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# Identifying and Correcting Students' Misconceptions in Defining Angle and Triangle

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**Abstract:** Misconceptions are one of the biggest obstacles in learning mathematics. This study aimed to investigate students' common errors and misunderstandings they cause when defining the angle and the triangle. In addition, we investigated the metacognition/ drawing/ writing/ intervention (MDWI) strategy to change students' understanding of the wrong concepts to the correct ones. A research design was used to achieve this goal. It identified and solved the errors in the definition of angle and triangle among first-year students in the Department of Mathematics Education at an excellent private college in Mataram, Indonesia. The steps were as follows: A test instrument with open-ended questions and in-depth interviews were used to identify the errors, causes, and reasons for the students' misconceptions. Then, the MDWI approach was used to identify a way to correct these errors. It was found that students generally failed in interpreting the concept images, reasoning, and knowledge connection needed to define angles and triangles. The MDWI approach eliminated the misconceptions in generalization, errors in concept images, and incompetence in linking geometry features.

**Keywords:** *Angle and triangle, cause, common errors, misconception correction.*

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## Introduction

Reports from the international organization and the Indonesian Ministry of Education describe that the mathematical achievement of secondary school students in Indonesia has been trending downward in recent years. This achievement includes concept definition, measurement (quantity), relationships, reasoning, and computation skills. The Program for International Student Assessment (PISA) tests and assessments conducted between 2015 and 2016 by the Organization for Economic Co-operation and Development (OECD, 2016) are considered less complete. The tested material includes geometry, and students did not fully master the crucial topics of geometry, including quantity, relation, and uncertainty (Lemke et al., 2004). In 2019, junior high students failed the national mathematics exam, namely in the mark position 45 on the 0-100 score interval. In high school, students' performance in answering all geometry and trigonometry test questions correctly was only 37% in 2017 and 34% in 2018. Students' mastery of mathematics was still not good, i.e., an average score of 45 from the interval of 0 to 100 (Ministry of Education and Culture of the Republic of Indonesia, 2019). These failures may be repeated in students' future studies, especially in the first year of an undergraduate program.

When students have a poor understanding of the definitions and concepts of geometry, it has implications for future mastery of geometry, difficulty, and failure. One factor that may occur in students is misconceptions of geometry when they focus only on physical form and geometric images rather than recognizing the essential geometric properties of the figures represented (Biber et al., 2013; Poon & Leun, 2016). Based on the geometry questions asked to students about the definitions of angle, measure, and shape, it was found in this study that students lack background knowledge, resulting in many learners making errors in reasoning and basic operation mistakes (Özerem, 2012). These facts indicate that students need to develop their understanding of geometry concepts and related abilities.

A theoretical and passive teaching approach that provides very few visuals and tends to ask the students to memorize does not guarantee that students can master the definitions and geometry concepts. It was reported that the images of geometrical objects could be used to illustrate the relationship between one and another concept. These images would help the students understand the abstract ideas and motivate them to acquire the needed knowledge. Moreover,

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concept images affected the students by transforming the situation model into a mathematical model (Battista et al., 1991; Phillips et al., 2010; Şahin et al., 2020). Discovering geometric figures' properties should be a process directed by definitions, axioms, or theorems (Karpuz & Güven, 2022). Besides, stimulating and challenging geometry concepts are required to improve students' understanding. A lack of a formal definition could cause problems for students as they will be unable to test their conception of the idea against the formal theory (Hogue & Scarcelli, 2020). Several studies (Cunningham & Roberts, 2010; Gal & Linchevski, 2010; Özerem, 2012; Ubi et al., 2018) informed that students failed to complete dimensional deconstruction of images to obtain mathematical properties. They found it difficult to determine the characteristics of figural elements relevant to the concept, transformations and construction, and 3-D shapes. This misunderstanding can occur because of the teachers themselves or their environment, i.e., the proficiency and inadequate book facilities. It also appears due to incomplete reasoning and wrong intuition (Kamid et al., 2020). Although the teachers' role in developing knowledge is essential, they also need to play an active role in dealing with misunderstanding problems. Through metacognitive activities, such as their own written work error analysis, students can find and try to align their conceptual inconsistencies with more formally accepted mathematical constructs (Tirosh, 1990, as cited in Kembitzky, 2009). This conceptual change needs some strategies and metacognitive skills. Referring to Stepans' model of conceptual change, changing students' alternative concepts requires the nature of learning tasks that can help students exchange their understandings with the right ideas. The nature of the learning environment can involve social dialogs and negotiations among students (Sarar & Al-Migdady, 2014).

These studies show that the role of teachers in developing knowledge is essential. Challenging concept mastery is necessary for students to improve their conceptions. Using illustrations of geometric objects in teaching and learning helps to understand abstract concepts and acquire the required knowledge. However, some students still have difficulty understanding geometry concepts. Students focus only on physical form and lack prior knowledge of geometry. They also fail to grasp mathematical properties through the visual media. To help students change their misconceptions, they need strategies and metacognitive skills. Due to these obstacles and limitations in learning geometry, we need to make students aware of and strengthen the conceptual understanding of error correction in teaching geometry. Consequently, a new learning strategy is required to improve the correct geometry concepts and logical thinking. For these reasons, this study identifies the errors and the cause of misconceptions and develops plans to reconstruct students' misconceptions of geometry concepts.

## Literature Review

Metacognition is crucial to support the performance of cognitive tasks in mathematics learning. From some research, metacognitive understanding includes aspects of cognitive knowledge and cognitive regulation. This knowledge consists of the cognitive abilities, processes, resources, and the influence of a person, task, or strategy factors on performance (Brown et al., as cited in Garofalo & Lester, 1985, p.164). The regulation of metacognition concerns making strategic decisions in a course on cognitive tasks. These activities include planning studies, monitoring processes, and evaluating and revising results. Sternberg (2002) explained that metacognition is diverse. It explains and controls cognitive processes, including planning, monitoring, and evaluating activities. This process of understanding must, of course, be effective action. It must also be remembered that metacognition interacts with many other aspects of the learner, i.e., skills, personality, and learning styles. Magiera and Zawojewski (2011) used metacognitive awareness, regulation, and assessment approaches for students in small groups. They identified and characterized the social and self-based contexts related to their metacognitive activities in mathematical modeling. The metacognition approach supports the learning process in mastering concepts and geometric reasoning (Nahmias & Teicher, 2021; Wonu & Charles-Ogan, 2017). Using metacognition prevents students from thinking they are just memorizing concepts. It can be a tool to make students aware of and correct their weaknesses in reasoning when mastering the concept of geometry. In the learning process, the teacher can use it to identify the weak points of the students' thinking, develop learning strategies, and establish some levels of error correction by the students.

Visual geometry objects help build people's experiences, beliefs, and understanding of an item through a cognitive process. Logically, the images of geometry objects are usable to explain the relationship between one and another concept. The efficacy of visualization will help understand concepts and support students to acquire educationally the required knowledge (Phillips et al., 2010). On the other sides, as a tool in learning geometry, images of geometry objects are widely used to explain definitions and concepts of geometry, but some students still have misconceptions. They fail to match both concept's formal definition and the geometrical figures (Vinner & Hershkowitz, 1980). Berthelot and Copy (as cited in Poon & Leun, 2016) stated that one of the students' misconceptions factors in geometry learning is an incapability to identify various shapes (symbolic, visual, etc.) of the same geometry concept. Özerem (2012) found the student feebleness of measures, angles and shapes, transformations and construction, and 3-D shapes. This finding is a real challenge for university educators, who generally refuse to be corrected.

Providing personalized interventions to help students resolve misunderstandings in this context is a difficult challenge. Educators must work with their students to identify, recognize, and correct commonly held misconceptions to attain the best learning outcomes. Any student misconceptions critically need to be evaluated, revised, and changed with information consistent with the accepted concepts (Verkade et al., 2016). Generally, there were five causes of errors: language misconceptions, spatial information difficulties, deficient mastery of prerequisite skills, facts, and concepts;

fallacies of thinking; and the application of irrelevant rules or strategies (Radatz, 1979, as cited in Kim, 2011). Ay (2017) reviewed the errors and stated that collecting qualitative data through interviews or observations is one of the most appropriate ways to detect these students' misconceptions apart from the test. These facts provide in-depth information about the students' difficulties in learning. However, the researchers did not prefer going on further steps. Research about the remedial misconceptions of mathematics, particularly in understanding geometry concepts, is still limited.

Teachers' teaching and intervention strategy can make a difference in students' comprehension, which is essential in instructional practice and student learning. In geometry learning, Lim (2011, as cited in Luneta, 2015) states that the information communication at the different levels of reasoning among the teacher and student becomes a common cause of misconception. When teachers explain different geometry thinking levels to students, the concepts are not fully understood or acquired. Teachers must know their students' level of geometrical understanding. Battista et al. (1991) reported that developing the students' meaningful comprehension of geometry concepts requires an appropriate instructional task and assessment in teaching and learning geometry. Clarke et al. (1993, as cited in Kembitzky, 2009) found that writing allows a teacher to see the kind of thinking and understanding that is not easy and accessible via the computational and proficiency test. Teachers can examine the sense-making process when students explore and work with mathematics. Therefore, teachers' intervention and students' writing assignments will direct the achievement of conceptual understanding following curriculum objectives. It can help the students to use previous experiences correctly and provide a new comprehension of the shortcomings of prerequisite material that students do not yet have and avoid understanding concepts via rote but by understanding processes.

In summary, the discussion of these research findings provides essential clues for resolving the misconception problems of the students. The metacognition approach supports the learning process in mastering concepts and geometric reasoning. It can also be a tool to make students aware and correct their reasoning weaknesses in understanding geometry concepts. Using geometry images can help students identify, recognize, and remedy misconceptions to attain the best learning outcomes. Writing geometry ideas can employ to see the kind of students' thinking and understanding in which the lecturers can direct the achievement of conceptual understanding following instructional objectives. Taking into account this thought, we conducted this research.

*Problems and Purposes of Research*

The ability of students to define geometric concepts is one of the main goals in achieving mathematical competencies in the first year of undergraduate study. Unfortunately, using geometric figures and tools to help students understand the concepts and definitions leads to some errors and misunderstandings. They find it difficult to determine the features of the figural elements relevant to the idea and often fail to order the words to construct the alternative definitions. Natural and appropriate learning methods are needed to replace students' misconceptions with the right ideas. In this study, we aimed to answer the following problems.

1. What are the common errors and roots of misconceptions in defining angles and triangles among undergraduate students of mathematics education?
2. How can strategies resolve students' misconceptions about the definitions of angles and triangles and turn them from incorrect concepts to correct concepts in the teaching-learning process?

This study aimed to identify students' common errors and the causes of misconceptions in defining angles and triangles. It also presented strategies to change students' understanding from the wrong concepts to the right concepts about angles and triangles.

*Framework of Research*

*Stage 1: Identification of Misconception*

Student misconceptions of geometry concepts can arise from many factors such as student experience and learning approach, teachers' roles, and equipment (Cunningham & Roberts, 2010; Gal & Linchevski, 2010; Özerem, 2012; Poon & Leun, 2016). The studies found that some students make mistakes related to a lack of understanding of geometric figures and insufficient knowledge of the significance of proofs, and they fail to state the particular polygons and features of polygons (Alamian et al., 2020; Cirillo & Hummer, 2019; Herholdt & Sapire, 2014; Junus, 2018). Students' difficulties in understanding geometry concepts vary, and this research needs to identify student misconceptions to strategize the remediation process.

*Stage 2: Misconceptions Diagnosis*

Research studies have shown that the causes of concept misunderstanding should be eliminated. Since mathematical materials are usually interconnected, students' misconceptions about previously discussed topics should be eliminated before introducing a new topic (Al-Khateeb, 2016; Ozkan et al., 2018). This study aims to diagnose students' misconceptions to determine their error rate and find the causes and roots of their difficulties in teaching and learning geometry. It also aims to investigate the weaknesses of the teaching approach and the teachers' capabilities. Using the interview method to diagnose errors can provide up-to-date information about students' weaknesses and flexibility in testing; meanwhile, open-ended testing methods allow students to write their answers in their own words, and they are likely to provide some new valuable answers (Gurel & Eryılmaz, 2015).

*Stage 3: Strategy and Correction of Misconceptions*

There are three treatment steps to conduct the strategy and errors corrections i.e., student awareness; defragmentation, reconstruction, and geometry concepts connection of students' knowledge; revision and decision to exchange from the wrong to the right ideas. The ways are as follows.

*Step 1: Student Awareness of Errors and Difficulties*

The first step in changing the misconception is to make the student aware from the beginning that there is a mistake. Removing the obstacles that stand in the way of students' learning must first and foremost come from them, including their beliefs and prior knowledge (Verkade et al., 2016). Kruger and Dunning (2009) also warned that students who do not know their abilities are doubly burdened: First, they will only draw incorrect conclusions, and second, developing metacognitive skills to recognize this is problematic. A study by Taylor and Kowalski (2004) informed that the power of belief is a crucial transition variable that can alter the change process in mind. In addition, Hughes et al. (2013) concluded that it is easier for students to ignore, reinterpret, or reject new information than to change their beliefs. In this study, the change in students' conceptual thinking was brought about by motivational activities, namely, dissatisfaction with their previous beliefs and providing a clear alternative explanation and rationale.

*Step 2: Defragmentation, Reconstruction, and Connection Treatments of Incorrect Concept*

- *Metacognitive Regulation*

Student error correction must be supported and guided as part of the learning process. By applying the approach of metacognitive strategies: Planning Strategies, Monitoring, and Evaluation, students' misconceptions are expected to be resolved through defragmentation or substitution, reconstruction, and connection processes of thinking structures (Artzt & Thomas, 1998; Garofalo & Lester, 1985). These processes involve reconstructing thinking fragments of a false concept, linking knowledge and correcting a minor thinking error, and reorganizing their knowledge structures and logical thinking error. This research makes students aware of the problems they know in the first place. Then the educator encourages students to think actively to replace the wrong concept with correct ideas. Metacognitive regulation guides students and teachers to design, control, evaluate, correct, and structure the understanding of geometry features and the words to define angles and triangles.

This study's scheme of metacognitive regulation involves students' understanding of prior geometric knowledge to construct an angle and triangle, i.e., point, line, position, and direction. Students must recognize the formal definitions of a line segment and a ray using undefined terms. By drawing, connecting, or combining the points, lines, segments, and rays, they must try to find the shapes of the angle and triangle. Using these constructed images, students practice explaining and putting together the definitions in their own words. Briefly (Figure 1), in the process of correcting misconceptions, they have to go through four stages, i.e., they have to recognize primitive concepts and complete their prior knowledge related to the defined geometry concepts (M1); draw and demonstrate the figures of the concepts (M2); represent and write down geometric ideas related to the prior knowledge (M3); write formal definitions (M4). These treatment series help students naturally identify, reflect on, evaluate, and correct misconceptions about angle definitions and triangles. Students' errors in these stages can be quickly identified and corrected from the teacher's side. This way, teachers can be more targeted and effectively correct students' errors.

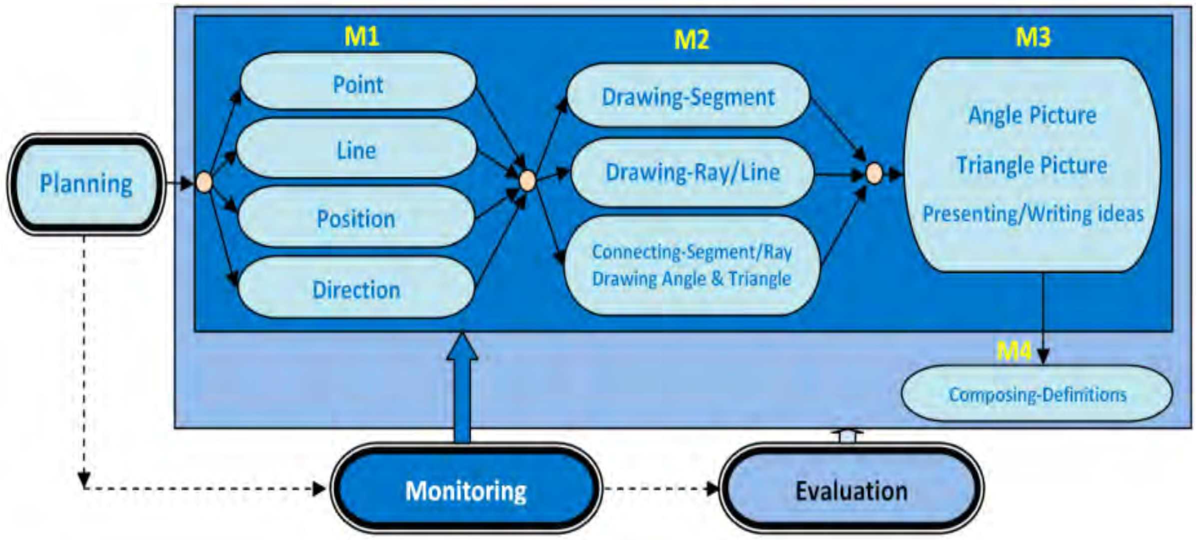



*Figure 1. Metacognitive Regulation Scheme*

- *Drawing Geometry Figures Based on Recognizing Primitives Concepts*

Using figural and visual representations of a geometry concept should help students understand abstract ideas. However, there are many cases of misunderstanding due to a lack of prior knowledge related to the development of these geometry concepts (Battista et al., 1991; Özerem, 2012). Students made errors caused by the concept image, the set of all mental images associated with the concept name in the student's mind, and its characterizing properties (Şahin et al., 2020). Consequently, these students failed to construct the definitions and misunderstood the mathematical concepts. When students' knowledge of geometry properties is incomplete, their concept image deviates from the required formal definition (Kembitzky, 2009; Poon & Leun, 2016). In this case, the researchers used students' thinking processes in defining a geometry concept, which was emphasized to avoid memorization of picture concepts and guided by the educator. They must strictly start from undefined geometry concepts (primitive) and experience geometry figures to build a geometry concept in the standard definition form and other geometry concepts in the broad sense.

- *Writing Task to Present Ideas and Write Definitions*

In this research, students' writing stimulates dialog for direct and indirect communication between students and the teacher in the teaching-learning process. Pugalee confirmed that writing enhances mathematical thinking and helps students internalize productive communication and relationships (cited in Urquhart, 2009). Students' writing can identify and assess their mathematical knowledge's correctness. The studies showed that students' understanding improved, and they exchanged their arguments and reevaluated their answers. They became more developed in the skills of thinking and ideas. In addition, they were better able to make a connection between abstract mathematics and the questioned context. (Barbara et al., 2016; Freeman et al., 2016; Wilson & Nebraska, 2009).

- *Intervention with Students with Social Engagement*

One of the most common conceptual changes made using classroom intervention and instructional strategies was to induce cognitive conflict by presenting unusual (strange) facts or contradictory information (Limo´n, 2001). The primary goal of cognitive conflict is to disconfirm students with their current beliefs (Ozdemir & Clark, 2007, as cited in Kabaca et al., 2011). The educator can use these strategies with students who lack prior knowledge (lack of knowledge) and incomplete knowledge or knowledge gaps (Chi, 2008, 2013). On the other hand, Kowalski and Taylor's study suggested that the educator uses a critical thinking method to predict the change in students' misconceptions. The change in student misconceptions can occur for each ability level and correct students who are critical thinkers (Kowalski & Taylor, 2004). Through small group discussions, the educator can use Stepans' model to help students think contrary to their previous beliefs. In this group, students become accustomed to the new concept and resolve the existing contradictions. Then, they further develop the concepts by connecting the thought learned in class with other related concepts and ideas (Stepans, 2011,as cited in Sarar & Al-Migdady, 2014). In this study, the main priority of the educator's interventions and challenges is how to connect and correct the student's previously learned material with their new knowledge.

*Step 3: Deciding How to Replace Incorrect Concepts with Correct Concepts*

After the treatment in Step 2, we evaluate the results of the measures. If the student's work is considered satisfactory and coherent with the explanatory replacement concept, and the student believes that the replacement concept has value in solving problems, the educator can give the student a chance to pursue a new idea under guidance. If not, troubleshooting must restart as soon as possible.

## Methodology

This study design was observed with quantitative and qualitative descriptive research. The steps of the method were as follows: gathering data, interpreting and analyzing data, and reporting the findings (Creswell, 2013; Nassaji, 2015). This approach was utilized because it helped us understand students' in-deep misconceptions about defining an angle and the triangle before attending a geometry course at the beginning of the first semester. The investigation had the following three main objectives. (1) To identify students' common errors, the roots, and the causes of misunderstanding in defining the angle and the triangle. (2) To investigate the metacognitive regulation scheme in guiding and leading student thinking of geometry concepts. (3) To introduce the strategies for defragmenting, reconstructing, or linking students' knowledge from the incorrect geometry concepts to be correct concepts about angle and triangle definition.

*Participants and Times*

The research involved two mathematics education department groups of students with 40 students per class from a private university in Mataram, Indonesia. Both groups were students who graduated from public and private high schools and passed the national mathematics examinations, including the geometry lesson. The research was held from March until December 2021.

*Instruments*

The main instruments of researchers and auxiliary instruments of lecturers were used in this research. The main instruments were observing, collecting, analyzing, and interpreting research data. Then, the auxiliary instruments made the definition questions of angles and triangles and composed the unstructured interview guidelines validated by two experts. The researchers used open-ended question tests instrument about angle and triangle concepts to investigate students' errors and misconceptions of the geometry ideas. The test instrument consisted of three questions based on the content areas related to the prior knowledge, drawing the shapes, writing ideas, and defining angles and triangles. The reliability of this instrument was evaluated using the test-retest method for the test scores interval 0-100 in the range of 20 days from 52 students (two classes) on the students' previous batch. Calculating the correlation of the successive test-retest results through the formula of Pearson's product moment correlation coefficient $r$, it obtained a considered good coefficient with the reliability index $r \geq 0.7$ at 0.05 significant. The validity of each test item was determined by the correlation value $r$ between the item score values and the total item score. It was the values $r$ in the interval $0.6 \leq r \leq 0.8$. On the other hand, the content validity of the unstructured interview guidelines was determined using experts' agreement through scoring the items according to a graded scale. It was calculated with Aiken's formula to obtain the item validity index $V$ of the high category, i.e., the value of $V$ was found in the score range 0.6-0.8.

*Procedure*

The technical analysis of this qualitative research data consisted of processing and preparing the data for analysis, summarizing and coding the data from the answer sheet, describing the types of errors and correction strategies, and drawing conclusions.

- *Data Analysis*

The quantitative data was collected from the students' written responses to the test questions about angle and triangle definition (in Appendix) with scores of 0-100. Two researchers, separately and independently, identified the students who got low scores and had difficulties answering the test items. Each researcher made a list of the error (misconception) types conducted by students and computed the frequency number of each error type found. Based on these error types and their frequency number, the obtained data of both researchers were compared and re-examined. At the end of this evaluation, the researchers agreed that it was identified and determined five types of general errors of the students, from high error levels to low mistakes.

- *Identifying common errors and diagnosing the roots and the causes of misunderstanding*

Referring to the students' test answer errors included in these five error types, the researchers interviewed each student to discover their existing concept errors and misconceptions dealt with their formal definitions composed. Finding common mistakes and the roots of misunderstanding were classified and coded from high common errors level ($E_1$) to low errors ($E_5$). We also calculated the number of students who made the errors for each test item, as shown in

Table 1. Discovering the roots of their misconception, ensuring students' mistakes from the test answers (whether it was an error or misunderstanding), and improving awareness of students from the thinking, we interviewed them 1-1 through a personal approach related to their incorrect answers. The interview content was related to their experiences with the students' geometry pre-knowledge, ways of thinking, oral describing the definition of angle and triangle, and admitting their errors. The results of these activities are presented in Table 2. The students' understanding and inadequate understanding of minor misconceptions are treated by defragmenting and connecting the incorrect geometry concepts with the right ideas. Other, it was treated by using reconstruction actions as shown in Table 3.

- *Misconceptions' Correction and Concepts Exchange Decision*

The treatment of the group of students with the minor error was performed individually according to the metacognitive regulation mechanism in Figure 1. At the beginning of the misconception correction tasks, students were asked to find all primitive terms and some supporting concepts used to define angle and triangle (point, line, ray, segment, position, and direction). Using these elementary geometry objects, they were to attempt to draw and represent any angle and triangle shapes. They then explained each construction process and wrote down their results in the formal definitions in their language. In this case, the educator's intervention and instruction focused on improving and revising the following aspects of knowledge. (1) recognizing the prior knowledge for defining the geometry concepts; (2) demonstrating the process of constructing the concepts using pictures; (3) presenting and writing down this demonstrated concept idea; and (4) writing the formal definitions. The corrective function of the treatments is to fill the gap (hole) in the students' conceptual understanding or to revise and reconstruct the wrong parts of the concepts.

On the other hand, the treatment of the students from the group with critical errors was the same as that of those from the group with minor errors. Nevertheless, we divided them into some groups (5 students per group). The task of the groups was to discuss and evaluate the alternative definitions that emerged from the group members. Finally, each group was to create some alternative definitions of geometry that best fit the formal concept discussed. During this step, the teacher used counterexamples and cognitive conflict strategies to make students dissatisfied with their ideas. These methods were also used to correct students' thinking, guide them, and help them replace students' incorrect concepts with correct ones. The scheme and results of these treatments are presented in Table 4.

The educator evaluated the students' difficulties in correcting the results of minor and major misconceptions. If he considered that the work of both groups was correct, clear, and coherent with the formal concepts, they could pursue a new idea. If not, they should repeat the work in depth.

## Results

*Problem 1: Identifying Common Errors and Diagnosing Roots and Causes*

Based on the analysis results of student answers to the test items in the Appendix, it was founded that thirty-seven students made errors in defining angles and sixty-one students in explaining triangles. This research identified five types of students' common mistakes in this case. First, students assumed that an angle was a point ($E_1$). Second, they thought that an angle was the area part of a plane between two legs of this angle ($E_2$). Third, students argued that an angle was a figure represented by two line segments combined at one endpoint of both line segments ($E_3$). Fourth, they stated that a triangle was a part of a plane piece that forms the triangle ($E_4$), and, fifth error, they concluded that any three line segments defined a triangle ($E_5$). On the other side, in answering Test Item 1, there were 14 students (17.5%) of the error $E_1$, 11 students (13.8%) of the error $E_2$, and 12 students (15%) of the error $E_3$. For Test Items 2 and 3, respectively, there were 28 students (33.8%) of the error $E_4$ and 33 students (41.3%) of the error $E_5$. Thus, the total errors of Test Items 1, 2, and 3 were successively 37 students (46.3%), 28 students (33.8%), and 33 students (41.3%). The misconceptions frequencies (*f*) of these thirty-seven students in solving geometry Test Items 1, 2, and 3 see in Table 1.

*Table 1. Frequencies of Students' Misconception in Understanding Angle and Triangle*

| **Problems** | **Error 1 ($E_1$)** | | **Error 2 ($E_2$)** | | **Error 3 ($E_3$)** | | **Error 4 ($E_4$)** | | **Error 5 ($E_5$)** | | **Total** | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **f** | **%** | **f** | **%** | **f** | **%** | **f** | **%** | **f** | **%** | **f** | **%** |
| Test Item 1 | **14** | 17.5 | **11** | 13.8 | **12** | 15 | - | - | - | - | **37** | 46.3 |
| Test Item 2 | - | - | - | - | - | - | **28** | 33.8 | - | - | **28** | 33.8 |
| Test Item 3 | - | - | - | - | - | - | - | - | **33** | 41.3 | **33** | 41.3 |

The errors' roots and causes of students' misconceptions have resulted from the student works analysis and in-depth interviews. The interview content was related to their experiences with geometry pre-knowledge, ways of thinking, oral describing and writing the definitions of angle and triangle, and admitting errors. From the interview results, we also evaluated the missing and incomplete knowledge structures, the connection among the learned material of the students, and the logical consequences as follows.

*Misconception 1: Over-specializing that an angle ∠ABC is the point B.*

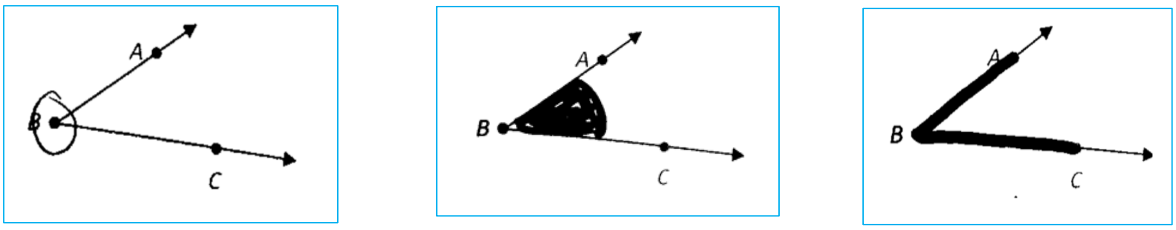
Students who underwent this misconception could memorize that an angle definition $\angle ABC$ was the union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Intersected at the common endpoint $B$, they marked point $B$ as an angle (Figure 2a). Regarding results of interviews with the students of the misconceptions $E_1$, they did not generally understand some undefined terms and geometry's elementary objects used for defining rays, line segments, and drawing an angle. They could not explain the relationship between these concepts to construct the angle. Moreover, they ignored the starting point position and the direction for a line ray and habited the writing angle symbol with only one capital letter. Due to the students defining the angle with rote, they argued that this angle is the point $B$.

*Misconception 2: Over-generalizing that an angle is the area part of a plane bounded by two legs of this angle.*

This misunderstanding appeared in the students' answers to Test Item 1 about the angle problem. Although the students already knew an angle constructed by two rays intersecting at the endpoint, they stated an angle figure as the area bounded by their angle's legs. For example, Figure 2b shows the work of a student with this second misconception type. The in-deep interviews could inform the students' misconceptions from two causes. They did not understand that two rays met at the starting point would consistently result in the ray's pieces connection (not a cut of plane), and they lacked the prior knowledge to differentiate between an angle and its measure. As a result, they believed that an angle was the area part of a plane bounded by two legs of this angle.

*Misconception 3: An angle is a figure formed by two-line segments that meet at one endpoint of the segments.*

In the third misconception, the students recognized that the angle $\angle ABC$ was a union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ coincided at its starting point $B$, but they stated that an angle $\angle ABC$ was two line segments $\overline{BA}$ and $\overline{BC}$ That met at point $B$. For example, Figure 2c shows the works result of students that the line segments $\overline{BA}$ and $\overline{BC}$ as an angle. They argued that an angle $\angle ABC$ was a set of points of line segments $\overline{BA}$ and $\overline{BC}$. Appertaining to the results of in-depth interviews, the students who committed the misconceptions had not consistently differentiated between segments and rays to define an angle. They over-specialized this angle represented with three points and three capital letters. Because of this, the students said an angle was a figure formed by two-line segments that met at one endpoint of the segments.




(a) (b) (c)

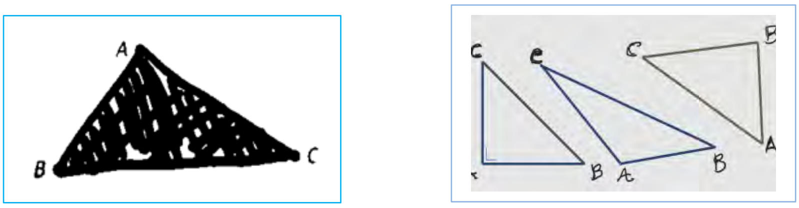
*Figure 2. Student work results in the completion of Test Items 1*

*Misconception 4: A triangle is a part of a plane piece that forms the triangle.*

Students considered a triangle the triangle's interior or the area bounded by the triangle. This misconception appeared in the students' works of Test Item 2 relating to the triangle concept. In Test Item 2, the students who had a misunderstanding could define a triangle as a polygon of three sides but pointed out the graph that the triangle was an area bounded by the sides of the triangle (interior of the triangle). For example, Figure 3a shows the works of students experiencing the fourth misconception. Referring to the results of the interviews, the students' error $E_4$ did not know that the merging three-line segments at its endpoints for constructing a triangle would produce three line-segments connection picture. They also used their primary school experiences in which a triangle was made from cutting paper through three noncollinear points. Consequently, these students declared that a triangle was a part of a plane piece that forms the triangle.

*Misconception 5: Any three-line segments define a triangle.*

Students concluded that any three-line segment could form a triangle. This misconception happened from Test Item 3 about three-line segments as data for triangle sides; students understood that triangle sides had three-line segments. Relating to the solution of Test Item 3, Figure 3b, the students explained that any three-line segments could form a triangle, i.e., a right triangle or other triangles. Because of these triangle images and without examining the measure of these three-line segments data, they made wrong conclusions.



(a) (b)

*Figure 3. Students' Work Results in the Completion of Test Items 2 and 3*

Appertaining to these students' interview results, errors $E_1$ to $E_5$ found ten information about the roots of students' misconceptions. The list of these roots and causes of students' misunderstanding presents in Table 2.

*Table 2. Common Errors, the Roots and the Causes of Students' Misunderstanding*

| Code | Common errors | The Roots and the Causes of Misunderstanding (Code) |
|---|---|---|
| $E_1$ | An angle $\angle ABC$ is a point $B$. | 1. The students do not understand some undefined terms and geometry's elementary objects used for defining rays, line segments, and drawing an angle ($E_{11}$).<br>2. They do not know the relation between these definitions to construct an angle ($E_{12}$).<br>3. They define the angle with the rote and habit of the writing angle symbol with only one capital letter ($E_{13}$). |
| $E_2$ | An angle is an area part of a plane bounded by two legs of the angle. | 1. The students do not understand that if two-line rays meet at their starting point will consistently result in the rays' pieces' connection ($E_{21}$).<br>2. They lack the prior knowledge to differentiate between an angle and its measure or the area between their angle legs ($E_{22}$). |
| $E_3$ | An angle is a figure formed by two-line segments that meet at one endpoint of the segments. | 1. The students cannot consistently differentiate between segment and ray to define an angle ($E_{31}$).<br>2. They over-specialize an angle represented with three points and three capital letters ($E_{32}$). |
| $E_4$ | A triangle is a part of a plane piece that forms the triangle. | 1. The students do not know that merging three-line segments at their endpoints for constructing a triangle will produce a line-segments connection picture ($E_{41}$).<br>2. They have a misconception from primary school experiences in which a triangle makes from cutting paper through three noncollinear points ($E_{42}$). |
| $E_5$ | Any three-line segments define a triangle. | 1. The students state the conclusion for defining a triangle using some triangle images without counting and comparing the length of three-line segments ($E_{51}$). |

*Problem 2: Misconceptions' Correction and Concepts Exchange Decision*

In general, we found the sources of these misconceptions of students were the lack of prior knowledge or missing knowledge (MK) of geometry concepts, the existence of knowledge gap or incomplete knowledge (IK), interpretation deviation (ID) of concept images, feeble-logical thinking (FT), and low connection (LC) of students' knowledge. These causes characterize the misconceptions in the following three types. In the case of causes of MK and IK, we call inadequate understanding with minor errors (10 students). For the causes of ID and FT, we state vague understanding with major mistakes (18 students), another (LC) calls almost understanding with minor errors (9 students). Resolving these misconceptions' causes, we introduce the approach to reconstructing thinking fragmentations of MK and IK, rearranging knowledge structures and logical thinking of ID and FT, and linking knowledge of students' LC. For this solutions approach, we believe, in the terms: defragmentation, reconstruction, and connection solutions, as shown in Table 3.

*Table 3: Treatment Approaches of Students' Misunderstanding*

| Concepts Mastery Achievement | Number and kinds of Students' Misconception and Treatment Types | |
|---|---|---|
| | **Minor** | **Major** |
| **Inadequate understanding** | 10 Students of<br>MK and IK *Defragmentation* | 18 Students of<br>ID and FT *Reconstruction* |
| **Almost understanding** | 9 Students of<br>LC *Connection* | - |

This section reports the error correction of the students. It cures the students' misunderstanding roots in Table 2 for their error cases in Table 1. The stages were as follows (Table 4). Implementing the metacognitive regulation scheme presented in Figure 1 is labeled M. The instructor directed the students to learn primitive concepts of the angle or triangle (activity M1) and design figures of the angle or triangle (activity M2). Evaluating these students' activities was focused on drawing an angle or a triangle idea connected with primitive concepts. The stage of activities M1 and M2 is called the drawing concept and labeled D. Then, the students presented pictures, wrote geometry ideas based on prior knowledge (activity M3), and composed formal definitions (activity M4). The instructor helped the students to recognize the geometrical characteristics and connections of the angle and triangle elements, the logical thinking for constructing an angle or triangle and write the definitions of angle and triangle through the pictures. The stage of activities M3 and M4 is called the writing task and labeled W. During the learning activities D and W; the instructor provided interventions and instructions to develop the students' knowledge structure and induce cognitive conflicts. These cognitive conflicts were designed to resolve the causes of students' misunderstanding of MK, IK, ID, FT, and LC. The cognitive conflict's intervention actions, i.e., are marked by the code $I_{MK}$, $I_{IK}$, $I_{ID}$, $I_{FT}$, and $I_{LC}$ presented in columns 4-7 in Table 4. Students with minor misconceptions were treated individually, in contrast to the major ones, remedied in groups of five students. To assess students' achievement with scores interval 0–100 and in-depth interview, the treatments found the average result scores shown in column nine of Table 4. These remedial activities M, D, W, and I are called metacognition/drawing/writing/intervention (MDWI) strategy.

Some cognitive conflict examples associated with the treatments of students' misunderstanding MK, IK, ID, FT, and LC in Table 2 and often used by the instructor for intervening students in this research were as follows.

1. An angle $\angle ABC$ is just a point B that is an undefined geometry object; versus a set of points consisting of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Intersects at the endpoint $B$.
2. An angle $\angle ABC$ is a measure of arc degree or an area between two angle legs $\overrightarrow{BA}$ and $\overrightarrow{BC}$; contra to the joint of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ coincides at point $B$.
3. An angle $\angle ABC$ is a joint of two-line segments $\overline{BA}$ and $\overline{BC}$ meet at the endpoint $B$; against the union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Intersects at the endpoint $B$.
4. An angle $\angle ABC$ is just three points $A$, $B$, and $C$; versus a union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ meets at point $B$.
5. The joining of three-line segments $\overline{BA}$, $\overline{BC}$, and $\overline{AC}$ at their endpoints $A$, $B$, and $C$ will form a cut of a plane $ABC$ called a triangle $ABC$; contra to they produce a picture of three-line segments called a triangle $ABC$.
6. Every three-line segments $\overline{BA}$, $\overline{BC}$ and $\overline{AC}$ can form a triangle $ABC$; contra to the line segments of the measures $\overline{BA}$ = 3 cm, $\overline{BC}$ = 5 cm, and $\overline{AC}$ = 15 cm will not construct a triangle $ABC$.

*Table 4: Misconceptions' Remediation Process*

| Errors Roots | Students Number | Treatment Types | Metacognitive Regulation (M) | | | | Correct Students Number | Average Result Scores (0 – 100) |
|---|---|---|---|---|---|---|---|---|
| | | | Drawing Concept (D) | | Writing Task (W) | | | |
| | | | M1 | M2 | M3 | M4 | | |
| $E_{11}$ | 7 | Defragmentation, Reconstruction, & connection. | $I_{MK}$ | $I_{IK}$ | $I_{ID}$ | $I_{FT}$-$I_{LC}$ | **5** | **71.43** |
| $E_{12}$ | 4 | Reconstruction. | - | - | $I_{ID}$-$I_{FT}$ | $I_{FT}$ | 3 | **75.00** |
| $E_{13}$ | 3 | Reconstruction & connection. | - | - | $I_{ID}$ | $I_{FT}$-$I_{LC}$ | 2 | **66.67** |
| $E_{21}$ | 6 | Reconstruction | - | - | $I_{ID}$ | - | 4 | **66.67** |
| $E_{22}$ | 5 | Defragmentation & reconstruction. | $I_{MK}$-$I_{IK}$ | $I_{ID}$ | $I_{ID}$ | - | 4 | **80.00** |
| $E_{31}$ | 7 | Defragmentation & reconstruction. | $I_{MK}$ | $I_{ID}$ | $I_{ID}$ | - | 5 | **71.43** |
| $E_{32}$ | 5 | Reconstruction. | - | - | $I_{ID}$ | $I_{FT}$ | 4 | **80.00** |
| $E_{41}$ | 17 | Reconstruction & connection. | - | - | $I_{ID}$-$I_{LC}$ | - | 13 | **76.47** |
| $E_{42}$ | 11 | Reconstruction. | - | - | $I_{ID}$ | - | 8 | **72.73** |
| $E_{51}$ | 33 | Reconstruction & connection. | - | - | $I_{ID}$ | $I_{LC}$ | 24 | **72.73** |
| **Total** | **98** | **Number of Interventions** | **4** | **3** | **12** | **7** | **72** | **73.31** |

## Discussion

Based on Table 1 and Table 2 informs that only 43 students (53.7%) understood the concept correctly, 37 students (46.3%) could not define the angle, and 33 of the 61 students (76.3%) failed to understand the triangle idea. Moreover, we found that no more than 54% of the students could correctly define the angle and triangle terms. The main impediments and feebleness of the students were that they did not know the geometry primitive terms, the function, and the role of the ray and line segment in defining an angle or triangle. In this case, they might learn the geometry concepts and definitions partially. They also underwent the visual deviation between a point and an angle, an angle measure and an angle, and between a triangle and a plane cut triangle. These results were relevant to the studies of Ozkan et al. (2018), Al-Khateeb (2016), and Özerem (2012). They informed that one of the misconceptions causes was the lack of prior knowledge and insufficient students' knowledge of the geometry concepts. These finding results were also in line with the studies of Poon and Leun (2016), Biber et al. (2013), Cunningham and Roberts (2010), and Gal and Linchevski (2010). They found that students faced difficulties selecting the characteristics of figural elements relevant to the concepts. Because students focused only on the physical shapes and the geometry images, they had difficulty identifying the essential geometry properties of represented figures and fundamental logical reasoning abilities.

The mathematical material is generally interrelated. Students who do not understand the geometric relationships between points, lines, rays, and line segments will have difficulty determining an angle or triangle. Thus, to correctly define an angle and triangle, they must first understand these elementary geometric objects' ideas, functions, and roles.

Table 4 shows that at least 26 interventions are needed from the teacher to address the students' misconceptions. Students' misconceptions are the interpretation deviation of the picture concept to define angles or triangles. The cause of this main problem is that students cannot recognize the geometric objects to draw the angle or triangle and cannot logically clarify how to construct this angle or triangle using these objects. As a result, they incorrectly assumed that an angle was a point, three points, or a combination of line segments. They also made an incorrect generalization (overspecialization and overgeneralization).

By correcting the students' misconceptions with the metacognition rule scheme (M), the students' actions to draw all the figural concepts (D), the writing ideas (W), and the instructor's intervention (I) shown in Table 4, the students' performance score improved to 73 (very good) and the students' performance score on the correct answers improved to 72/98 = 73%. It was found that this MDWI strategy could help students to change their wrong ideas in defining angles and triangles into correct concepts. It could also address students' errors in generalization, concept images, and incompetence in linking geometry features discussed by Gutiérrez and Jaime (1999), Özerem (2012), Poon and Leun (2016), Ozkan et al. (2018), and Şahin et al. (2020).

This MDWI approach encourages students to revise their preconceived notions and incorporate new ideas. We hope that it will allow students to reflect on their misconceptions and negotiate true mathematical meaning so that they become stronger. They will be more able to replace their incorrect geometry concepts with correct concepts and decisions. Using this MDWI strategy will help defragment and reconstruct or replace the student's misconceptions and a gap in understanding nature and considering the following. These include student motivation and beliefs, prior knowledge, and cognitive engagement; teacher content knowledge, interests, and teaching strategies; the role of peer learning; and the student-teacher relationship in the social context. The errors corrected by students include misconceptions related to generalization, concept images, geometry features, and properties (Figure 4).

The corrections made by these students result in average scores showing that their work was corrected and consistent with formal definitions. Each student's score was above 60, so they were able to pursue a new learning theme.

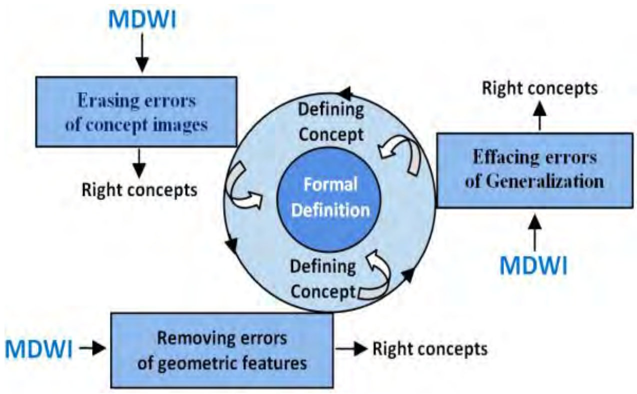



*Figure 4. Errors Elimination*

## Conclusion

The most common errors occurred in the definition of angle and triangle among students of mathematics, that is, the assumption and generalization that an angle was a point, a plane portion of a plane between two legs of the angle, and a union of two-line segments meeting at an endpoint of the segments. Then they made the mistake that a triangle is a part of the plane that forms the triangle and that all three-line segments define a triangle. The causes of these errors generally lie in the basic ideas. They did not know the undefined terms, the function, and the role of the ray and the line segment in defining an angle or triangle. Students generally failed in interpreting concept images, reasoning, and connecting knowledge needed to draw, construct, and write the definitions of angle and triangle.

The MDWI approach could guide students to learn primitive concepts, draw the geometry notion associated with primitive ideas, and capture and write these images in formal definitions. On the other hand, the instructor could provide interventions and instructions to develop their knowledge structure, trigger cognitive conflicts, and address the causes of their misunderstanding in terms of generalization errors, interpretation deviations of concept images, and connection inability of geometry features. During the recovery process, these treatment effects could motivate students to decrease and reduce their errors in recognizing geometric objects, generalization, interpretation of concept images, and linking geometry features.

This study provides new insights related to identifying the errors, roots, and causes of students' misconceptions in defining geometry concepts, especially angles and triangles. This study provides educators and researchers with insights into correcting students' misconceptions through using metacognitive skills, drawing geometry concepts, writing down geometry ideas in conjunction with prior knowledge, and student intervention during the learning process. The research highlights that using the MDWI approach can help correct students' misconceptions in defining these geometry concepts.

## Recommendations

Considering that error correction in a geometry object definition is rare, this MDWI treatment provides a guide to rid students of misconceptions in geometry concepts step by step based on metacognitive regulation. According to the error rate of college students in these generalizations and concept images, M2 and M3 treatment activities are more effective in helping them avoid misconceptions in interpreting and reasoning concept images and geometry features.

In general, the learning materials for geometry at the secondary level emphasize the mastery of geometry concepts related to the definition of geometric objects, measurements, and calculating the area and volume of these objects. On the other hand, assessment reports were incomplete in mathematics instruction, especially geometry. Based on the study results, further research should be conducted with middle and high school students to determine the cause and roots of the difficulties in mastering geometry concepts. In addition, the MDWI treatment application helps students correct misconceptions caused by lack of prior knowledge, incomplete knowledge, interpretation deviations of conceptual images, weak logical thinking, or low linkage of students' prior knowledge.

## Limitations

This research was conducted on students elected from a private university in NTB Province, Indonesia. Thus, the generalization of the results in this research has limitations. Another limitation of the study was that it only focused on defining the terms of geometry concepts.

### Authorship Contribution Statement

Kusno: Concept and design, data analysis, writing, editing, review, final approval. Sutarto: Data acquisition, data analysis.

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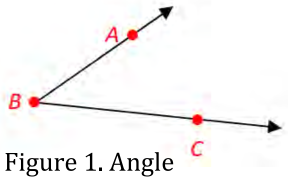
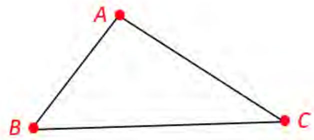
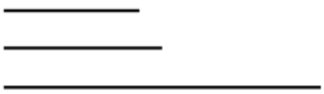
## Appendix

*Instruments for Identifying Angle and Triangle Misconceptions and Types of Problems*

| Problem | Problems Types |
| --- | --- |
| 1. Consider the points *A*, *B*, and *C* in Figure 1. Give a mark using a colored pen, which is a part of Figure 1 called an angle? Give your reason in detail!<br><br>Figure 1. Angle | Angle Problem |
| 2. Let three points *A*, *B*, and *C* in Figure 2. Give a mark using a colored pen, which is a part of Figure 2 called a triangle? Give your reason in detail!<br><br>Figure 2. Triangle | Triangle Problem |
| 3. Given any three line segments in Figure 3. Can these line segments form a triangle and explain why!<br><br>Figure 3. Three line Segments | Line Segments Problem on Triangle |