

Teachers' Conceptions in Training on Mathematics of Medical Students

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Abstract: *The goal of the paper is to pay attention to some important techniques and approaches including adequate designations as a tool for unambiguous understanding and a key to success in solving problems, vivid visual images as a mnemonic techniques, and special formulas as a universal tool for solving typical problems, when teaching medical students of mathematics.*

The motivation for this paper is to help non-mathematics students understand complicated mathematical topics in an easy, natural, and simple way.

1. INTRODUCTION

Mathematics is the fundamental science giving language means to other sciences. This has been noted by many outstanding scholars who asserted that “The book of nature is written in the language of mathematics” (Galileo Galilei), “In any science it is so much true, how many in it of mathematics” (Immanuel Kant), “Mathematics is a basis of all exact natural sciences” (David Hilbert).

Today a great variety of mathematical methods is applied in biology, medicine, and other biological sciences. Using mathematics in public health services in world space occurs promptly, new technologies and the methods based on mathematical achievements in the field of medicine are entered. Mathematical methods are widely applied in medicine. The modern medicine cannot do without the most complicated techniques, therefore the role of mathematics appreciably grows.

An understanding of mathematical calculus leads to better comprehension of chemistry and physics because calculus offers new ways of thinking that are quite useful to a medical practitioner. A knowledge of different parts of mathematics is very important for most future physicians (Nusbaum, 2006). But, unfortunately, medical students often find it difficult to study mathematics (Chasteen-Boyd).

Real-life applications of mathematics provide a great deal of stimulation for various kinds of research in the subject matter field, involving professional mathematicians and students of different majors alike (Abramovich & Grinshpan, 2008).

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In this paper, we would like to share the operational experience at carrying out of employment on mathematical disciplines. The data reported in this paper occurred in a mathematical course taught by the authors at the Immanuel Kant Baltic Federal university (IKBFU) in Kaliningrad (Russia) for medical students. Students seeking a medical degree were required to complete a course in mathematics, which is typical of university degree programs in medicine (Khobragade & Khobragade, 2015; Voltmer et al., 2019). We will describe some features and the curious moments which arose from authors' experiences while teaching medical students at IKBFU.

It is important for a teacher to help a medical student master the necessary sections of mathematics in a short time. And we suggest using methods that are successfully applied in our classes for teaching mathematics to medical students.

The purpose of this study is to characterize the best moments in the growth of medical students' mathematical understanding. This is very relevant since the need for mathematics in medical research is growing rapidly.

Of course, at our lectures and practical classes we have no intention to distract the medical students from their main field of activity and to train them as competent mathematicians. Our aim is rather to prepare them for an understanding of the basic mathematical operations and to enable them to communicate successfully with mathematicians in case they need help of the last.

Usually we use many illustrations and some historical notes to encourage the medical and biological students who are perhaps somewhat reluctant to be involved with the abstract side of mathematics. This is not surprising, after all, the course of Mathematics includes the following sections: Precalculus, Linear algebra, Limits, Differentiation, Integration, Differential equations, Probability, and Statistics (Batschelet, 1979).

Thirty students in two groups from India enrolled in the one-year course in Mathematics and Computer Science for medical students at Immanuel Baltic Federal University (Fig. 1 and 2). At the beginning of the first semester, all students were told that research was being conducted on this experimental class; all students agreed that the data that was collected in the study could be used for research purposes.



Figure 1: Medical students from the group 1



Figure 2: Medical students from the group 2

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2. ADEQUATE DESIGNATIONS AS A TOOL FOR AN UNAMBIGUOUS UNDERSTANDING AND A KEY TO SUCCESS IN SOLVING PROBLEMS

It is very important to respond to pedagogical situations around notations that students might encounter. This study explored an importance of notations of logarithmic function and inverse trigonometric functions in their calculations.

2.1. Logarithms $\ln x$ vs $\log_a x$

Researchers continue to report that many students struggle to develop coherent understandings the idea of logarithm (i.e., logarithmic notation, logarithmic properties, the logarithmic function). Of course, developing students' understanding of the idea of logarithm requires much more than being introduced to and applying Euler's definition (Kuper & Carlson, 2020; Hirsch & Pfeil, 2012).

Recall the definition of logarithm.

Definition 1. If a is any number such that $a > 0$ and $a \neq 1$ and $x > 0$, then $y = \log_a x$ is equivalent to $a^y = x$. We usually read this as “log base a of x ”.

If students do not hold the foundational understanding, they may struggle to envision the relationship between a and x .

It is also important to mention that there are special notation for the logarithmic function for different bases: the common logarithm $y = \log_a x$; the natural logarithm $y = \ln x = \log_e x$; the Briggsian logarithm $y = \lg x = \log_{10} x$.

Remark 1. Of course, if it is necessary, one can always change the base of a logarithm. The logarithm $\log_a b$ can be computed from the logarithms of b and a with respect to an arbitrary base c using the following formula:

$$\log_a b = \frac{\log_c b}{\log_c a}. \quad (1)$$

There have been a number of studies that have examined students' difficulties in understanding of logarithms. Our data collection and analysis focused on understanding and characterizing the meanings students constructed as they engaged in tasks and responded to questions that provided opportunities for reflection.

Our medical students from India, for the most part, prefer to write more short notation $\ln x$ for all logarithms and do not use more long notation $\log_a x$. If $c = e$ formula (1) has the form $\log_a b = \frac{\ln b}{\ln a}$. But it can make formulas and expressions very bulky. Undoubtedly, the common logarithm is more universal and convenient and students should not avoid it.

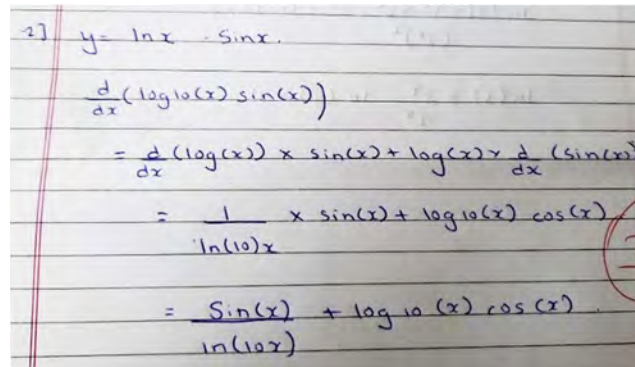
Some students use incorrect notation “log” without any base at all (Fig. 3a) or confuse designations “ln” and “lg” (Fig. 3b)

$$\int_3^8 \frac{1}{u} du$$

$$= [\log u]_3^8$$

$$= \log 8 - \log 3.$$

a)



2] $y = \ln x \cdot \sin x.$

$$\frac{d}{dx} (\log_{10}(x) \sin(x))$$

$$= \frac{d}{dx} (\log(x)) \times \sin(x) + \log(x) \times \frac{d}{dx} (\sin(x))$$

$$= \frac{1}{\ln(10)x} \times \sin(x) + \log_{10}(x) \cos(x)$$

$$= \frac{\sin(x)}{\ln(10x)} + \log_{10}(x) \cos(x)$$

b)

Figure 3: Students' works

2.2. Inverse trigonometric functions ($\sin^{-1}x$ vs $\arcsin x$)

Inverse function is an important concept in secondary and post-secondary mathematics (Paoletti, 2020). Although inverse functions play an important role in many secondary mathematics curricula, but unfortunately students' understanding of inverse functions is limited. Many authors try to give the mental constructions using a unit circle approach to the sine, cosine, and their corresponding inverse trigonometric functions (Martinez-Planell & Delgado, 2016). Inverse trigonometric functions do the opposite of the “regular” trigonometric functions. For example, for the regular function $y = \sin x$ (Fig. 4) the inverse function is $y = \sin^{-1}x$ (Fig. 5) (see, e.g., (Weber et al., 2020)).

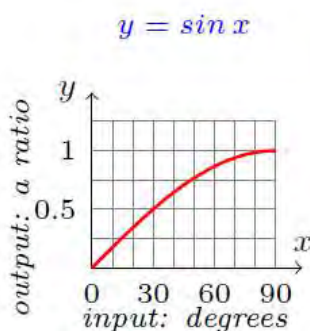


Figure 4: The graph of the function

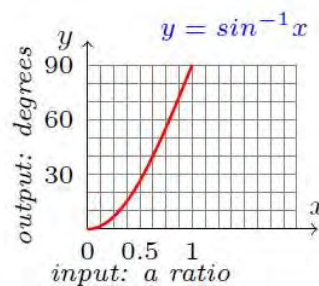


Figure 5: The graph of the inverse function

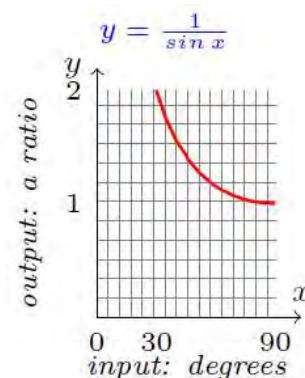


Figure 6: The graph of the function

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$$y = \sin x$$

$$y = \sin^{-1}x$$

$$y = \frac{1}{\sin x}$$

One particular source of confusion of students is the symbol “−1” in the inverse notation f^{-1} . Many students simply conflate the meanings of the superscript “−1” in functional and numerical settings, regarding $f^{-1}(x) = 1/(f(x))$ (see, e.g., (Zazkis & Kontorovich, 2016)).

Example 1. *It is well known that $\sin 30^\circ = 0,5$. Consequently, we have the true equalities*

$$\sin^2 30^\circ = (\sin 30^\circ)^2 = 0,25.$$

However, the inverse sine does not work that way: $\sin^{-1} 30^\circ \neq (\sin 30^\circ)^{-1}$ because $\sin^{-1} 30^\circ = \text{error}$ whereas $(\sin 30^\circ)^{-1} = 2$.

There are two alternate notations for inverse trigonometric functions. The inverse sine can be expressed as $\arcsin x$ or $\sin^{-1}x$. It means “The inverse sin of x ”. The expression $\sin^{-1}x$ is not the same as $\frac{1}{\sin x}$ (Fig. 6). In other words, the −1 is not an exponent. Instead, it simply means inverse function.

The inverse trigonometric functions are also called arcfunctions, as they return the unit circle arc length (in radians) for a particular value of sine, cosine, etc. and denoted by $\arcsin x$, $\arccos x$, etc. These notations enable us to avoid the above-mentioned ambiguity.

In Table 1 one can see inverse trigonometric functions, which are used more often.

Notation 1	Notation 2
$\arcsin x$	$\sin^{-1}x$
$\arccos x$	$\cos^{-1}x$
$\arctan x$	$\tan^{-1}x$
$\text{arccotan} x$	$\text{cotan}^{-1}x$

Table 1: The table of various designations for inverse trigonometric functions

Remark 2. Carl Gauss also objected to this particular notational inconsistency. He proposed that $\sin^2 x$ ought to mean $\sin(\sin x)$, whereas $(\sin x)^2$ should be written in that way.

In our opinion, the form of these functions when denoted by Notation 1 (the first column in the Table 1) is much more unambiguous and correct than in the second. If we use Notation 2 we must say that here −1 is not an exponent, but it is not always convenient. And it can create additional problems, as if −1 is an exponent then $\tan^{-1}x = \text{cotan} x$, and $\text{cotan}^{-1}x = \tan x$. Another convention used by a few authors is to use a upper-case first letter along with a −1 superscript: $\text{Sin}^{-1}x$, $\text{Cos}^{-1}x$, $\text{Tan}^{-1}x$, $\text{Cotan}^{-1}x$. This potentially avoids confusion with the multiplicative

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inverse, which should be represented by $\sin^{-1}x$, $\cos^{-1}x$, etc. But it is not so successful way out as the difference in upper-case and lower-case letters can be not so appreciably in hand-written texts of students. Teachers should be careful with the notation for inverse trig functions.

Remark 3. The notations $\arcsin x$, etc. are common in computer programming languages. Moreover, this notation arises from the following geometric relationships: When measuring in radians, an angle of x radians will correspond to an arc whose length is rx , where r is the radius of the circle. Thus, in the unit circle, “the arc whose sine is x ” is the same as “the angle whose sine is x ”, because the length of the arc of the circle in radii is the same as the measurement of the angle in radians.

Remark 4. The notations $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, and $\cotan^{-1}x$ introduced by John Herschel are often used as well in English-language sources.

Remark 5. In the mathematical literature of Russia it is used $tg x$ and $ctg x$ instead of $\tan x$ and $\cotan x$, and named inverse trigonometric functions using an arc- prefix: $\arcsin x$, $\arccos x$, $\arctg x$, and $\text{arcctg } x$.

We develop a more productive understanding of inverse functions in our work with students and draw students’ attention to the best ways to denote inverse functions. In order to avoid ambiguity and not to be confused it is better to use the notation for the inverse functions with the prefix “arc” (see., e.g., Edelstein-Keshet, 2020; Brenner & Lacay, 2016/17).

From our pedagogical experience, when studying the actions of students with inverse functions, we received the following pictures:

1. Before an explanation for what the *arcsine/arccosine* function represents (Fig. 7a).
2. After an explanation for what the *arcsine/arccosine* function represents (Fig. 7b-7e).

a)

$$\begin{aligned}
 y &= \cos^{-1} \sqrt{-3x+1} \\
 &= \frac{-1}{\sqrt{1-(-3x+1)^2}} \cdot \frac{1}{2} (-3x+1)^{-1/2} \cdot (-3) \\
 &= \frac{-1}{\sqrt{1+3x-1}} \cdot \frac{1}{2} (-3x+1)^{-1/2} \cdot -3 \\
 &= \frac{-1}{\sqrt{3x}} \cdot \frac{1}{2} \frac{(-3x+1)}{\sqrt{(-3x+1)}} \cdot -3
 \end{aligned}$$

b)

$$y = x \times \arcsin x.$$

$$(\sin^{-1}x)' = (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$= 1 \times \arcsin x + \frac{1}{\sqrt{1-x^2}} \times x$$

$$= \arcsin x + \frac{x}{\sqrt{1-x^2}}$$

c)

$$y = \arctan \sqrt{e^x - 1}$$

$$= (\arctan \sqrt{e^x - 1})' = \frac{1}{1 + (\sqrt{e^x - 1})^2} \cdot (\sqrt{e^x - 1})'$$

$$= \frac{1}{1 + (e^x - 1)} \cdot \left[\frac{1}{2} (e^x - 1)^{-1/2} \right] \cdot (e^x - 1)'$$

$$= \frac{1}{e^x} \cdot \frac{1}{2\sqrt{e^x - 1}} \cdot e^x = \frac{1}{2\sqrt{e^x - 1}}$$

d)

$$\textcircled{2} \int \arctan \sqrt{2x-1} \, dx = \left. \begin{array}{l} \sqrt{2x-1} = t \\ \frac{x}{2\sqrt{2x-1}} \, dx = dt \\ \frac{1}{t} \, dx = dt \\ dx = t \, dt \end{array} \right\} = \int t \cdot \arctan t \, dt$$

$$U = \arctan t \quad dU = \frac{1}{1+t^2} \, dt$$

$$dV = t \, dt \quad V = \frac{t^2}{2}$$

$$vU = \int v \, dU$$

$$\frac{t^2}{2} \arctan t - \int \frac{t^2}{2} \cdot \frac{1}{1+t^2} \, dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} \int \frac{t^2 + 1 - 1}{1+t^2} \, dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} \int 1 \, dt - \frac{1}{2} \int \frac{1}{1+t^2} \, dt$$

$$\frac{t^2}{2} \arctan t - \frac{1}{2} t - \frac{1}{2} (-\arctan t) + C$$

$$\frac{1}{2} (t^2 + 1) \arctan t - \frac{1}{2} t + C$$

$$\frac{1}{2} ((\sqrt{2x-1})^2 + 1) \arctan \sqrt{2x-1} - \frac{1}{2} \sqrt{2x-1} + C$$

$$x \arctan \sqrt{2x-1} - \frac{1}{2} \sqrt{2x-1} + C$$

e)

$$y = \sqrt{2x} \cdot \arccos x$$

$$y' = \frac{1}{2\sqrt{2x}} \cdot \arccos x + \sqrt{2x} \cdot \left(-\frac{1}{\sqrt{1-2x^2}} \right)$$

Figure 7. Inverse trigonometric functions in students' works

3. VIVID VISUAL IMAGES AS A MNEMONIC TECHNIQUE

Many authors (Arcavi, 2003; David & Tomaz, 2012; Kadunz & Yerushalmy, 2015) investigate how visual representations can structure maths activity in the classroom and discuss teaching practices that can facilitate students' visualization of mathematical objects. Some concepts simply should be presented visually rather than verbally (Clements, 1982; Leppink, 2017).

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We would like to highlight the importance of various visual mathematics approaches that are effective in mathematical education, e.g., bright memorable images for complicated mathematical notions. Some mathematical concepts are difficult to understand by words. Everyone knows that a picture is worth a thousand words. For students it is very important to understand mathematics intuitively. Ways must be found for them to learn mathematics that will promote intuitive understanding (Aso, 2001).

A theory of Howard Gardner (Gardner, H. (1983)) about multiple intelligences suggests that people have different approaches to learning, such as a visual, kinesthetic or logical approach. Thus, along with rigorous mathematical methods including formulas and proofs, mathematics teachers should use visuals, manipulative and motion to enhance students' understanding of mathematical concepts. All these aids help learners to boost their confidence and performance in maths (Boaler et al., 2016).

Maths classes are often composed entirely of symbol manipulation and the idea that visuals or manipulative are a mere prelude to abstract mathematics becomes instantiated. Calculus is often taught as a technical subject with rules and formulas (and occasionally theorems). Students are made to memorize maths facts, and plough through worksheets of numbers, with few visual or creative representations of mathematics or invitations to work visually. When non-mathematics students learn through visual approaches they are given access to deep and new understandings. Most of students reported that the visual activities enhanced their learning of mathematics. Normally, most of our students tell that they feel that mathematics is inaccessible and uninteresting when they are plunged into a world of abstraction and numbers. Someone might develop the idea that visuals and manipulative are babyish, and mathematical success is about memorizing numerical methods, but undoubtedly, visual aids are highly effective. A goal of any teacher is to make every student (especially non-mathematician) see that mathematics is not just a subject of numbers and symbols (Boaler et al., 2016; Matic, 2014).

3.1. Limits (Squeeze Theorem vs Sandwich Rule)

To calculate limits, we need the following

Theorem 1. *Squeeze Theorem*

If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains x_0 (except possibly at x_0) and

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = A,$$

then

$$\lim_{x \rightarrow x_0} g(x) = A.$$

Proposition 1 *Sandwich Rule*

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For the best remembering Theorem 1 it is possible to use a picture of a sandwich (see Fig. 8). Here two functions $f(x)$ and $h(x)$ are pieces of bread, and $g(x)$ is ham.

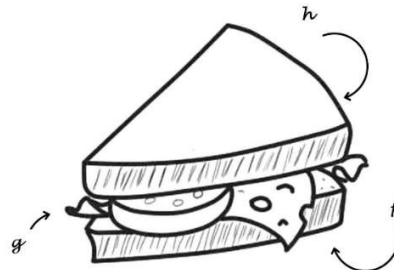


Figure 8: Sandwich

3.2. Differentiation of composite function (Chain Rule vs Nesting Doll Rule)

Differential calculus is an indispensable tool in every branch of science and engineering. Differential calculus is about describing in a precise fashion the ways in which related quantities change. In day to day life we are often interested in the extent to which a change in one quantity affects a change in another related quantity. This is called a rate of change.

The major motivations for introducing the differential calculus are problems of growth rate, reaction rate, concentration, velocity, and acceleration. There exists another group of problems, equally important for life scientists, which leads to the integral calculus. An integration in medicine is applied for describing, for example, cardiac output and Poiseuille's law.

Differential calculus is a procedure for finding the exact derivative directly from the formula of the function, without having to use graphical methods. In practise we use a few rules that tell us how to find the derivative of almost any function that we are likely to encounter (Thomas, 1997).

The composite function rule (also known as the chain rule) reads as follows: differentiate the "outside" function, and then multiply by the derivative of the "inside" function, i.e., if y is a function of g and g is a function of x then

$$\frac{dy}{dx} = \frac{dy}{dg} \cdot \frac{dg}{dx}$$

This makes the rule very easy to remember. The expressions $\frac{dy}{dg}$ and $\frac{dg}{dx}$ are not really fractions but rather they stand for the derivative of a function with respect to a variable. However, for the purposes of remembering the chain rule we can think of them as fractions, so that the dg cancels from the top and the bottom, leaving just $\frac{dy}{dx}$.

Of course, the first step is always to recognise that we are dealing with a composite function and then to split up the composite function into its components.

To find the derivative of composite functions, we need the *Chain Rule*.

Theorem 2 Chain Rule

If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function

$$(f \circ g)(x) = f[g(x)] \tag{2}$$

is differentiable at x , and

$$(f \circ g)'(x) = f'[g(x)]g'(x). \tag{3}$$

In (Bittinger, M.L., Ellenbogen, D.J., Surgent, S.A. (2012)), authors give a very interesting visualization the composition of functions as a composition machine for functions f and g . To find $(f \circ g)(x)$ we substitute $g(x)$ for x in $f(x)$. The function $g(x)$ is nested within $f(x)$.

Sometimes for non-mathematicians it is not so easy for comprehension. For better understanding and visual memorization, we propose to use the following rule. We call this rule the *Nesting Doll Rule*. As is known, a nesting doll (Russian Dolls, Stacking Dolls, Matryoshka) is a Russian wooden toy as a painted doll inside which there are dolls of the smaller size similar to it (see Fig. 9).

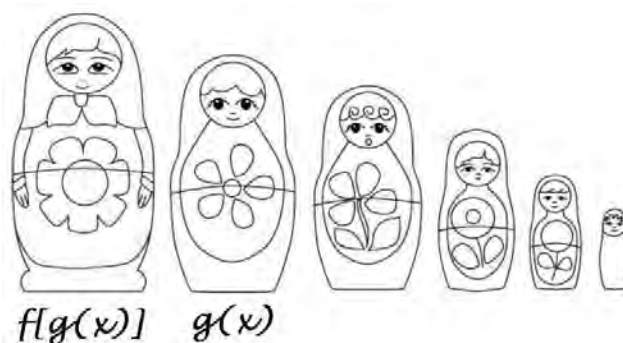


Figure 9: Nesting doll

Readers can ask: what is the connection between mathematics (namely, differentiation) and Russian souvenir? Let us consider an example.

Proposition 2 Nesting Doll Rule

Let in (2) f be the first (the biggest) nesting doll and g is the second one. First of all we can see only the biggest nesting doll. And we must take derivative of the function f . Then we open the first nested doll and we see the second one. And we must take derivative of the function g , etc. We should multiply all founded derivatives, thus we use formula (3).

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Example 2 Find the derivative of the function $y = \sin(3x + 1)$.

Step 1. The function $y = \sin(3x + 1)$ is a composite function. Let $\sin(\dots)$ be the first nesting doll. The derivative of $\sin(\dots)$ is equal to $\cos(\dots)$.

Step 2. And $3x + 1$ is the second nesting doll. We have $(3x + 1)' = 3$.

Step 3. Finally we get: $y' = (\sin(3x + 1))' = \cos(3x + 1) \cdot 3$.

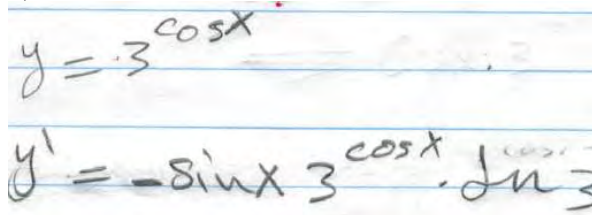
Mathematically the answer will be more correct in the form: $y' = 3\cos(3x + 1)$.

In other words, it looks like the outside function is the sine and the inside function is $3x + 1$, i.e.,

$$y' = \underbrace{\cos}_{\substack{\text{derivative of} \\ \text{outside function}}} \underbrace{(3x + 1)}_{\substack{\text{leave inside} \\ \text{function alone}}} \cdot \underbrace{3}_{\substack{\text{derivative of} \\ \text{inside function}}}$$

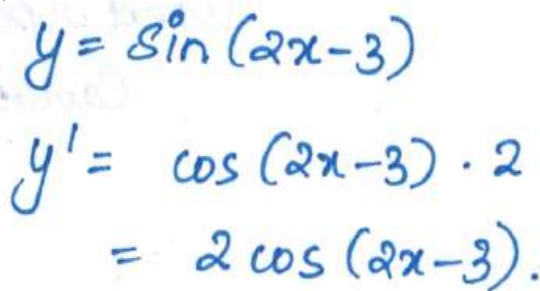
Students showed good results using the Nesting Doll Rule. Their results are presented in Fig. 10.

a)



$y = 3 \cos x$
 $y' = -\sin x \cdot 3$

d)



$y = \sin(2x - 3)$
 $y' = \cos(2x - 3) \cdot 2$
 $= 2 \cos(2x - 3)$

b)

e)

$$4) \ y = \sqrt{1+2x}$$

$$y = (1+2x)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(1+2x)^{-\frac{1}{2}} \cdot 2$$

$$= (1+2x)^{-\frac{1}{2}}$$

$$3) \ y = \sin(4x+3)$$

$$\frac{dy}{dx} = \frac{d}{dx} [\sin(4x+3)]$$

$$\frac{dy}{dx} = \cos(4x+3) \cdot 4$$

$$\frac{dy}{dx} = \cos(4x+3) \cdot 4$$

c)

$$4) \ y = 2\sqrt{\sin x}$$

$$= 2(\sin x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2 \left[\frac{1}{2} (\sin x)^{-\frac{1}{2}} \cdot \cos x \right]$$

f)

$$y = \sin^2(7x+3)$$

$$y = 2\sin(7x+3) \cdot \cos(7x+3)$$

Figure 10: Students apply the Nesting Doll Rule

3.3. Concave functions (Concavity Theorem vs Smile Rule)

Graphs are a common method to visually illustrate relationships in data. Skill to read graphs is very important for the future physicians. It is known, that the second derivative of a function is related to the shape of its graph.

The important result that relates the concavity of the graph of a function to its derivatives is the following one.

Theorem 3 Concavity Theorem

Let a function f be twice differentiable at $x = x_0$. Then the graph of f is concave upward at $(x_0, f(x_0))$ if $f''(x_0) > 0$ and concave downward if $f''(x_0) < 0$.

A good way for explaining the notions of concavity up and concavity down (or convexity down and convexity up), which are very confusing, is to recall a parabola. Any student knows the

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branches of the parabola $y = x^2$ look up. Indeed, it is well-known if the coefficient of x^2 is positive, then the branches look up. On the other hand, it is easy to evaluate the first derivative $y' = 2x$ and the second one $y'' = 2$. That is, the positive second derivative implies the branches look up, i.e., “ \cup ”. Similarly, the negative second derivative implies the branches look down, i.e., “ \cap ”. Positive (+) means up (\cup), negative (−) means down (\cap).

The best way for understanding distinctions between the sign of the second derivative (positive “+” or negative “−”) and the direction of concavity (up “ \cup ” or down “ \cap ”) is the following *Smile Rule*.

Proposition 3 Smile Rule

If the second derivative of a function $y = f(x)$ is positive on (a, b) , then the graph of $y = f(x)$ on this interval “smiles”, i.e., “ \cup ”.

If the second derivative of a function $y = f(x)$ is negative on (a, b) , then the graph of $y = f(x)$ on this interval “frowns”, i.e., “ \cap ”.

Students remember distinctions between two types of concavity if they use the “smile” and “frown” pictures (see Fig. 11).

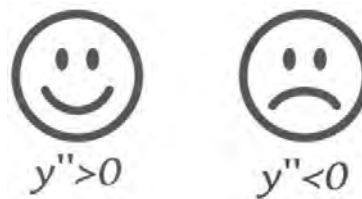


Figure 11: a) Concavity up b) Concavity down

Indeed, positivity of the second derivative corresponds to positive emotion, since the symbol “ \cup ” for concavity up is similar to “smile” (Fig. 11a). Negativity of the second derivative corresponds to negative emotion, since symbol “ \cap ” for concavity down is similar to “frown” (Fig. 11b).

4. SPECIAL FORMULAS AS A UNIVERSAL TOOL FOR SOLVING TYPICAL PROBLEMS

At last, we promote universal formulas replacing lengthy multi-step algorithms.

Algebra provides us with the ability to deal with formulas that always work. This can relieve us from the burden and messiness of having to muck about with the numbers every single time we do the exact same thing. The examples of square equations and systems of linear differential equations demonstrate the advantage of the formula over the algorithm for medical students.

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4.1. Factorization (Grouping vs Discriminant)

All the way through tertiary level mathematics, quadratic expressions routinely appear and so being able to quickly factor them is a basic skill.

Quadratic equations are considered important in mathematics curricula because they serve as a bridge between different mathematical topics. In general, for some students, quadratic equations create challenges in various ways such as difficulties in algebraic procedures (particularly in factoring quadratic equations) (Didis & Erbas, 2015; Kachapova et al., 2007).

The general quadratic function has the form

$$y(x) = ax^2 + bx + c \quad (a \neq 0) \quad (4)$$

with three constants $a, b, c \in \mathbb{R}$. The right-hand expression in (4) is a polynomial of the second degree in x three termed quadratic (i.e., trinomial). The graph of this function is a parabola.

Sometimes it is necessary to find for what values of x the equality $y(x) = ax^2 + bx + c = 0$ holds? Or, where does the quadratic parabola intersect the x axis? In order to answer these questions we have to solve the quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0). \quad (5)$$

In many mathematical tasks it is important to represent a quadratic polynomial as a product of two linear polynomials. To factorize a quadratic equation is to find what to multiply to get the quadratic one. There are a number of different techniques for factoring this type of expression.

1. The solutions of (5) can be found by the Quadratic Formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (6)$$

The expression the square root of which must be taken is called *the discriminant* of the quadratic equation. It is denoted by D . Thus, $D = b^2 - 4ac$.

When the discriminant D is

1. positive, there are two different real solutions $x_1 \neq x_2$;
2. zero, there are two equal real solutions $x_1 = x_2$;
3. negative, there are two complex conjugate solutions x_1 and x_2 .

Remark 6. Vieta's formulas can be also helpful. The roots x_1 and x_2 of the quadratic polynomial (4) satisfy the following relations:

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1 \cdot x_2 = \frac{c}{a}. \quad (7)$$

From (7) we can find the roots x_1 and x_2 of polynomial (4).

Using (6) one can find two roots x_1 and x_2 and rewrite (4) in the following way

$$y(x) = ax^2 + bx + c = a(x - x_1)(x - x_2).$$

2. The factorization (6) can be accomplished also by grouping. First of all it is necessary to check if there exist any common factors. But it is not always easy.

We may apply the following method: to find two numbers that multiply to give ac , and add to give b . Let α and β be the real numbers. Then

$$ax^2 + bx + c = ax^2 + \alpha x + \beta x + c.$$

We factor first two and last two terms and find the common factor.

Example 3 Factorize $6x^2 + 5x - 6$.

Step 1. $ac = 6 \cdot (-6) = -36$ and $b = 5$. List the positive factors of $ac = -36$: 1, 2, 3, 4, 6, 9, 12, 18, 36. One of the numbers has to be negative to make -36 , so by playing with a few different numbers we find that -4 and 9 work nicely: $-4 \cdot 9 = -36$ and $-4 + 9 = 5$.

Step 2. Rewrite $5x$ as $-4x$ and $9x$: $6x^2 - 4x + 9x - 6$.

Step 3. Factor first two and last two terms: $2x(3x - 2) + 3(3x - 2)$.

Step 4. Common factor is $(3x - 2)$, that is $(2x + 3)(3x - 2)$.

The nice thing about the Quadratic Formula is that the Quadratic Formula always works. As compared to completing the square, we're just plugging into a formula. There are no "steps" to remember, and thus there are fewer opportunities for mistakes.

But it is important to apply the Quadratic Formula correctly. For this purpose there are the following helpful recommendations for students. 1) Take care not to omit the \pm sign in front of the radical. 2) Don't draw the fraction line as being only under the square root, because it is under the initial $-b$ part, too. 3) Don't forget that the denominator of the Formula is $2a$, not just 2. That is, when the leading term is something like $5x^2$, you will need to remember to put the $a = 5$ value in the denominator. 4) Use parentheses around the coefficients when you're first plugging them into the Formula, especially when any of those coefficients is negative, so you don't lose any "minus" signs. 5) When using the Formula, take the time to be careful because, as long as you do your work neatly, the Quadratic Formula will give you the right answer every time.

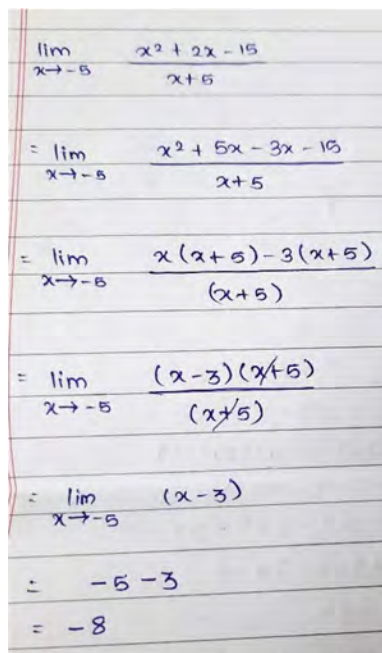
Specially for auditory learners there is a song to help remembering the Quadratic Formula, set to the tune of “Pop Goes the Weasel”:

- x is equal to negative b
- Plus or minus the square root
- Of b -squared minus four ac
- All over two a .

In the real world though, we always use the quadratic formula. Factoring by grouping is almost always completely useless in any kind of an experimental or scientific scenario. The reason why we teach factoring by grouping is to give students at least some exposure in high school to Diophantine Equations. As such, it is an important part of their education.

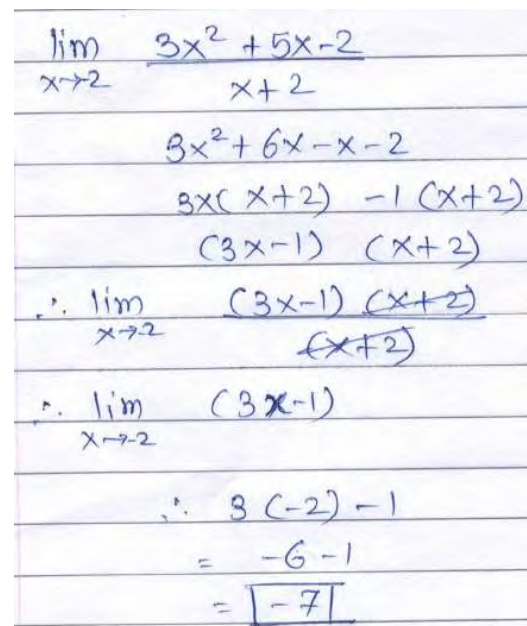
Remark 7. Unlike Russian tradition our medical students from India prefer finding the common factor for computing D .

More than 90 percents of Indian students from the selected group do not find roots of a quadratic function for its factorization (compare Fig. 12 and 13).



$$\begin{aligned} \lim_{x \rightarrow -5} \frac{x^2 + 2x - 15}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{x^2 + 5x - 3x - 15}{x + 5} \\ &= \lim_{x \rightarrow -5} \frac{x(x + 5) - 3(x + 5)}{(x + 5)} \\ &= \lim_{x \rightarrow -5} \frac{(x - 3)(x + 5)}{(x + 5)} \\ &= \lim_{x \rightarrow -5} (x - 3) \\ &= -5 - 3 \\ &= -8 \end{aligned}$$

a)



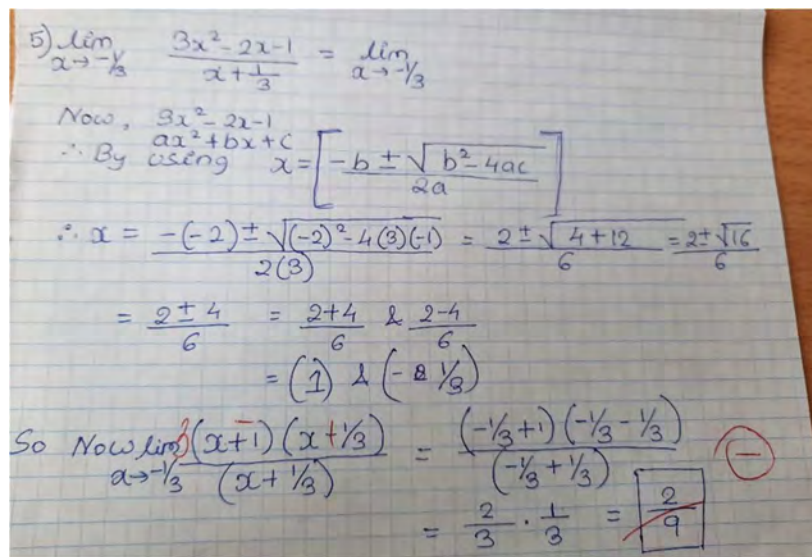
$$\begin{aligned} \lim_{x \rightarrow 2} \frac{3x^2 + 5x - 2}{x + 2} \\ &= \frac{3x^2 + 6x - x - 2}{x + 2} \\ &= \frac{3x(x + 2) - 1(x + 2)}{(x + 2)} \\ &= \frac{(3x - 1)(x + 2)}{(x + 2)} \\ \therefore \lim_{x \rightarrow 2} (3x - 1) \\ &= 3(-2) - 1 \\ &= -6 - 1 \\ &= \boxed{-7} \end{aligned}$$

b)

Figure 12: Usual solutions for Indian students

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5) $\lim_{x \rightarrow -\frac{1}{3}} \frac{3x^2 - 2x - 1}{x + \frac{1}{3}} = \lim_{x \rightarrow -\frac{1}{3}}$

Now, $3x^2 - 2x - 1$
 \therefore By using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)} = \frac{2 \pm \sqrt{4 + 12}}{6} = \frac{2 \pm \sqrt{16}}{6}$

$= \frac{2 \pm 4}{6} = \frac{2+4}{6} \text{ \& } \frac{2-4}{6}$
 $= (1) \text{ \& } (-\frac{1}{3})$

So Now $\lim_{x \rightarrow -\frac{1}{3}} \frac{(x+1)(x+\frac{1}{3})}{(x+\frac{1}{3})} = \frac{(-\frac{1}{3}+1)(-\frac{1}{3}-\frac{1}{3})}{(-\frac{1}{3}+\frac{1}{3})}$

$= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

Figure 13: Unusual solution for Indian students

4.2. Differential equations in medicine and biology

Almost every real-world system can be modelled by differential equations (Chasteen-Boyd). Differential equations occur frequently in the analysis of physiological systems and of ecological systems (see Table 2).

The form of equation	Application
$y' = ay$	growth of a cell, a birth process, a birth-and-death process, radioactive decay, living tissue exposed to ionizing radiation, radioactive tracer, dilution of a substance, chemical kinetics;
$y' = ay + b$	restricted growth, a birth-and-immigration process, cooling, a diffusion problem, nerve excitation;
$y' = ay^2 + by + c$	restricted growth, spread of infection, chemical kinetics, autocatalysis;
$\frac{dy}{dx} = k \frac{y}{x}$	relative growth of parts of a body, metabolism, dose-response problems, racial differences, evolutionary history;

<p>A system of linear differential equations</p> $\frac{dx}{dt} = ax + by,$ $\frac{dy}{dt} = cx + dy,$ <p>a, b, c, d are given constants.</p>	<p>ecology, passage of food in ruminants, excretion of a drug.</p>
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Table 2: On application of differential equations in medicine

There are lists of differential equations and their solutions available (Kamke, 1942, Kamke, 1956). Solutions of some differential equations cannot be written in a manageable form. We teach our student to solve these equations by computers.

4.3. Systems of linear differential equations (Substitutions vs Determinants)

Let us consider a systems of linear first-order differential equations with constant coefficients

$$\begin{aligned} y' &= a_1y + b_1z + f_1, \\ z' &= a_2y + b_2z + f_2, \end{aligned} \quad (8)$$

where $y = y(x)$, $z = z(x)$ are unknown functions, a_1, a_2, b_1, b_2 are constants, $f_1 = f_1(x)$, $f_2 = f_2(x)$ are given functions. Differentiating the first equation with respect to x we obtain

$$y'' = a_1y' + b_1z' + f_1'.$$

By substituting the expression of z' (8) we can eliminate z' from the last equation

$$y'' = a_1y' + b_1(a_2y + b_2z + f_2) + f_1'. \quad (9)$$

Express z from the first equation of (8)

$$z = (y' - a_1y - f_1)/b_1. \quad (10)$$

Eliminate z from equation (9) substituting (10) into equation (9). Then

$$y'' = a_1y' + b_1a_2y + b_1b_2(y' - a_1y - f_1)/b_1 + b_1f_2 + f_1'.$$

Regroup the terms and rewrite the last equation as follows

$$y'' - (a_1 + b_2)y' + (a_1b_2 - b_1a_2)y = f_1' + b_1f_2 - b_2f_1. \quad (11)$$

This algorithm can be applied for every system of the form (8). Also for system (8) written in matrix notation $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ one can easily make up the linear non-homogeneous second-order differential equation with constant coefficients (11) written in the form

$$y'' - (a_1 + b_2)y' + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = f'_1 + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix}, \quad (12)$$

based on the coefficients and functions of system (8). For finding z we use (10).

Hence, in order to solve (8) one can either take all the above-mentioned multi-step algorithm obtaining (9), (10), and (11) consistently or use the universal formula (12) in the stated below example. The first path is no doubt longer, more tricky and bulky than the second one based on (12). The formula (12) is easier than the multi-step algorithm (9)–(12).

Example 4 Solve the system $y' = 8y - 9z + 3x$, $z' = 7y - 8z + 2x$.

Our matrices have the form $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 8 & -9 \\ 7 & -8 \end{pmatrix}$, $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 3x \\ 2x \end{pmatrix}$.

Calculate all coefficients for the final equation (12):

coefficient of y' and y : $a_1 + b_2 = 8 - 8 = 0$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 8 & -9 \\ 7 & -8 \end{vmatrix} = -1$,

the right-hand side: $f'_1 + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix} = 3x' + \begin{vmatrix} -9 & 3x \\ -8 & 2x \end{vmatrix} = 6x + 3$.

Make up the final equation $y'' - y = 6x + 3$.

Then one should write auxiliary (or characteristic) equation $k^2 - 1 = 0$, find the general solution for the homogeneous equation $y'' - y = 0$ and a particular solution for non-homogeneous equation $y'' - y = 6x + 3$, and at last the general solution for the non-homogeneous equation (see (Adams & Essex, 2013)).

In particular, for solving a system of homogeneous linear first-order differential equations with constant coefficients

$$y' = a_1y + b_1z,$$

$$z' = a_2y + b_2z$$

one can easily make up the linear homogeneous second-order differential equation with constant coefficients

$$y'' - (a_1 + b_2)y' + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = 0.$$

The last formula is memorized easily and it is very simple to use.

5. RESULTS AND DISCUSSION

In teaching medical students, different methods of explanation were chosen in two different groups. The content of the first method was strictly mathematical, while the second method allowed and welcomed all kinds of funny images from the world around us. Our students struggled to understand the concepts through definitions, but that embodied, visual ideas proved a valuable adjunct to their thinking. The results are presented in the form of the histogram (see Fig. 14). Here “Traditional explanation” means theorems prevail over vivid rules, and “Special explanation” means vivid rules prevail over strict theorems.

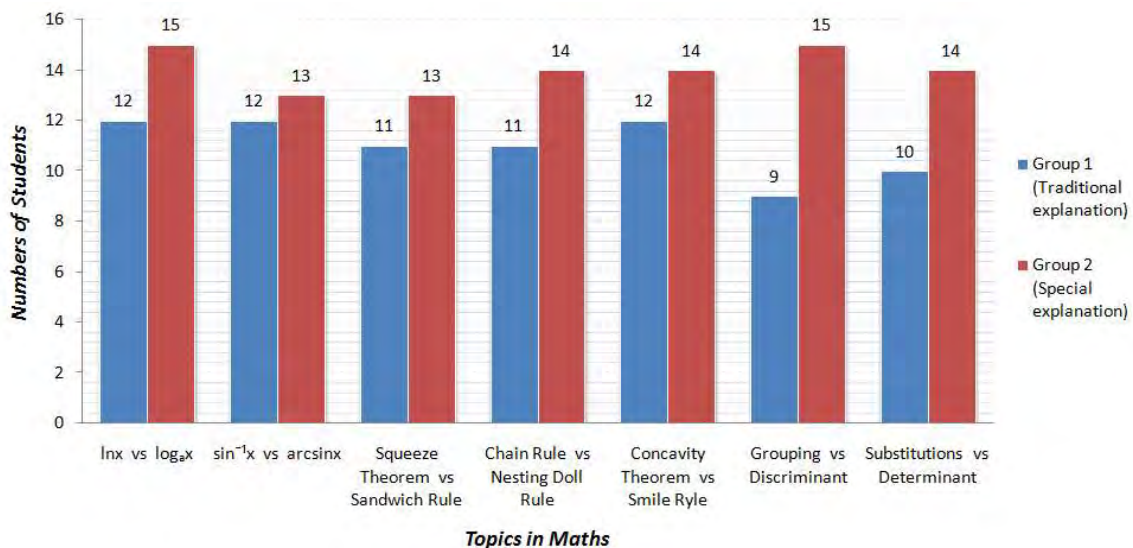


Figure 14: Comparative analysis of the results of mastering topics

93.3% of our students were able to solve problems using some special techniques, as compared to 73.3% for students who followed the standard course of study. In the first group we had taught maths in the conventional way. In the second group, we had implemented, among other things, images which encouraged students to favour visualization. Thus, we see a difference of 20%.

6. CONCLUSIONS

The main purpose of teaching is to stimulate students to perceive mathematics as an indispensable part of the medical curriculum and profession and to provide an education adapted, as far as possible, to the needs and demands of the students and to interest as many of them as possible. It

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requires from teachers a steady improvement in their own culture and teaching methods. This, in turn, calls for the regular updating of the content of the teaching and the regular exchange of experiences.

Our study has shown that performance of students of the first group was poor in mathematical calculations compared to the students from the second group. It was found that students in the second group demonstrated the better mathematical knowledge. Thus, the alternative ways of memorizing formulas and procedures improve multimodal learning abilities of students.

From this comparative analysis, it can be concluded that all kinds of techniques and approaches including adequate designations, vivid visual images and universal formulas enhance students' understanding and improve their performance in the study of mathematics (see Kachapova et al., 2007).

Mathematics plays a significant role in the advancement of medicine. Mathematical reasoning can contribute to medicine in many ways. It may enable physicians, on the one hand, to obtain quantitative estimates in situations where their information has previously been only qualitative, and on the other hand, to find qualitative interpretations for purely quantitative measurements. And sometimes pure mathematical research may produce beneficial practical applications, some of which may lead to new innovations and even life-saving technologies. For this reason it is very important for medical students to study mathematics. And an essential and crucial factor is to give mathematical knowledge in the vivid, accessible, and clear image. We hope that our experience, a small collection of techniques and amusing tricks presented above will turn out to be useful to colleagues and students. Now we know the answer to the question: "How mathematics should be taught to non-mathematicians?"

The authors' experience indicates that famous theorems and conjectures with origins in both pure and applied mathematics have the potential to trigger the imagination and thought process of those whose minds are open to challenge, and thus can be utilized appropriately as useful didactical tools (Abramovich & Grinshpan, 2008).

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