.039), with a small size effect (r = 0.18).

The higher achievements of both groups of students in computing the perimeter and the area, as well as the weak correlation between the sketch and the procedural parts of the tasks, indicate that the students were more successful in solving the procedural parts of the tasks and that they performed the adequate strategies for computing the perimeter/area independently of the sketch. Presumably, these students approached the solving of the geometric task primarily as a computational rather than a geometric problem that addresses the relations between geometric objects. Similarly, Hegarty and Kozhevnikov (1999) also noted in their research that the visualisation of the concept or problem is more demanding for students than the procedural part itself.

Example: Task 11

Draw a geometric figure with data A (0, 3), B (1, 0), C (2, 3) and D (1, 4) in the coordinate grid with a given unit. Name the geometric figure. Read the necessary data from the GR and calculate the area of the figure.

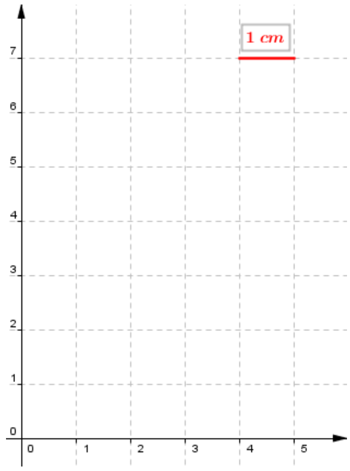

Figure 12. Task 11

Task 11 is an example of a task without visual GR, where the GR, which contains data for the procedural part, is created by the student (the student is the creator, GR the solution). The success of creating GR is a condition for calculating the area. The objectives of task 11 are to plot the given points in the coordinate plane, to draw the shape, to read the characteristics of the plotted shape and to apply an adequate strategy for computing the area.,

CG students were more successful than EG students in entirely solving task 11. Students achieved an average rank of 63.86 in the EG, and 62.13 in the CG. The differences were not statistically significant (U = 1889, 2p = 0.780). In the application of the area calculation strategy, students achieved an average rank of 70.20 in the EG, and 55.69 in the CG. The differences were statistically significant (U = 1499.5, 2p = 0.010), with a small size effect (r = 0.23).

Students in the EG were more successful at calculating the area ($\bar{R}$ = 70.20) than at drawing the image ($\bar{R}$ = 57.73) (among the students in the EG, 14% applied an adequate strategy for the calculation of the emerged shape in spite of the incorrectly drawn image); in the following, they were, however, more successful at analysing the plotted image, and they selected an adequate strategy for computing the area, which is evident from five different examples of images (Figure 13A, Figure 13B, Figure 13C, Figure 13D, Figure 13E). The correlation between the image and the procedural part of the task is moderate (EG: [r = 0.619, 2p = 0.000], statistically significant medium correlation).

Students in the CG were more successful at drawing the image ($\bar{R}$ = 68.35) yet less successful in the continuation, at the analysis of the image and at selecting an adequate way of solving the problem, that is, at calculating the area ($\bar{R}$ = 55.69). The correlation between the image and the procedural part of the task is low (CG: [r = 0.344, 2p = 0.006], statistically significant weak correlation).

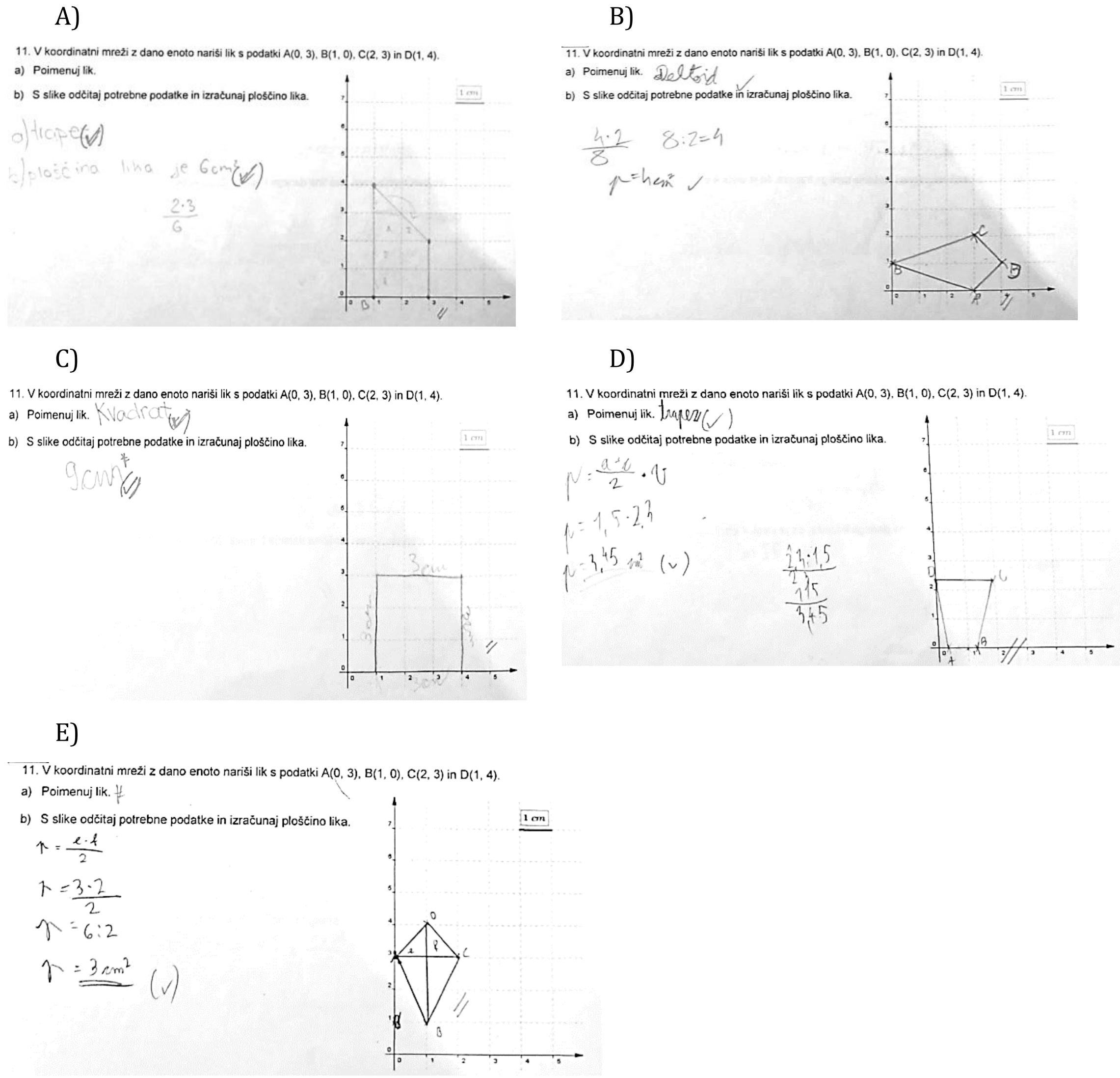

Figure 13. Examples of Students' Solutions

The above examples show that students in the EG experienced greater problems plotting the points in the coordinate plane than that in student in the CG; they did, however, correctly identify the plotted shape and using this, found adequate strategies for calculating the area.

In summary, the statistical analysis of the data in individual tasks without GR shows that for the students of the two groups combined, a significant low or moderate correlation was found between the sketch and the procedural parts of the tasks and higher achievements of students of both groups in calculating the volume and the area than in drawing the sketch. Similar to the third task, the results suggest that these students solved the geometric task primarily as a computational problem and initially not as a geometric problem that addressed the relationships between geometric objects.

Discussion

We will provide an interpretation of the results of our study and answers to the research questions:

1. Is geometry learning with the use of digital learning resources reflected in higher student achievements in solving geometric problems?

Results of the survey show a positive impact of the model of teaching on the achievements of the students in the EG. At the post-test, the students who received the teaching of geometry with the use of digital technologies in virtual learning environments were significantly more successful in total number of points.

Hassidov's (2017) findings show that the teaching method is a decisive factor in student achievement in math and that full coordination of classroom teaching with computer practice is of prime importance, which requires a change from traditional teaching methods, incorporating attention to the differing needs and achievements of students. Relating students' achievements to learning and teaching approaches (Hassidov, 2017) and taking into account that in contrast to students in the CG, students in the EG received a different model of geometry lessons, we can assume that the learning with digital resources in virtual learning environment also contributed to the high achievement of the students in the EG. In the virtual learning environment (rich with different digital resources), the geometric situations were pictorially supported with applets, animations, simulations, video clips and so on, which were not provided in the students taught in the traditional way. Using applets and generating additional new cases (with static situations, only a limited number of new cases are generated), the students investigated the relations between geometric objects, which is crucial for understanding geometric concepts. Jancheski (2011) points out that paper-based materials support only fixed visual representations, learning by trying is not possible, whereas the applet enables students to manipulate objects to learn by identifying an appropriate solution. With the use of digital technologies and resources students in the EG had both static and dynamic representations at which they could investigate the relations between geometric objects from diverse aspects.

Rau (2017) points out that physical and virtual representations have complementary cognitive affordances for student conceptual learning (de Jong et al., 2013; Klahr et al., 2007). Physical representations have been shown to be particularly effective in helping students learn concepts that build on movement or real-world experiences. Virtual representations have been shown to be effective in helping students learn concepts that describe invisible processes, summarizing data or when removing concrete details can make concepts more salient (de Jong et al., 2013; Rau, 2017). Similar findings that the use of technology can improve mathematics learning, that the use of digital tools has a positive impact on students' learning outcomes, and that intelligent tutoring and simulation systems have significant effects on student learning were also reached by Hillmayr et al. (2020), who investigated how the use of technology can improve the learning of mathematics and science in secondary school.

Similarly, researchers who have studied the effectiveness of digital technologies in learning and teaching have a similar conclusion. Adelabu et al. (2019), in a study of the use of Dynamic Geometry Computer Software and other relevant digital geometrical tools in geometry teaching and learning, found their positive contribution to students' geometric thinking.

With the model of geometry teaching provided in EG, choice was ensured to the students, as in the virtual environment, opportunities to encounter diverse physical and virtual situations were at disposal. Atanasova-Pachemska et al. (2016) also point out that adequate choice and application of modern technologies contribute to the development of the ability to visualise and develop conceptual representations.

In a virtual learning environment, prompt feedback on the correctness of solving was further assured to the students in the EG both by the teacher and by digital technology, which meant ongoing monitoring and guidance of the students. Thus, the students (co-)created their own learning path and were also allowed less linear, hierarchic or systematic progression that included recommendations for further work.

Students in the CG mainly gained experience via static representations that were either provided or created by themselves. Moreover, they usually received feedback only in the process of revising solutions. They primarily used classical materials and textbooks that mainly contain static images, and fewer opportunities were available for independent research in the sense of observing relations between geometric elements from various aspects than in virtual environments (rich with diverse digital learning materials and activities, including dynamic geometry activities) that provide simulations and representations.

2. Are there differences in the achievements of solving geometric tasks with or without visual GR between the experimental group (EG) and the control group (CG) students?

The results of the study have elucidated relevant aspects and the role of visual GRs in solving geometric tasks. The analysis of the results calls attention to neuralgic points of students in recognising relations between geometric objects in a picture and in independent creation of GRs. Although the achievements of students in visualising GRs of both groups were lower compared to the procedural parts of tasks (where students had to calculate the perimeter and the area of a triangle or a quadrilateral), EG students achieved higher scores than that in the CG in the analysis of GRs (example: task 10) and in creating GRs (example: tasks 7 and 3). We can attribute this finding to the learning process with digital resources, where EG students had opportunities to learn how to transform the selected quadrilateral into a figure for which they can calculate the area by observing a simulation through which they generalise the findings and write them in symbolic form as a formula. Simulations are more than just an interactive model or a collection of facts with which the learner interacts. Computer simulations are becoming more generally recognised as efficient learning environments where students can explore, experiment, question and hypothesise real-life situations (Jancheski, 2011, p. 177). In the learning process, the students also worked with animations—they triggered the animation process and observed the transformation of shapes. Animations and interactive computer graphics methods provide new insights into a problem or GR. In the process of finding appropriate solving strategies, students also used applets, which enabled them to manipulate objects to learn by trying and consequently to find an appropriate solution.

In tasks with added GRs, where they were asked to identify data from pictures and analyse and recognise relations between geometric objects in the picture or recognise adequate data and subsequently continue solving the problem with them, students in both the EG and CG solved these tasks less successfully than in solving other parts of tasks. In the case of difficulties analysing images, for example, the students resorted to other options to access the data (such as simply measuring the data in the picture in task 10) to be able to realise the second part of the task or its procedural part.

Lower achievement is also detected in both groups with tasks (e.g., task 7) involving independent creation. The results indicate that, in the present study, independent visualisation of a geometric concept, as well as identification of relationships between geometric objects, were neuralgic points—to a larger extent for students in the CG, yet with difficulties noted for both students in the EG and CG. Presumably, both visualisation of geometric concepts and analysis of relations between geometric concepts are mentally more demanding for students or—alternatively—that they lack experience with such situations. Similar findings were made by Hegarty and Kozhevnikov (1999), who state that the visualisation of a concept or problem is often more difficult for students than the procedural part itself.

Similarly, Arcavi (2003), Bishop (1989) and Hershkowitz (1989), who emphasise that students often have not developed the ability to shape adequate visual representation and solve problems, cannot arrive at an adequate solution. Visualisation is not an innate ability; however, in elementary education, the first mathematical processes and symbols are quite often represented with the aid of images. Encouraging the use of visualisation in mathematics is therefore extremely important at all levels of schooling (Hoffmann, 1998; Whiteley, 2004), whereas, according to Antolin Drešar and Lipovec (2015) and Güler and Çiltaş (2011), it is also an efficient teaching strategy.

From the outcomes of the qualitative analysis, we can further conclude that the students participating in the study were more successful in solving the procedural parts of tasks using forms than geometric tasks that addressed spatial characteristics and their mutual relations. This finding has been proven illustratively by students' achievements in tasks where they were expected first to independently create a sketch. The analysis of the results shows that frequently, the sketch did not serve them as support in further solving; they solved the task independently from the sketch—possibly also successfully. Notwithstanding the deficient or incorrect sketch part of both the students in the EG and CG used an adequate strategy for computing the perimeter (EG 33%, CG 32%) or the area (EG 22%, CG 13%); this strategy indicates that the students applied the two types of knowledge separately or that they do not put them in relation—it can be assumed they address a geometric task as an algebraic rather than geometric problem that deals with relations between geometric objects.

The outcomes indicating a low correlation between the sketch and the procedural parts of a task and higher achievements of both groups in computing the perimeter and the area than in drawing the sketch point out that students' performance in tasks where, for example, a sketch and computing the perimeter, the area of a shape, and so on, is expected does not depend on successful creation of a sketch. We can conclude/assume from the outcomes that in case the image is not in the role of the conveyor of information but as a requirement or part of the task, some students see this kind of problem as algebraic problems. The causes of this outcome may be found in the ways of learning and teaching that traditionally do not sufficiently rely on the exploration of relationships between geometric elements.

Conclusion

The outcomes of the study show that teaching geometry in a virtual learning environment (rich with diverse learning materials and activities, including dynamic geometry activities) can significantly contribute to the success of students in developing geometric representations, procedural and problem-solving competences in the area of geometry and solving geometric problems. EG students performed significantly better than the CG students in calculating the area and the perimeter of the triangles and quadrilaterals. EG students also achieved higher scores in analysing and reading data from GRs than CG students; however, the differences are not statistically significant.

Mešinović et al. (2017) state similarly that the use of additional aids in the formation of geometric concepts positively affects achievement in geometry. Similarly, the baselines of large-scale studies Trends in International Mathematics and Science Study [TIMSS] 2011 (Mullis, 2012) and Programme for International Student Assessment (PISA) 2012 (Organisation for Economic Co-operation and Development [OECD], 2014) state that in the last educational cycle, attention should be focused on visualisation of geometric concepts. Only in this way can students apply spatial representations to transit between three-dimensional shapes and their illustrations. Analysis of outcomes draws attention to students' neuralgic points both in the recognition of relations between geometric objects in a picture as well as in independently creating visual GRs.

The amendment of such difficulties with students can also be achieved by introducing digital technologies and resources into the process of learning and teaching mathematics, which has also been proven by the outcomes of the present study. Similarly, Klančar et al. (2019), Clements et al. (2008) and Hillmayr et al. (2020) assert the positive effects of using digital technologies in learning and teaching mathematics are shown in the development of students' skills for solving problems, the development of numeric and geometric representations and the exploration of patterns and relationships through the process of visualisation.

Recommendations

Digital technologies and resources can play an important role in the development of spatial representations because they foster the process of visualisation and enable the development of spatial and geometric reasoning. Teaching geometry with digital resources in a virtual learning environment can significantly contribute to the success of students in developing geometric representations, procedural and problem-solving competences in the area of geometry.

To effectively integrate digital technologies into mathematics teaching and learning, which is supported by the findings of this study, teachers need to be trained and the use of digital technologies needs to be integrated into mathematics curricula and in the national mathematics assessment. Based on the results of other research (Hillmayr et al., 2020; Viberg et al., 2020), it is reasonable to consider that digital tools are used in addition to other teaching methods and not as a substitute, both in the planning of the teacher training programme and in the implementation of the programme. In further research, it would be sensible to test the model in other areas of mathematics and explore its contribution and weaknesses areas.

Limitations

The teaching experiment was conducted in six existing classes of three randomly selected Slovenian schools. These constraints could be used as a starting point for future research.

Acknowledgements

The article was written as part of project activities of national project NA-MA POTI—Natural Science and Mathematical Literacy: Promoting Critical Thinking and Problem Solving and efficient use of ICT (Information and Communication Technology). Investment is cofinanced by the Republic of Slovenia and the European Union under the European Social Fund. Under the Operation Programme for the Implementation of the European Cohesion Policy in the 2016–2022 period.

Authorship Contribution Statement

Žakelj: Concept and design, data analysis / interpretation, drafting manuscript, critical revision of manuscript, supervision. Klančar: Concept and design, data acquisition, data analysis / interpretation, drafting manuscript, critical revision of manuscript, statistical analysis.

References

Adelabu, F. M., Makgato, M., & Ramaligela, M. S. (2019). Enhancing learners' geometric thinking using dynamic geometry computer software. *Journal of Technical Education and Training, 11*(1), 44–53. https://bit.ly/3OzR2XH

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education, 33*, 131–152. https://doi.org/10.1016/S0360-1315(99)00029-9

Antolin Drešar, D., & Lipovec, U. (2015). Shematske vizualne reprezentacije potence [Schematic visual representations of exponents]. *Revija za Elementarno Izobraževanje*, *8*(4), 143–158. https://bit.ly/3vb8mKZ

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215-241. https://doi.org/10.1023/A:1024312321077

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM – The International Journal of Mathematics Education*, *34*(3), 66–72. https://doi.org/10.1007/BF02655708

Atanasova-Pachemska, T., Gunova, V., Koceva Lazarova, L., & Pachemska, S. (2016). Visualization of the geometry problems in primary math education: Needs and changes. *Istraživanje Matematičkog Obrazovanja, 8*(15), 33–37. https://bit.ly/3rR3BnQ

Bishop, A. (1989). Review of research in visualization in mathematics education. *Focus on Learning Problems in Mathematics, 11*(1–2), 7–16.

Borwein, J. M., & Bailey, D. H. (2003). *Mathematics by experiment: Plausible reasoning in the 21st century.* AK Peters.

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research: A systematic review of recent trends. *Computers & Education, 114*, 255–273. https://doi.org/10.1016/j.compedu.2017.07.004

Bruner, J. (1966). *Toward a theory of instruction*. The Belknap press of Harvard University press.

Cencič, M., Cotič, M., & Medved-Udovič, V. (2010). Spremembe pouka in kompetence učiteljev za uporabo informacijsko-komunikacijske tehnologije [Changes to teaching and teacher competences for the application of ICT]. *Pedagoška Obzorja, 25*(2), 19–34.

Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM Mathematics Education*, *52*(7), 1223–1242. https://doi.org/10.1007/s11858-020-01196-0

Clements, D. H., Sarama, J., Yelland, N. J., & Glass, B. (2008). Learning and teaching geometry with computers in the elementary and middle school. In M. K. Heid, & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 1, research syntheses* (pp. 109–154). Information Age Publishing, Inc.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Taylor and Francis.

Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining an apparent paradox. *American Educational Research Journal, 38*(4), 813–834. https://doi.org/10.3102/00028312038004813

de Jong, T., Linn, M. C., & Zacharia, Z. C. (2013). Physical and virtual laboratories in science and engineering education. *Science*, *340*(6130), 305–308. https://doi.org/10.1126/science.1230579

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education, 1*(2), 1–16. https://doi.org/10.1007/s10649-006-0400-z

Fritz, C. O., Morris, P. E., & Richler, J. J. (2012). Effect size estimates: Current use, calculations, and interpretation. *Journal of Experimental Psychology, 141*(1), 2–18. https://doi.org/10.1037/a0024338

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education*. *Monograph, 3*, i-196. https://doi.org/10.2307/749957

Gerlič, I., & Jaušovec, N. (1998). *Spoznavni procesi prisotni pri multimedijsko posredovanem gradivu* [Cognitive processes present in the multimedia delivery of materials]. *Sodobna Pedagogika*, *49*(2), 197–206.

Güler, G., & Çiltaş, A. (2011). The visual representation usage levels of mathematics teachers and students in solving verbal problems. *International Journal of Humanities and Social Science, 1*(11), 145–154. https://bit.ly/3vPoZey

Hartas, D. (2010). *Educational research and inquiry, qualitative and quantitative approaches.* Continuum International Publishing Group. https://doi.org/10.5040/9781474243834

Hassidov, D. (2017). The link between teaching methods and achievement in math in computer-assisted elementary schools. *Creative Education, 8*(14), 2293-2311. https://doi.org/10.4236/ce.2017.814157

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 648–689. https://doi.org/10.1037/0022-0663.91.4.684

Hershkowitz, R. (1989). Visualization in geometry – two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61–76.

Hillmayr, D., Ziernwald, L., Reinhold, F., Hofer, S. I., & Reiss, K. M. (2020). The potential of digital tools to enhance mathematics and science learning in secondary schools: A context-specific meta-analysis. *Computers & Education, 153,* 103897*.* https://doi.org/10.1016/j.compedu.2020.103897

Hoffmann, D. (1998). *Visual Intelligence: How we create what we* see. W. W. Norton.

Jancheski, M. (2011). The importance of animations and simulations in the process of e-learning*.* In A. Madevska-Bogdanova, V. Dimitrova, & D. Spasov (Eds.), *Proceedings of the Eighth Conference on Informatics and Information Technology (CIIT 2011)* (pp.177-181). Springer.

Klahr, D., Triona, L. M., & Williams, C. (2007). Hands on what? The relative effectiveness of physical versus virtual materials in an engineering design project by middle school children. *Journal of Research in Science Teaching, 44*(1), 183–203. http://doi.org/10.1002/tea.20152

Klančar, A., Cotič, M., & Žakelj, A. (2019). *Učenje in poučevanje geometrije z uporabo informacijsko-komunikacijske tehnologije v osnovni šoli* [Learning and teaching geometry using ICT in elementary school]. Založba Univerze na Primorskem. https://doi.org/10.26493/978-961-7055-63-4

Kmetič, S. (2008). *Vloga računalniške tehnologije pri pouku matematike* [The role of ICT in teaching mathematics]. *Vzgoja in Izobraževanje, 39*(5), 52–58.

Kmetič, S., Miholič, T., & Zobec, V. (2014). Do višine trikotnika po več poteh [Arriving at the height of a triangle in diverse ways]. In S. Kmetič, J. Bone, S. Rajh, A. Sambolić Beganović, M. Sirnik, & M. Suban. (Eds.), *Mednarodna Konferenca o učenju in poučevanju matematike KUPM 2014 [2nd International Conference on Learning and Teaching Mathematics KUPM 2014]* (pp. 303–325). Zavod Republike Slovenije za šolstvo. https://bit.ly/3vbvVn0

Kokol-Voljč, V. (2006). Razvoj osnovnih matematičnih pojmov z uporabo programov za dinamično geometrijo – dinamična ponazoritev [Development of basic mathematical concepts using dynamic geometry programs - dynamic illustration]. *Pedagoška Obzorja, 21*(1), 34–47.

Kosslyn, S. (1996). *Image and brain: The resolution of the imagery debate*. The MIT Press.

Lameras, P., & Moumoutzis, N. (2015). Towards the gamification of inquiry-based flipped teaching of mathematics a conceptual analysis and framework. *In Proceedings of 2015 International Conference on Interactive Mobile Communication Technologies and Learning (IMCL)* (pp. 343-347). IEEE. https://doi.org/10.1109/IMCTL.2015.7359616

Lee, H., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education, 8*(4), 326–341. https://bit.ly/3kerrFK

Lipovec, A., & Podgoršek, M. (2016). Risba kot orodje za vpogled v matematično razumevanje [Drawing as a tool for insight into mathematical understanding]. *Psihološka Obzorja, 25*, 156–166. https://doi.org/10.20419/2016.25.452

Mešinović, S., Cotič, M., & Žakelj, A. (2017). Učenje in poučevanje osnovnih geometrijskih pojmov [Learning and teaching basic geometric concepts]. *Pedagoška obzorja: časopis za didaktiko in metodiko, 32*(2), 49–66.

Mullis, I. V. S. (Ed.). (2012). *TIMSS 2011 international results in mathematics*. TIMSS & PIRLS International Study Center.

Oates, G. (2011). Sustaining integrated technology in undergraduate mathematics. *International Journal of Mathematical Education in Science and Technology, 42*(6), 709–721. https://doi.org/10.1080/0020739x.2011.575238

Ollesch, J., Grünig, F., Dörfler, T., & Heidelberg, M. (2017). Teaching mathematics with multimedia-based representations-what about teachers' competencies? In T. Dooley & G. Gueudet (Eds.), *Tenth Congress of the European society for Research in Mathematics Education (CERME10)* (pp. 3976-3983). DCU Institute of Education and ERME. https://bit.ly/3EUc1Ag

Organisation for Economic Co-operation and Development. (2014). *PISA 2012 results in focus: What 15-year-olds know and what they can do with what they know*. https://bit.ly/36KOUeD

Presmeg, N. C. (2014). Contemplating visualization as an epistemological learning tool in mathematics. *ZDM – The International Journal on Mathematics Education*, *46*(1), 151–157. https://doi.org/10.1007/s11858-013-0561-z

Pustavrh, S. (2014). Od zelene table do tabličnih računalnikov. In S. Kmetič, J. Bone, S. Rajh, A. Sambolić Beganović, M. Sirnik & M. Suban (Eds.), *2. mednarodna Konferenca o učenju in poučevanju matematike KUPM* 2014 [2nd International Conference on Learning and Teaching Mathematics KUPM 2014] (pp. 230–238). Zavod Republike Slovenije za šolstvo. https://bit.ly/3ELNmO1

Raphael, D., & Wahlstrom, M. (1989). The influence of instructional aids on mathematics achievement. *Journal for Research in Mathematics Education*, *20*(2), 173–190. https://doi.org/10.2307/749281

Rau, M. A. (2017). How do students learn to see concepts in visualizations? Social learning mechanisms with physical and virtual representations. *Journal of Learning Analytics*, *4*(2), 240–263. https://doi.org/10.18608/jla.2017.42.16

Rebolj, V. (2008). *E-izobraževanje: Skozi očala pedagogike in didaktike* [E-education through the lens of pedagogy and didactics]. Didakta.

Reed, H. C., Drijvers, P., & Kirschner, P. A. (2010). Effects of attitudes and behaviours on learning mathematics with computer tools. *Computers & Education, 55*(1), 1–15. https://doi.org/10.1016/j.compedu.2009.11.012

Republiški izpitni center. (2014). *Nacionalno preverjanje znanja iz matematike v 9. razredu* [National Assessment of Knowledge – mathematics (9th grade)]. Državni izpitni center. https://bit.ly/3xT9AfD

Sagadin, J. (1993). *Poglavja iz metodologije pedagoškega raziskovanja* [Chapters in methodology of educational research]. Zavod Republike Slovenije za šolstvo in šport.

Selwyn, N. (2011). Editorial: In praise of pessimism—the need for negativity in educational technology. *British Journal of Educational Technology, 42*(5), 713–718. https://doi.org/10.1111/j.1467-8535.2011.01215.x

Seufert, T. (2003). Supporting coherence formation in learning from multiple representations. *Learning and Instruction*, *13*(2), 227–237. https://doi.org/10.1016/S0959-4752(02)00022-1

Sočan, G. (2011). *Postopki klasične testne teorije* [The procedures of classical testing theory]. Znanstvena založba Filozofske fakultete.

Thurm, D., & Barzel, B. (2022). Teaching mathematics with technology: A multidimensional analysis of teacher beliefs. *Educational Studies in Mathematics, 109*, 41–63. https://doi.org/10.1007/s10649-021-10072-x

Tratar, J., Mahnič, B., Lešnik, V., Štaher, A., Pev, M., Miklavčič-Jenič, A., Hauptman, A., & Tadina Bence, V. (2014). *Matematika 7: i-učbenik za matematiko v 7. razredu osnovne šole* [7th Grade Math Interactive Notebook]. Zavod Republike Slovenije za šolstvo. https://eucbeniki.sio.si/matematika7/3300/index.html

Viberg, O., Grönlund, Å., & Andersson, A. (2020). Integrating digital technology in mathematics education: A Swedish case study, *Interactive Learning Environments*. Advance online publication. https://doi.org/10.1080/10494820.2020.1770801

Volk, M., Cotič, M., Zajc, M., & Istenic Starcic, A. (2017). Tablet-based cross-curricular maths vs. traditional maths classroom practice for higher-order learning outcomes. *Computers & Education*, *114*, 1–23. https://doi.org/10.1016/j.compedu.2017.06.004

Whiteley, W. (2004, July 4-11). *Visualization in mathematics: Claims and questions towards a research program* [Paper presentation]. 10th International Congress on Mathematical Education, Technical University of Denmark, Denmark.

Wright, D. (2010). Orchestrating the instruments: Integrating ICT in the secondary mathematics classroom through handheld technology networks. *Technology, Pedagogy and Education, 19*(2), 277–284. https://doi.org/10.1080/1475939X.2010.491239

Zbiek, R. M. (2003). Using technology to foster mathematical meaning through problem solving. In H. L. Schoen, & R. I. Charles (Eds.), *Teaching mathematics through problem solving* (pp. 93–104). The National Council of Teachers of Mathematics, Inc.