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Semiotic analyses of actions on digital and analogue material when sorting data in primary school

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Abstract

This paper focuses on the actions of learners on digital and analogue materials while dealing with a statistical problem. To investigate the learners' actions, a semiotic perspective of mathematical learning according to C. S. Peirce is used, since in this perspective learning mathematics is described as visible activities on diagrams. Through a qualitative semiotic analysis of the actions of two third-graders working with given data, the statistical diagram interpretations of the learners can be reconstructed. A comparison of the reconstructed diagram interpretations reveals whether different movements lead to similar diagram interpretations. In addition, it is of interest whether the diagram interpretations are the same when acting on digital and analogue diagrams because the same mathematical relationships have to be observed. Through this comparison, conclusions can be drawn about the similarities and differences in working with digital and analogue materials and how these materials may be used profitably in statistical learning.

Keywords: diagrammatic reasoning, mathematical actions, primary school children, semiotic perspective on learning mathematics, statistical learning

INTRODUCTION

In recent years, the practice and research in mathematics education have shown that the use of digital material can play an important role in helping learners "represent, identify, and explore behaviors of diverse mathematical relationships" (Moreno-Armella & Sriraman, 2010, p. 221). Therefore, much research focuses on the digital itself. However, there is little research available on how such a process of exploring mathematical relationships with digital technologies differs from the processes with paper and pencil. Moreno-Armella and Sriraman (2010) provide an indication that a tool, whether analogue or digital, can have an impact on human action, while the mathematical ideas and the mathematical processes can be totally different.

Tools, such as a compass, change the mathematical action of drawing a circle; the tool establishes the mathematical relationship between the center and this relationship does not have to be considered in the actions when drawing a circle. Drawing a circle without a compass means that during the action one has to make sure that every point has the same distance to the center. With a tool, the relationships can be observed during and have to be interpreted after the action, whereas, without a tool, the relationships have to be recognized before to be able to perform the action. Thus, a tool, whether digital or analogue, can change the way one acts with the signs and the relationships between the signs.

The aim of this paper is to explain, by comparing actions on digital and analogue material, how learners' mathematical actions are influenced by digital and analogue materials, whether the digital or analogue material functions as tools and whether a possible change in actions has an impact on the learners' mathematical interpretations. This is achieved by using a Peircean (1931-1935) semiotic perspective on mathematical learning, which describes mathematics learning as a visible activity with diagrams. In this line, Dörfler (2006) describes learning mathematics as

> "something like a reflected handicraft of working productively with diagrams. This underlines the materiality of mathematics and mathematical activities versus its purported abstractness […]." (p. 105)

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Contribution to the literature

- In the paper, the actions of primary school children on digital and analogue materials are analyzed to reconstruct their mathematical interpretations. The compared analyses provide information about whether comparable digital and analogue learning situations lead to different interpretations.
- The analysis, which is specifically adapted to the semiotic perspective according to Peirce, enables a focus on mathematical activities on materials rather than the pure evaluation of didactically prepared materials.
- The results show how the materials function as tools that can influence the learners' actions and interpretations, and refer to how digital and analogue material can be used profitably in primary mathematics lessons.

A diagram, according to Peirce can be understood as "a representation of relations that is constructed by means of 'system of representation.' Such a system is defined by a set of rules, conversations, and a certain ontology" (Hoffmann, 2010, p. 42). A 'system of representation' can be a specific order of material, something written down on paper or related gestures. In this paper, adopting this semiotic view, the actions on digital and analogue material arrangements are analyzed to reconstruct which rules and relations the learners can recognize and establish in the 'system of representation'. These reconstructed mathematical diagram interpretations of the learners are compared to describe possible differences or similarities between them.

With this goal in mind, firstly, the theoretical background describing the semiotic perspective according to Peirce on the learning of mathematics is presented. Secondly, important terms required for teaching statistics in primary school are briefly highlighted. Thirdly, a link is made between the theory presented and the aims of this paper which are used to formulate the research questions. This is then followed by the description of the data collection and the presentation of the semiotic qualitative analysis. Subsequently, the diagram interpretations of two third-graders (9 year-olds) are reconstructed and compared. At the end of the paper, the results are summarized and discussed with regard to the research questions.

THEORETICAL FRAMEWORK

The theoretical framework clarifies what signs, their usage, and the term of diagrammatic reasoning mean and, thus, elaborates a semiotic definition of actions. It also relates to the mathematical content of the empirical example by focusing on statistical learning.

The Semiotic Perspective on Learning Mathematics

In the semiotic sense, with a mathematical activity, we perform "means of visible signs, and by interpreting and transforming signs we develop mathematical knowledge" (Hoffmann, 2006, p. 279). Thus, working with visible signs, which includes the interpretation and transformation of signs, is the core of doing mathematics. By using signs, we have the means to think about mathematical relations and thereby develop new signs. Peirce (CP 2.228), describes a sign as something that stands for someone in a certain sense. A sign is directed to someone and evokes in that person an *interpretant*, for example, an equivalent or perhaps more developed sign (CP 2.228). This "interpretant is determined by the concepts, theories, habits, and skills of the observer […]" (Schreiber, 2013, p. 57). Therefore, the recognition of relations between signs, or the perception of a sign as a sign, depends on the previous mathematical experiences of the learners (Billion, 2021a). In other words, "[P]attern recognition depends partly on the already memorized patterns" (Dörfler, 2006, p. 108).

Peirce distinguishes three types of signs: icons, indices and symbols. An icon is characterized by its similarity to its object and has the function of representing relationships (Bakker & Hoffmann, 2005). However, icons are not similar themselves; the impression of similarity is because one can do possible activities with the icons (Kadunz, 2006). An index draws the sign reader's attention to something, while the meaning of a symbol is determined by its usage or a rule (Bakker & Hoffmann, 2005). The described types of signs can be placed in relationship to each other and, in this way, signs that are more complex can be generated; diagrams are described as such, being complex signs which "are primarily of an iconic character but contain indexical and symbolic elements as well" (Dörfler, 2006). A definition that focuses less on the three types of signs states that signs which are in a contradiction-free context and have certain rules for usage are considered diagrams in the Peircean sense (Dörfler, 2016). Based on these definitions, diagrams are not only geometric figures but can also be equations, algebraic theorems or a sentence in spoken language (Dörfler, 2016).

Diagrams, themselves, do not have a fixed meaning or sense; the (mathematical) meaning unfolds through the activities with diagrams which include the learners' interpretations (Dörfler, 2006). In this sense, mathematical reasoning is diagrammatic and "all reasoning depends directly or indirectly upon diagrams" (Peirce, NEM IV, p. 314):

> "By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms" (Peirce, NEM IV, p. 47-48).

Mathematical rules and relations determine how to construct, read and experiment with a diagram (Hofmann, 2010). Depending on the activities, according to the different possible rules, the signs "might give rise to essentially different diagrams" (Dörfler, 2016, p. 25). For instance, Dörfler (2016) draws a comparison to an everyday example in which the same cards of a playing card deck perform different roles in different card games. Thus, the use of diagrams, which means activities according to certain rules and relations between the signs, is of great importance for learning mathematics.

In the empirical example of this paper, the learners have to construct a statistical display to answer given questions. In relation to the diagrammatic reasoning mentioned above, the construction of a diagram could be the creation of a plot with a sorting according to the values of a characteristic on the data card. Generating such a diagram must be motivated by the need to represent relations that are significant to the answer the given question. To explore the constructed diagram, actions defined by the relationships between the signs can be performed, for example, modifying the scale of the plot. Changing the scale of the plot is bound to certain relationships:

> "a dot has to be put above its value on the x-axis and this remains true if the scale is being changed." (Bakker & Hoffmann, 2005, p. 341).

When transforming the diagram, other relationships come to the fore and it becomes possible to make more detailed assumptions about the distribution. Diagrammatic reasoning also includes the observation of the results of exploring and reflecting on them. It is primarily through this reflection that new mathematical relationships can be recognized and expressed in general terms.

The Understanding of Actions in a Semiotic View

According to the semiotic theory, actions on diagrams are of major importance in learning mathematics (e.g., Dörfler, 2006; Peirce, NEM IV). These actions allow the learners to construct and explore diagrams and, once performed, the actions bring to light other mathematical relations that enable (mathematical) knowledge processes. The difficulty of performing the appropriate actions on a diagram depends on recognizing and observing the relationships that the diagram represents. Learning mathematics can be described as a mutual process in which learners notice already known relations and observe them in their actions, and can recognize further relations by observing the results of these actions. In this way, the more adept the learners are in using a diagram, the more mathematical relations they know and the better they can decide which actions are possible and which are not. This suggests that actions on the diagram are determined by the learner's interpretation, which, in turn, depends on their mathematical experiences and interaction with other learners (Dörfler, 2006). Huth (2022) shows that gestures, like actions, can also indicate possible transformations on diagrams and even the gestures themselves can have a diagrammatic character.

In terms of the types of signs described by Peirce, the focus is especially on indices as an index can be regarded as a reference to action,

> "An index represents an object by virtue of its connection with it. It makes no difference whether the connection is natural, or artificial, or merely mental" (Peirce, MS [R] 142).

For indices, the connection to an object is of great importance as it refers to the object through this connection. Concerning the actions on a material, the sign that arises from such actions can be regarded as an index of these actions. In the process of the action, the sign tells the actor whether he or she is right or wrong, thus, the actor can adjust their actions accordingly (Kadunz, 2016). Through this reciprocal process, the actor can notice relationships during the action, which, in turn, are expressed through the action. Tools can shorten the actions and, hence, the relationships may not be obvious in the actions. Kadunz (2016) argues that when a digital tool is used there can be a complete separation of the action and the relation. In the statistical example considered in this paper, during the action of assigning a characteristic carrier to a value on the scale, the resulting sign (arrangement of material) reports back to the actor whether the assignment was correct. In the action, the relationship between the characteristic carrier and its matching value is expressed by the positioning of this characteristic carrier in the plot.

By using a digital tool such as TinkerPlots (Konold & Miller, 2011), a click on the *separate* button assigns all characteristic carriers to the appropriate values on the scale. With the action of clicking, the relationships represented by the diagram are no longer observed. TinkerPlots also permits a drag-movement over the plot to assign a characteristic carrier to a value on the scale. By using this action, some relationships are established, such as the relationship between the characteristic and the axis on which the scale is plotted. However, the relationship between the characteristic carrier and its matching value is not expressed in the action.

Statistical Learning in Primary School

To live as an independent citizen in a data-driven community "today's students need to learn to work and think with data and chance *from an early age*" (Ben-Zvi, 2018, p. vii, emphasis in the original). It is primarily important that learners develop an understanding of data rather than teaching learners different skills independently of each other (Ben-Zvi & Garfield, 2004). To learn a comprehensive understanding of data, the terms of statistical literacy, statistical reasoning, and statistical thinking are often used in research (e.g., Ben-Zvi & Garfield, 2004; Frischemeier, 2020). Ben-Zvi and Garfield (2004) distinguished these three terms as follows:

> "**Statistical literacy** includes basic and important skills that may be used in understanding statistical information or research results. These skills include being able to organize data, construct and display tables, and work with different representations of data. […] **Statistical reasoning** may be defined as the way people reason with statistical ideas and make sense of statistical information. This involves making interpretations based on sets of data, representations of data, or statistical summaries of data. […] **Statistical thinking** involves an understanding of why and how statistical investigations are conducted […and] statistical thinkers are able to critique and evaluate results of a problem solved or a statistical study." (p. 7, emphasis in the original)

Sriraman and Chernoff (2020) concede that the described perspective on statistical and probabilistic learning, in distinguishing the three terms above, has a psychological origin and not an epistemological one. Subsequent to this, Kollosche (2021) draws attention to the following:

> "Thus, the questions how probabilistic and statistical reasoning relates to other forms of reasoning, how it justifies assertions, and how it contributes to our understanding of the world are not yet part of the academic reflections of the field" (p. 482).

Due to this need for research, in this paper an attempt is made to establish a link between diagrammatic reasoning and statistical learning. In addition, many researchers see modelling as a way to work with complex data (e.g., English, 2018; Gravemeijer, 2002; Wild & Pfannkuch, 1999). Modelling includes statistical processes such as

> "[...] posing, and refining questions; collecting and organizing data [...] and drawing conclusions and informal inferences from models generated [...]" (English, 2018, pp. 296-297).

Doerr and English (2003) define models as

> "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (p. 112).

In a similar way, Hestenes (2013) defines a model as

> "a representation of structure in a given system. A system is a set of related objects, which may be real or imaginary, physical or mental, simple or composite. The structure of a system is a set of relations among its objects" (p. 17).

These definitions of a model are similar to the above definitions of a diagram in the Peircean sense, hence, this supports the idea of this paper in attempting to establish a link between statistical learning and diagrammatic reasoning.

Following diagrammatic reasoning (Peirce, NEM IV) described above, it can be assumed in this paper that statistical literacy is important for constructing a diagram, such as a plot. It is necessary to understand the statistical information to organize the data and construct a diagram. To experiment with or investigate a statistical diagram, such as a plot, learners must identify and interpret relations. In addition, different diagrams, such as the data card and the plot, can be combined to render further relations between values visible. For this, it is important that the learners can make sense of statistical information, thus, statistical reasoning is in the foreground when transforming diagrams. Observing the transformations of the diagrams, reflecting on the results and expressing these results in general terms can be described by statistical thinking. Statistical literacy, statistical reasoning and statistical thinking can be found beyond the proposed classification in all steps of diagrammatic reasoning. Attempting to establish a link between diagrammatic reasoning and statistical learning it can be assumed in this paper that diagrammatic reasoning contains the central terms of statistical learning. To continue in the semiotic perspective, this paper talks about diagrammatic reasoning because it encompasses all the important aspects of working with data.

In the example analyzed in this paper, the focus of the learners' task is on answering given questions using the statistical diagrams they have created with different materials. Questioning is an important aspect of understanding data (Friel et al., 2001). However, not every question is the same as another, so

> "[s]hallow questions address the content and interpretation of explicit material, whereas deep questions involve inference, application,

synthesis, and evaluation [...]" (Graesser et al., 1996, p. 23).

Friel et al. (2001) distinguish between three question levels: the elementary level focuses on extracting data from a plot (read the data), the intermediate level characterizes finding a relationship in the data (read between the data) and the advanced level requires analyzing the relationships implicit in the plot (read beyond the data). In the statistical learning situation considered in this paper, learners are asked questions that can be categorized into the first two levels. For example, the question '*How many children have named purple and how many children have named orange as their favorite color?*' posed to the third-graders can be classified as reading the data. Another question, the third-graders were asked to answer in another statistical learning situation is '*Do more boys or more girls attend the class?*' focusing on reading between the data.

Research Focus

In the following, implications for the research interest of this paper are described from the theoretical explanations and the research questions are formulated.

Relationship between the theory & the aim of the paper

The semiotic perspective on mathematical learning emphasizes the rule-governed (mathematical) activities with and on diagrams, which can be seen as the core of doing mathematics. In order to act with a diagram, learners need to consider the relationships between signs. This means that the perception of relationships influences the learners' actions on the diagram. For this paper, it is important that, by analyzing the learners' actions, it is possible to reconstruct the interpretation of the relations that the diagram shows. In this manner, different actions can be investigated to show whether different movements of the learners (e.g. on digital or analogue diagrams) arise from different diagram interpretations. Moreover, following the semiotic perspective, it becomes clear that the activities with and on the diagrams contain gestures that need to be analyzed to achieve an approximately complete description of the diagram interpretations.

Dörfler (2006) states that diagrams have an iconic character and that they are primarily characterized by representing relationships. Due to this main characteristic of diagrams, relationships between the signs and the resulting rules for using the signs are of greater importance than the appearance of the signs (Dörfler, 2015). Unless the appearance of the signs changes the relational structure of the diagram, the appearance is subordinate to the structure (Shapiro, 1997). If diagrams are constructed with the same relations differing only in the materiality of the signs, then the learners are likely to focus on the same relations rather than the materiality of the sign. When acting on digital and analogue diagrams, whose signs are connected by the same relations, these relations must be recognized and observed regardless of whether the signs are represented digitally or analogously. Concerning this paper, it is probable that the same diagram interpretations can be reconstructed when analyzing the learners' actions on the statistical digital and analogue diagrams.

Following Kadunz (2016) for the investigation in this paper, the assumption can be made that the digital tool leads to a shortening and separation of the actions and mathematical relationships. This means that the learners do not have to consider all the relationships of the diagram in their actions when manipulating the diagram. The tool automatically establishes the relationships that the learners do not have to do. In this case, the reconstruction of the diagram interpretation based on the actions alone may be impossible. Learners may re-establish the relationships established by the tool through subsequent actions, gestures or spoken language, which then provide information about the interpretation of the diagram.

In this paper, it is of major interest to determine whether the different actions lead to the reconstruction of different diagram interpretations. Another interest of the paper is whether the diagram interpretations are the same when acting on digital and analogue diagrams because the same relationships have to be noticed in the actions, or whether the digital tool shortens the actions and, thus, the learners do not have to notice all the relationships in their actions; in this case, the learners' reconstructed diagram interpretations would differ.

Research questions

With regard to the considerations of which implications arise from the theory for the research interest, research questions can be formulated as follows:

1. Which mathematical diagram interpretations can be reconstructed based on the actions on the statistical diagram implemented once with the digital and once with the analogue material?
2. Which possible differences exist between the reconstructed diagram interpretations, as it can be assumed that the actions are shortened with the digital tool?

METHOD AND DESIGN OF THE STUDY

In this section, first, the aims of the *MatheMat* study are described, followed by a detailed description of how the data collected in the study were analyzed.

Methods of Data Generation–The *MatheMat* Study

In the study *MatheMat–Mathematical learning with materials*, actions on the different materials (analogue or digital) form the center of interest. The goal of the study is to reconstruct diagram interpretations by analyzing the actions of the learners to draw conclusions about whether the different material has an influence on the learners' diagram interpretations. To achieve these goals, geometric and statistical learning situations are designed that deal with the same mathematical relationships; these are realized once with analogue and once with digital material. The learners work in pairs on one geometric and one statistical learning situation. When distributing tasks, care was taken to ensure that each pair worked once on a digital and once on an analogue learning situation. A total of 32 third- and fourth-graders (9-11 year-olds) from two German primary schools work on the learning situations for about 45 minutes. The pairs' work on the statistical and geometric problems is videotaped with two cameras; one focuses on the actions on the material, while the other camera records the overall situation. Comparable video passages in which the learners act on the material are transcribed for the analysis. In this way, the interpretations of the learners, reconstructed from the visible actions and gestures, can be compared in relation to the different materials used.

Preparation of Data

For the reconstruction of the diagram interpretations, all the learners' actions, gestures and spoken language are transcribed in detail. When working with analogue material, the separation of gestures and actions can be made through the definitions formulated by other researchers (e.g., Harrison, 2018; Kendon, 1984). However, when working with digital material, gestures and actions become more blurred, thus, the definitions of the actions and gestures can only help to distinguish between these with difficulty. Therefore, a definition of the actions on the digital material is provided in this paper. For the transcription of the actions on the digital material, the touch gestures on the tablet and the resulting manipulations in the program are interpreted together as actions. For the descriptions of the movements over the screen and the manipulations in the program, the touch gesture reference guide (Villamor et al., 2010) is adapted for the particular statistical learning situation. **Figure 1** describes the important movements for the analysis of the actions on the digital material.

Methods of Data Analysis

To reconstruct the learners' mathematical diagram interpretations, a semiotic specification (Billion, 2021b; Billion & Vogel, 2021) of Vogel's (2017) adaptation of Mayring's (2014) context analysis is made.

The context analysis according to Mayring (2014) aims to explain the meaning of a statement by adding further text passages from the data. Vogel's (2017; Vogel & Huth, 2020) adaptation for application in mathematics education focuses on the reconstruction of mathematical concepts by contrasting mathematical concepts and the learners' multimodal expressed individual concepts. For this purpose, Vogel (2017) uses the theoretical background of conceptual change (e.g., Carey, 1988).

For the semiotic specification of the qualitative analysis, the conceptual change theory is replaced by the Peircean theory of signs. As mentioned above, each sign evokes an interpretant in a person, and this interpretant depends on the knowledge, habits and experiences of the sign reader. The interpretant

> "can be a reaction to a sign or the effect in acting, feeling, and thinking […]" (Bakker & Hoffmann, 2005, p. 336).

The learners' actions and gestures working with the various material can be seen as the interpretant of them reading the signs. In the analysis, the learner's interpretant is contrasted with an interpretant based on current research that is close to the 'final logical interpretant'. Following Peirce, the 'final logical interpretant' can be defined as how "it comes out ideally 'in the long run' of scientific communication" (Bakker &

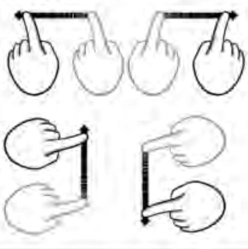
Movement on the screen	Description of movement	Manipulation in TinkerPlots
"Tap" 	"Briefly touch surface with fingertip" (Villamor et al., 2010)	Select value
"Drag" 	"Move fingertip over surface without losing contact" (Villamor et al., 2010)	Sorting data

Figure 1. Movements on the screen and triggered manipulations in TinkerPlots[1]

[1] Petra Tanoupoulo made the illustrations of the movements on the screen

Hoffmann, 2005, p. 336). Since this 'final logical interpretant' is something ideal that cannot be formulated, an interpretant is formulated based on research that should come close to the 'final logical interpretant'. The description of this interpretant includes rule-governed actions that can be performed on the signs based on the mathematical relations that the diagram shows. Firstly, the mathematical relations between the signs are described and then rule-governed actions are derived from them. Here, the focus is on the relations that are important with regard to the task to be solved. Other relations may also be recognized, but these are neglected. By the contrast of the interpretants, the learner's diagram interpretation can be formulated. In addition, by including increasingly more transcript and video passages, the learner's diagram interpretations can be made visible in the ongoing sign process.

The steps of analysis are structured in such a way that in explication 1, a very small transcript passage of a learner's action is focused on and contrasted with the research-based interpretant. In the course of analysis, the scope of the transcript or video passages considered is expanded. In explication 2, more of the same or similar passages are included in the analysis to reconstruct the learner's diagram interpretation. Explication 3 focuses on further video passages of the learning situation; these are similar to the transcript passages already found, and are, again, contrasted with the research-based interpretant. The passages that are important for analysis are those which are the same or similar to the first passage with which the analysis began. These same or similar passages can be found in the processing of the task both before and after the first transcript passage.

EMPIRICAL EXAMPLE

This section describes the statistical learning situation, outlines the diagram interpretations of Walerius and Matteo and compares their interpretations.

The Learning Situation

This paper focuses on a statistical learning situation in which German third-graders (9 year-olds) are asked to represent the values of one characteristic from different data cards in a plot to answer given questions. The focused question to answer is: *How many children have named purple and how many children have named orange as their favorite color?* This question can be classified as reading the data because the learners have to extract information from the data displayed in the plot. For the processing, the learners have data cards at their disposal. The data cards contain four nominally and ordinal scaled characteristics (gender, favorite color, grade in German and in mathematics) and the corresponding values of 14 children. In the *MatheMat* study, four third-graders use TinkerPlots to create a univariate plot with

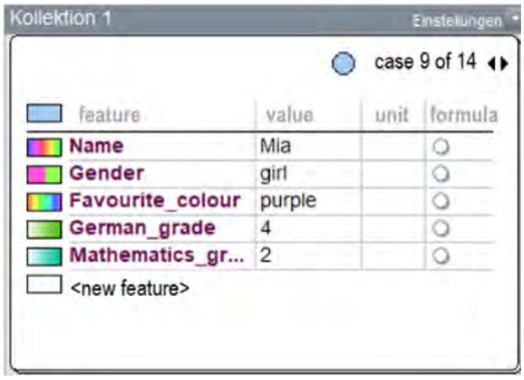


Figure 2. Data cards in TinkerPlots

these data cards, while another four third-graders use sticky notes and wooden cubes labelled with names.

To answer the question, the considered learner Walerius and his partner use TinkerPlots; this is a software toolkit for visualizing and simulating data (Konold & Miller, 2011). The software allows values of 14 children to be entered into data cards (see **Figure 2**). The data of the 14 children had already been entered into TinkerPlots at the beginning of the learning situation, allowing the learners to see available data on the data cards. The accompanying person explains to Walerius and his partner how to look at the data cards of the children entered into TinkerPlots.

In addition, a plot is opened in which the 14 children can be seen as dots (see **Figure 3**). At the beginning of the situation, all the dots are colored blue (see **Figure 3a**). By tapping on a characteristic listed on the data card, TinkerPlots colors the dots in the plot (see **Figure 3b**), thus the dots adopt the respective value of the clicked characteristic. However, the colors of the dots do not match the value of the characteristic *favorite color*. TinkerPlots automatically assigns different colors for nominally scaled data, therefore, unfortunately, it is not possible to set the colors to match the values of the characteristic *favorite color*. In the learning situation, the accompanying person discusses this with the learners. However, if one performs a drag-movement, starting from a dot, vertically or horizontally across the plot, TinkerPlots provides sorting of the dots according to the selected characteristic. TinkerPlots determines a scaling and allocates each dot to this scaling and, by performing another drag-movement in the same way, it provides a finer scaling. In relation to the question posed for the learners to answer, it can be seen that two children chose orange, and three children chose purple as their favorite color (see **Figure 3c**).

In the semiotic sense, the learners can interpret the data cards and the plot, each as a complex sign or a diagram. The plot itself is an icon because it represents relations between the values, the dots are indices to the 14 children whose values are considered and the signs on the scales can be seen as symbols or indices for the measurement or survey that was made. Similar to the

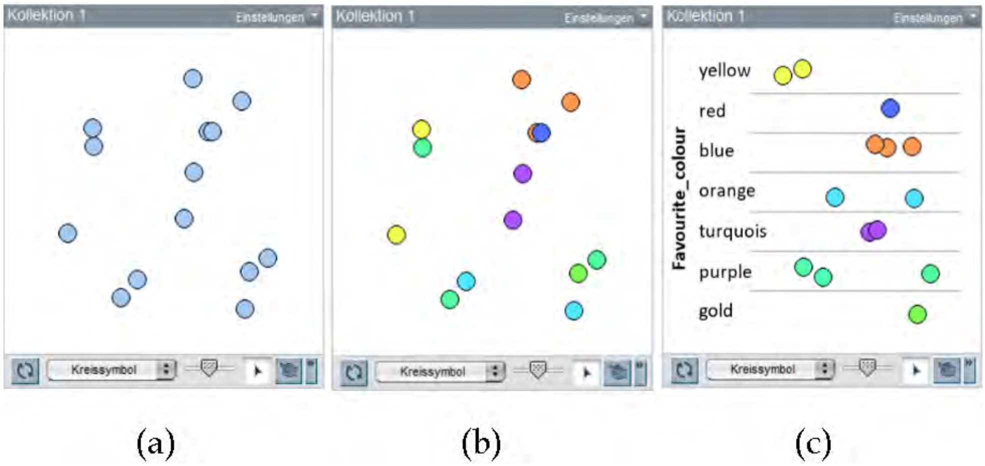


Figure 3. Data plots (a-c) in TinkerPlots

plot, all three types of signs according to Peirce can be found on the data card.

The other learner considered in this paper is Matteo; he and his partner work with analogue material. The learners receive 14 analogue data cards which they can order or sort independently according to certain relationships. Following Harradine and Konold (2006), the learners can sort the data cards flexibly by creating a plot from them. Such an idea gave rise to TinkerPlots, in which

> "[...] the construction of statistical displays is realized via the data card-operations stack, separate, order [...]" (Frischemeier, 2020, p. 42).

However, the semiotic view at TinkerPlots reveals that the diagram *plot* does not emerge from the data cards; it already exists. The learners who are working with the analogue data cards act exclusively on one diagram (data cards), establish relationships between the parts of that diagram and manipulate them, while the learners working with TinkerPlots manipulate two diagrams and have to recognize previously established relationships between the two diagrams (data cards and plot, see **Figure 2** and **Figure 3**).

To ensure that the learners who are working with the analogue material also manipulate two diagrams in a semiotic sense, they are provided with wooden cubes and blank sticky notes to accompany the analogue data cards (see **Figure 4**). Each wooden cube is marked with the name of a child whose values are noted on the data cards. As with TinkerPlots, relationships between the data cards and the wooden cubes are already predefined, making it possible to create a separate diagram (plot) with the wooden cubes and sticky notes. By labelling the sticky notes with the values of one characteristic and sticking them next to each other, the learners can determine a scale to which they can allocate the wooden cubes. In addition, by matching the cubes to the appropriate data cards, the learners can recognize the relationship between the cube and the data card. Subsequently, the learners can transfer a value from the data card to the cube and translate it into a position above the scale.

In **Figure 4**, above Matteo's hand, one can see sticky notes stuck next to each other as a scale. The wooden cubes above the sticky notes have been positioned according to their values on the data cards and the values on the sticky notes. This material arrangement can be interpreted as a plot. As in TinkerPlots, this plot helps the learner to see how many children have indicated orange or purple as their favorite color.

The analogue material has been chosen in such a way that the same relationships between the parts of the material are defined as in TinkerPlots. Even if different materials are used, the relations represented in the two digital and analogue diagrams (data cards and plots) are the same. To enable the reconstruction of the same mathematical diagram interpretations, the mathematical relationships observed when experimenting with the two digital or analogue diagrams must be the same. Nevertheless, when manipulating the diagrams, the assumption that the digital material can shorten the actions and separate them from the relationships must be considered.

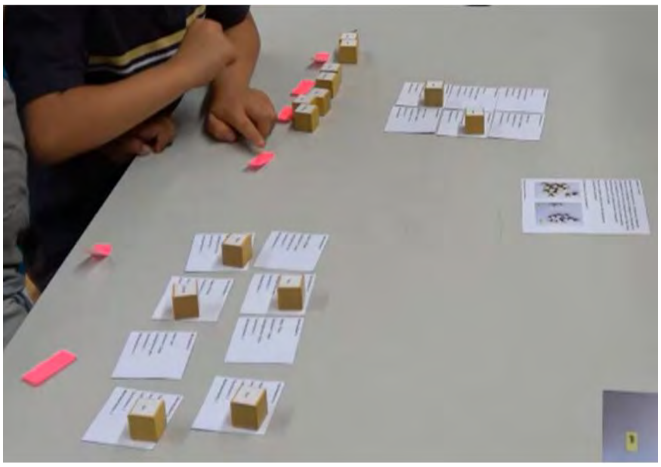


Figure 4. Analogue data cards, cubes marked with names and sticky notes

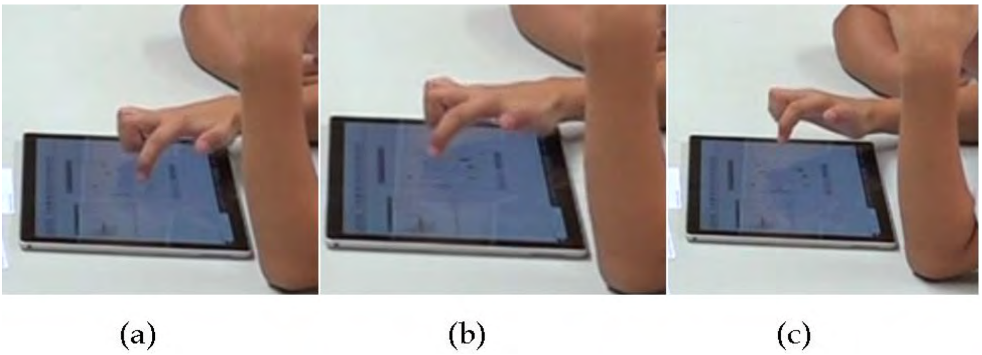


Figure 5. Drag-movement from the light blue dot upwards across the plot (a-c)

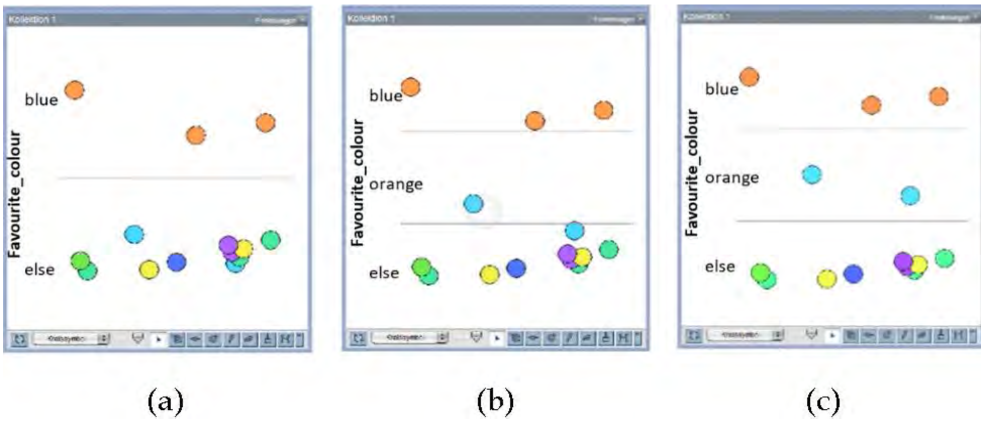


Figure 6. Manipulations in TinkerPlots initiated though the drag-movement (a-c)

Reconstruction of the Interpretation of Diagrams by Working with TinkerPlots

Walerius and his partner have not worked with TinkerPlots previously. They receive an introduction to the functions of the software relevant to the task from the accompanying person. The learners have also not previously sorted data with analogue material in their mathematics lessons.

For the analysis, mainly the actions, such as selecting a characteristic on the data card or making a drag-movement over the plot, but also the gestures and phonetic utterances made by Walerius are included. However, due to space restrictions here, the contrasts made in the steps of analysis between the learner's interpretant and the research-based interpretant are presented in summary. The analysis begins with the following transcribed video sequence:

Walerius:

1. *Makes a drag-movement with the right index finger, starting at a light blue dot and moving upwards* (see **Figure 5** and **Figure 6a**).

2. *The light blue dot moves upwards in the plot* (see **Figure 6b**).

3. *TinkerPlots separates the children in the plot who have indicated orange as their favorite color from the other children in the plot* (see **Figure 6c**).

4. Two\

5. *The plot now shows the children who indicated orange as their favorite color separately from the children who indicated another color* (see **Figure 6c**).

Explication 1: To sort the dots in the plot according to a characteristic, relationships between the signs have to be recognized. The research-based interpretant describes these relationships and the resulting actions on the material to contrast them with Walerius's actions. In this contrast, the diagram interpretations of Walerius can be formulated. The summarized research-based interpretant focuses on four main relationships and their resulting actions:

1. *Relationship between the data cards and the dots in the plot*: To establish this relationship, the learners have to make a tap-movement on a characteristic on the data card; consequently, TinkerPlots colors the dots according to the values of the characteristic. The relationship between a value and a dot is only partially expressed in the coloring of the dots. TinkerPlots does not color the dots according to the values for the characteristic *favorite color* but assigns the colors independently of the values.
2. *Relationship between the characteristic and the axes in the plot*: To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. Depending on the direction of

the drag-movement (horizontal or vertical), TinkerPlots plots the scaling on the x- or y-axis.

3. *Relationship between the values on one axis*: Since *favorite color* is a categorical characteristic, the individual values of the characteristic do not have to be placed in any particular order. Nevertheless, the distances between the individual values on the scale should be equal to enable a better interpretation of the plot. To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. TinkerPlots takes an equal distribution of the values on one axis.
4. *Relationship between the values on the scale and the positioning of the dot in the plot according to their values*: To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. TinkerPlots positions the dots in the plot according to the values on the scale.

Contrasting the research-based interpretant with Walerius's actions, his diagram interpretation can be summarized as follows: due to Walerius's drag-movement upwards across the screen, Walerius most likely establishes a relationship between the y-axis and the characteristic *favorite color*. Based on this action, TinkerPlots automatically scales the y-axis and positions the dots according to this scale. Thus, no relationship between the individual values of the scale or the positioning of the dots are observed in the actions. Accordingly, it cannot be reconstructed from Walerius's action whether he can interpret the relationship between the individual values of the scale, which TinkerPlots provided. Furthermore, it is unclear whether he can interpret the arrangement of the dots in the plot according to the scale given by TinkerPlots. Exclusively from the subsequent spoken language "two\", it can be reconstructed that he can interpret the positioning of the dots in the plot. He probably recognizes that two children have indicated *orange* as their *favorite color*. Following this assumption, Walerius can see that the two blue dots are children who have indicated orange as their favorite color. He recognizes the relationship between the scale and the dots and is not influenced by the color of the dots. Walerius has probably previously separated the orange dots from the others because he has assumed that the scale corresponds to the color.

Explication 2: In explication 2, Walerius's same and similar actions in the transcript are once again contrasted with the research-based interpretant. Walerius's diagram interpretations formulated in explication 1 can also be reconstructed in explication 2. In addition, by Walerius's tap-movement on the characteristic from the data card he likely recognizes a relationship between the diagram *data card* and the diagram *plot*. Since TinkerPlots automatically performs a translation of the values into coloring the dots, the relationship between a value on the data card and a dot in the plot cannot be explicitly reconstructed from Walerius's actions. Again, the relationships in the actions do not become fully clear because TinkerPlots, as a tool, shortens the action process. Since TinkerPlots does not adopt the favorite colors that are on the data cards when coloring the dots, the interpretation of the plot becomes more difficult for Walerius. Explication 1 shows that Walerius is unable to rely on the colors, but has to consider the relationship between the position of the dots and the scale to extract how many children have indicated *orange* as their *favorite color*.

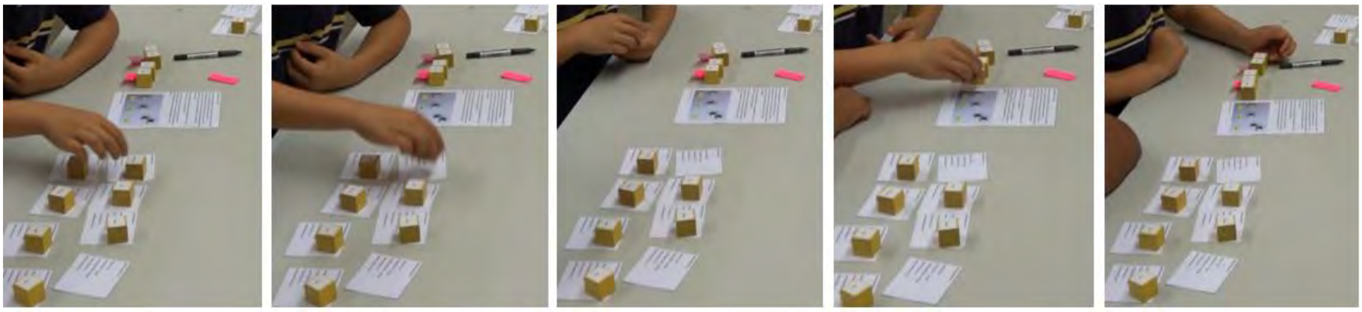
Explication 3: In explication 3, the same and similar actions of Walerius are identified in the recorded processing and contrasted with the research-based interpretant. In summary, Walerius's reconstructed diagram interpretation from explications 1 and 2 can be confirmed by certain passages. It is also possible to find actions where he assigns the values of a characteristic with his directions of movement to both axes. TinkerPlots sorts a dot on one axis according to the value *otherwise* and on the other according to a value of the characteristic *favorite color*. Walerius probably does not realize that the values of a characteristic can only be plotted on one axis and that the sorting made by TinkerPlots is not successful. The relationship between the characteristic and an axis does not become clear via his actions.

Summary of explications: In all steps of the analysis, TinkerPlots, which can be understood as a tool, shortens the actions. Not all of the relationships for creating the diagram *plot* have to be observed in separate actions and, thus, it is impossible to reconstruct from the one drag- or tap-movement whether or not Walerius recognizes all the relationships. Therefore, it remains open whether Walerius can already fully interpret the diagram *plot* during his actions on it. Only by looking at his subsequent phonetic utterances does it becomes clear that Walerius has succeeded in analyzing the diagram *plot*.

Reconstruction of the Interpretation of Diagrams by Working with Analogue Material

Matteo and his partner have not sorted data with digital or analogue material before this learning situation. Like Walerius, Matteo and his partner also receive a short introduction to the subject matter by the accompanying person.

For the second analysis, the focus is on Matteo's actions while working with the analogue material. As in the reconstruction of Walerius's diagram interpretation, Matteo's gestures and spoken language are included in the analysis.



(a) (b) (c) (d) (e)

Figure 7. Positioning of the cube labelled Wilhelm from the data card in the plot (a-e)

Matteo:

1. *Moves his right hand to the cube, which is marked with the name 'Wilhelm', and is lying on the corresponding data card* (see **Figure 7a**).

2. *Grabs the cube marked 'Wilhelm'* (see **Figure 7b**).

3. **purple**\

4. *Leads his right hand with the cube towards his left hand.*

5. *Grabs the cube marked 'Wilhelm' also with his left hand* (see **Figure 7c**).

6. *Moves the cube marked 'Wilhelm' with his right hand to the tabletop* (see **Figure 7d**).

7. *Places the cube marked 'Wilhelm' above the sticky note labelled 'purple' on the tabletop* (see **Figure 7d** and **Figure 7e**).

Explication 1: To sort cubes according to a certain characteristic in the plot, different relationships between the signs have to be considered. As in the work with the digital material, the summarized research-based interpretant focuses on four important relationships and the resulting actions:

1. *Relationship between the data cards and the cubes*: To establish the relationship between the data card and the cubes the learners have to relate the cubes to the corresponding data card. One such way is to place the cubes on the matching data cards.
2. *Relationship between the characteristic and the axes in the plot*: To establish the relationship between the characteristic and the axes, the learners can stick the sticky notes horizontally or vertically next to each other. It is important that the sticky notes be arranged in a line so that the distribution of the wooden cubes can be easily identified.
3. *Relationship between the values on one axis*: As already mentioned, for a categorical characteristic there is no particular order in which the values must be plotted on the scale. However, care should be taken to ensure that the values are equally spaced on the scale and, therefore, the sticky notes must be stuck next to each other at the same distance.
4. *Relationship between the values on the scale and the positioning of the cubes in the plot according to their values*: To establish a relationship between the cubes and the values on the scale, the learners have to recognize the scale as a part of the plot. To position the cube over the matching value, they need to read the value on the data card, transfer this value to the cube and position it over the matching value on the sticky note.

In contrasting the research-based interpretant with Matteo's actions, his diagram interpretation can be summarized as follows: Matteo has probably recognized the relationship between the data card and the cubes since the cube labelled *Wilhelm* is already placed on the corresponding data card before he started his action. With this assignment, it is later possible for Matteo to establish a relationship between the cube and a value on a sticky note. Based on his actions, he establishes a relationship between the data card and the positioning of the cube marked *Wilhelm*. He translates the value *purple* from the data card to the position of the cube above the sticky note labelled *purple*. To be able to place the cube, Matteo has to recognize the scale, which is part of the plot. By analyzing Matteo's actions, he probably recognizes the relationships between the *data card* and the *plot* and between the position of the cube and values on the scale. With regard to the question that Matteo and his partner are required to answer, he selects a suitable cube and makes a phonetic utterance to underline his choice. Due to the emphasis on "**purple**\", he likely deliberately chooses the cube marked *Wilhelm*.

Explication 2: In explication 2, the same and similar actions of Matteo are again contrasted with the research-based interpretant. Matteo's reconstructed diagram interpretation in explication 1 can also be reconstructed by analyzing his transcribed actions in explication 2. Further transcript passages substantiate the assumption that Matteo chooses the cubes based on the values noted on the data card. This can be reconstructed from his pointing gesture to the data card before choosing a cube and can be interpreted as an index in the semiotic sense. Matteo probably only selects cubes that can be assigned to the favorite color *orange* and *purple*. This can be interpreted as an effective selection of cubes with regard to the question to be answered.

Explication 3: By contrasting more of the same and similar video passages with the research-based interpretant, Matteo's reconstructed diagram interpretation from explication 1 and 2 can be confirmed. In the video passages, he probably translates the value of the data card into the position of the cube in the plot and succeeds by comparing the value on the data card and the sticky notes in pairs. At the beginning of the learning situation, Matteo already recognizes the relationship between the cubes and the data cards by assigning the cubes to the data cards. This diagram interpretation can be reconstructed very early in the situation based on Matteo's first actions. Furthermore, during the learning situation, Matteo writes down the values of a characteristic on the sticky notes and arranges them in a horizontal line with the same distance from each other. He can probably recognize the relationship between the characteristic *favorite color* and the x-axis and also the relationship between the values on the x-axis.

Summary of explications: Overall, Matteo can interpret the diagram *data card*, recognize a relationship between the cubes and the data cards and translate the values from the data card into the positioning of the cubes in the plot. Matteo succeeds in this translation by comparing the values on the data cards with the values on the sticky notes in pairs. He is also able to interpret the diagram *plot* and recognize a relationship between a characteristic and the x-axis, and the values among the others plotted on the x-axis. This relationship can be reconstructed based on his phonetic utterances but is also expressed in the actions made previously. Matteo observes all the necessary mathematical relationships in separate actions and, thus, Matteo's diagram interpretations can be reconstructed exclusively from his actions. The analogue material does not shorten Matteo's action as no relationships are established automatically.

Comparison of the Diagram Interpretations

By comparing the reconstructed diagram interpretations of Matteo and Walerius, it can be seen that the digital material leads to a shortening of Walerius's actions. With an action, TinkerPlots automatically observes several relationships. One drag-movement refers TinkerPlots to establish automatically the relationship between the characteristic and an axis, between the individual values on an axis and the dots in the plot and the values on the scale. TinkerPlots acts like a tool and, therefore, shortens the actions and separates them from the mathematical relationships.

Consequently, the reconstruction of Walerius's diagram interpretations through the analysis of the actions on the digital material is not successful in all places. When working with the digital material, only after TinkerPlots has performed the sorting does Walerius need to re-establish the relationship between the position of the dot and the value on the scale to interpret the diagram. In this way, there is a shift in when to interpret the diagram and how to express the relationships between the parts of the diagram. Walerius does not need to interpret the plot while sorting the dots according to a characteristic and, therefore, does not need to establish the relationships between the signs in his actions. After TinkerPlots has made a sorting, Walerius has to interpret the plot and re-establish the relationships between the signs; this is analyzed through his subsequent phonetic utterances and gestures.

In contrast, when working with analogue material, the relationships have to be established during the actions on the diagram because in *one* action, *one* relationship has to be observed. In this way, a reconstruction of Matteo's entire diagram interpretations is realizable through the actions on the analogue material. Matteo is required to form an interpretation during the sorting process, otherwise, he will not be able to do any sorting. In this way, the interpretations become visible in his actions and he observes the relationships between the signs to enable sorting.

However, it also becomes clear that in some places the same diagram interpretations could be reconstructed. Matteo and Walerius both show that they recognize a relationship between one axis and the characteristic *favorite color*: Matteo labels sticky notes with the different values of the characteristic *favorite color* and sticks them next to each other in a line, while Walerius makes a drag-movement from one point upwards. It is, therefore, possible to reconstruct the same diagram interpretations, although the actions on the digital and analogue material differ. This means that the actions can be different although the learners' have to observe the same relationships to act on the digital and analogue diagrams.

MAJOR FINDINGS AND DISCUSSION

This paper aimed to reconstruct mathematical diagram interpretations based on the actions on statistical diagrams that were carried out once with digital and once with analogue materials. For this purpose, the diagram interpretations of Matteo and Walerius were reconstructed with qualitative semiotic analysis. The goal of the comparison of the reconstructed interpretations was to find similarities or differences in the interpretations, as it was assumed that the digital material, as a tool, could shorten the actions.

In this paper, it could be shown that the diagram interpretations can be reconstructed by analyzing the actions of the learners. In contrast to the learners' interpretant expressed by gestures, actions and spoken language, and the research-based interpretant, the learners' diagram interpretations can be described.

In the comparison, the same diagram interpretations could be reconstructed, although the movements of the hands differed. It could be shown that different movements on analogue or digital material in which the same relationships were observed led to the reconstruction of the same diagram interpretation. In line with Dörfler (2015) and Shapiro (1997), analyses show that the different material has no influence on the diagram interpretations if the relationships between the signs are the same. Even if the same relationships must be recognized and observed when acting on the diagrams, the reconstructed diagram interpretations of Walerius and Matteo were not equal in all places.

The comparison shows that when working with analogue material, the necessary mathematical relationships in the actions become clear and the continuous reconstruction of the diagram interpretation is possible. This can be attributed to the fact that with the analogue material, one must already express the interpretation of the relations in the actions during the sorting. Due to the abbreviation of the actions by TinkerPlots, some diagram interpretations cannot be reconstructed by analyzing the actions on the digital material. As already mentioned, according to Kadunz (2016), it could be shown that the actions on the digital material were shortened because TinkerPlots functions as a tool. In this way, there is a shift between the expressions of the relationships interpreted by the learners. Working with the digital material, the relations have to be interpreted after TinkerPlots has made manipulations on the diagram; these results are comparable to the analogy of the compass at the beginning of the paper.

However, by shortening the actions, the digital material opens up the possibility of investigating more complex questions or large amounts of data. Consequently, in practice, the respective learning goal is significant for the choice of material type. If a basic understanding of sorting, as statistical literacy or statistical reasoning, is to be developed, this can be done through the actions on analogue material, as learners need to consider all the relationships between the signs in their actions. In the semiotic sense, the focus is in constructing and manipulating a diagram. For investigating large amounts of data or complex issues, requiring statistical reasoning or statistical thinking, it makes more sense to use digital material; TinkerPlots can help to sort through many data and, thus, learners can focus more effectively on interpreting the results of the manipulation of the diagram performed by TinkerPlots.

In this way, Biehler et al. (2013) make a suitable metaphor to these results, comparing statistical learning with travelling: "[W]hen travelling by plane or train we see fewer details along the road than when walking or cycling" but travelling by train or plane is faster and easier (Biehler et al., 2013, p. 678). This comparison takes into account that when acting on analogue material more relationships between parts of the diagram have to be observed, while some digital materials shorten the actions (and the relationships to be observed) to enable focusing on the entire diagram. However, learners should know by walking or cycling which way to take before they "arrive somewhere fast without knowing about all the decisions taken" (Biehler et al., 2013, p. 679). In relation to the results of the paper, it is important to know the relationships between the parts of the diagram, otherwise, the material arrangement cannot be recognized as a diagram and it is merely a picture for the learners.

Limitation and Outlook

In this paper, only a small data selection was considered. Further analyses of actions on statistical learning situations should show whether these results are also evident among other learners working with digital or analogue materials. In addition, the question of whether a shortening of actions can always be observed when working with TinkerPlots should be pursued.

If one compares the results with the analyses of actions in a geometric learning situation (Billion, 2021b), it is noticeable that no shortening of the actions by other programs was detected and the same diagram interpretations could be reconstructed at all points in the processing. These results can be attributed to the fact that in the geometric learning situation considered, the program did not function as a tool due to the design of the learning situation. If this program had been used differently in the learning situation, it would very likely have also functioned as a tool. It is likely that different programs and their different usage have different effects on the learners' diagram interpretations, as programs permit different actions. This assumption should be more precisely substantiated by undertaking further research.

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two schools where the survey took place were informed about the study in the school conference and approved it on February 21, 2019 and February 25, 2019.

Consent for publication: The consent forms from the parents of the two minor children in this paper were obtained and are available from the author. The consent forms take into account that data from the study may be published in scientific articles under anonymisation. In this paper, the names of the students are anonymised to avoid any conclusions about their identity.

Data sharing and reproducibility: The data of the study cannot be made available for other researchers and are not accessible for sharing.

Declaration of interest: No conflict of interest is declared by the author.

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