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## Identifying Preservice Teachers' Concept-based and Procedure-based Error patterns in Multiplying and Dividing Decimals

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## Identifying Preservice Teachers' Concept-based and Procedure-based Error patterns in Multiplying and Dividing Decimals

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### Abstract

The purposes of this study are to report current Preservice Teachers' (PTs) decimal abilities regarding the types of error patterns that they commit and to analyze characteristics of the errors in the problems posed for multiplication and division in decimals. We have determined common *concept-* and *procedure-based* errors made by the PTs. 37 PTs in the USA were asked to complete two decimal multiplication problems and two decimal division problems. Results indicated that four major types of error patterns could be detected: Misunderstanding of the place value, misunderstanding of mathematical equivalence, arithmetic errors, and reversing the positions of divisor and dividend. PTs were more likely to have *concept-* than *procedure-based* errors. This indicates that PTs may have limited knowledge on the definition of decimals and the meaning of mathematical equivalence, as well as a lack of understanding of place value. Consequently, we believe that the learning of the concept of decimals requires conceptual change. The *concept-based* errors found intersect with those of students in primary school, so this helps Teacher Education programs in ways that promote PTs' mathematical understanding of decimal operations on a conceptual basis to prevent their future students from making errors.

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### Introduction

When students solve problems incorrectly, an in-depth analysis of error patterns or misconceptions can help teachers understand whether errors are idiosyncratic, or they come from misunderstandings in their conceptual or procedural knowledge. NCTM (2000) stresses that teachers should recognize and respond to students' errors to understand their thinking. Error patterns can provide teachers with an initial understanding of student misconception (Ashlock, 2010). Brodie (2014) defines errors as "systematic, persistent and pervasive mistakes performed by learners across a range of context" and explains the characteristics of errors: "they are persistent even when corrected, thus seemingly resistant to instruction;" "they are systemic rather than the result of individual learner or teacher failure" (p.223). Also, the author notes that errors can arise from misconceptions that stem from learners' overgeneralization of a concept. It means that mathematical knowledge in one domain

(e.g., natural numbers) does not work for the other domains (e.g., rational numbers including decimals, fractions, and percents). Conceptual change is necessary in the process of knowledge development on rational numbers so one's prior knowledge of natural numbers can be compatible with rational numbers (McMullen, Laakkonen, Hannula-Sormunen & Lehtinen; 2015; Putra, 2019). In the same sense, an analysis of errors revealed by preservice teachers (PTs) allows teacher educators and researchers to understand PTs' misconception and help them improve their knowledge of teaching mathematics.

Understanding decimals is one of the most difficult concepts for students and teachers alike (Lortie-Forgues, Tian, & Siegler, 2015; Muir & Livy, 2012). Research studies have identified similar difficulties and misconceptions with decimals in both student and teacher populations, particularly, PTs (Kastberg & Morton, 2014; Muir & Livy, 2012). For example, children and PTs alike struggle to compare and sort, operate, and represent decimals. In particular, PTs had difficulty understanding that 0.7 and 0.70 are equivalent and they interpret decimals as composites of multiples of place value units. Such difficulties and misconceptions are largely derived from a fundamental lack of understanding of place value and decimal notation (Kastberg & Morton, 2014; Muir & Livy, 2012). As an anticipated result, the lack of PTs' knowledge of decimals leads to a lack of in-service teachers' decimal knowledge, which is therefore an explanation for children's misconceptions (Muir & Livy, 2012). As a consequence of such a vicious cycle, this study explores PTs' major error patterns in multiplying and dividing decimals, aiming to provide insights on how to support PTs' growth in knowledge of decimals. The central research questions of this study are: What error patterns emerge when PTs solve decimal multiplications and divisions? and to what extent do PTs reveal common errors in relation to their conceptual and procedural knowledge? This study uses the written responses obtained from 37 PTs to highlight the common error patterns in terms of their responses when carrying out operations with decimal multiplication and decimal division. We describe the research base of these error patterns to understand which errors emerge as persistent.

## **Theoretical Framework**

### **Conceptual Change and Misconceptions**

According to Putt's (1995) study, student's misconceptions with decimal knowledge can be explained by the framework theoretical approach of conceptual change (Durkin & Rittle-Johnson, 2014; Vosniadou & Verschaffel, 2004). Conceptual change refers to the learning process during which the new information conflicts with learners' prior knowledge, which is usually well established and based on their existing experiences (Vosniadou & Verschaffel, 2004). When students first encounter rational numbers, the knowledge of whole numbers does not fit easily into their current conceptions. Namely, the existing knowledge structure can be fragmented. Consequently, this may trigger misconceptions in conceptual and operational aspects when students are learning decimals (Moskal & Magone, 2000; Steinle & Stacey, 2003, 2004; Vamvakoussi & Vosniadou, 2004).

For example, Steinle and Stacey (2003, 2004) conducted a cross-sectional study and a longitudinal study investigating how students changed their misconceptions of decimal notations with age. Their research found

that students can extend their knowledge from learning whole numbers to learning decimals. However, students' prior knowledge and experience with whole numbers showed a negative effect on the interpretation of decimal numbers (Durkin & Rittle-Johnson, 2014; Lai & Murray, 2014; Lortie-Forgues et al., 2015). The misconceptions are referred to as 'whole number thinking' or 'whole number misconception,' thinking of decimals as if they demonstrate the properties of whole numbers (e.g., thinking .25 is greater than .7 because 25 is greater than 7).

Students' errors can provide a window into their learning progress and a clue for successful conceptual change. Thus, it is important for math educators to identify their students' error patterns as a step of the conceptual change process. To provide an effective intervention, educators can then distinguish between strongly held misconceptions that are difficult to change and weakly held misconceptions that could be changed through their instructions (Durkin & Rittle-Johnson, 2014).

### Common Errors with Decimals

According to Son's (2013) study, the *concept-based* errors stem from students' conceptual approach, focusing on understanding, meanings, and definitions of mathematical concepts. The *procedure-based* errors indicate students' procedural knowledge of symbol rules for computation but show arithmetic errors in calculation. Based on these two types of errors, we summarize examples of *concept-based* errors in relation to decimal multiplications and divisions that researchers have found as shown in Table 1. The examples of *concept-based* errors are listed: 1) misunderstanding of place values, 2) misunderstanding of mathematical equivalence, and 3) misunderstanding of multiplication rule or division rule.

Table 1. Types of *Concept-based* Errors in Decimal Multiplication and Division

Types of errors	Characteristics	Example/Reference	<i>Concept-based</i> errors
Multiplication makes bigger	Students often have such assumption because teachers often say that "multiplication makes bigger" in computation of whole numbers	$  \begin{array}{r}  0.12 \\  \times \quad 23 \\  \hline  240 \\  \quad 36 \\  \hline  27.6  \end{array}  $ (Lai & Murray, 2014)	Misunderstanding of place value: Using whole number thinking, students misplace the decimal point.  Correct: $0.12 \times 23 = 2.76$
Multiplying and dividing the power of ten	When multiplying by a power of ten, students multiply both sides of the decimal point by the power of ten.  When dividing by a power of ten, students divide both sides of the decimal point by the power of ten	$6.9 \times 10 = 69.90$ $70.5 \div 10 = 7 \frac{1}{2}$ (Jong et al., 2017)	Misunderstanding of mathematical equivalence: Incorrectly applying distributive property of multiplication/division over addition  Correct: $(6 + 0.9) \times 10 = 60 + 9 = 69$ Correct: $(70 + 0.5)/10 = 70/10 + 0.5/10 = 7 + 5/100 = 7 + 1/20$

Types of errors	Characteristics	Example/Reference	Concept-based errors
Fragile understanding of the place values after the decimal point	Students lost track in executing the multiplication and wrongly placed the decimal points in their final answers	23.00 $\times 0.12$ 46.00 <u>230.00</u> 276.00 (Lai & Murray, 2014)	Misunderstanding of place value: Using whole number thinking, students misplace the decimal point.  Correct: $23 \times 0.12 = 2.76$
Swapping the divisor and dividend in the algorithm	Students changed the divisor and dividend when dividing decimal division	$3 \div 0.12$ instead of $0.12 \div 3$ (Lai & Murray, 2014)	Misunderstanding of division operation - reversing the positions of the divisor and dividend

### PTs' Difficulties with Decimals

Many studies have shown that PTs experience difficulties with the concept of decimals, interpreting decimal notation and operating decimals, as well as comparing and sorting decimals (Kastberg & Morton, 2014; Muir & Livy, 2012; Stacey, Helme, teinle, Baturo, Irwin & Bana, 2001; Ubuz & Yayan, 2010). PTs also lack understanding in the relationships among decimals, whole numbers, fractions, zero and negative numbers (Graber & Tirosh, 1988; Graber, Tirosh, & Globber, 1989; Putt, 1995; Steinle, 2004a, 2004b; Thipkong & Davis, 1991). Specifically, Thipkong and Davis (1991) point out that PTs have difficulty in marking unfamiliar decimals rather than familiar decimals (i.e., 1.05 vs 1.5) on a number line.

In terms of interpreting decimals, PTs face problems when subunits are not based on ten and solving decimal word problems when the divisor or multiplier is less than one. Accordingly, Ubuz and Yayan (2010) indicate in their research that PTs have difficulty in scale reading, operating with decimals, and ordering decimals. Muir and Livy (2012) examine primary preservice and in-service teachers' content knowledge of decimals and stress that solving decimal division is difficult for them.

Although a growing body of research has focused on students' errors regarding basic decimals, there are few studies on PTs' decimal errors in terms of multiplication and division. Thus, the purpose of this study is to help Teacher Education programs in ways that promote PTs' mathematical understanding of decimal operations in the conceptual basis to prevent their future students from making errors.

### Research Questions and Method of the Present Study

To attain the purpose of this study, the following two research questions were addressed:

1. What error patterns emerge when PTs solve decimal multiplications and divisions?
2. To what extent do PTs reveal common errors in relation to their conceptual and procedural knowledge?

## **Method**

### **Participants and Setting**

Participants in the study were 37 PTs enrolled in a teacher education mathematics class for teaching kindergarten through 8th grade students at a mid-western university in the USA. There were 33 female students (89%) and 4 male students (11%). Students reported their status as follows; freshmen (38%), sophomores (32%), juniors (24%) and seniors (6%). Participants ranged in age from 18-32, with the majority aged 18-22 (86%). Most of the PTs are Caucasian (88%). They were taking a mathematics content and methods course (Course A) that emphasized multiple representations of mathematical concepts, procedural and conceptual mathematical knowledge, and pedagogical content knowledge for teaching elementary mathematics. They have not taken any mathematics content and methods courses prior to Course A.

### **Instrument**

This study used qualitative methods to analyze the PTs' written responses to four decimal items on a Decimal Knowledge Test (DKT). The DKT was specifically designed to measure the PTs' knowledge of solving decimal problems in different ways in areas related to multiplication (i.e.,  $2.5 \times 8$  &  $0.05 \times 0.8$ ) and division (i.e.,  $0.25 \div 5$  &  $0.96 \div 0.8$ ) to identify PTs' understandings, errors, and misconceptions. We divided multiplication and division into two different areas because students encountered more difficulties in solving the decimal multiplication problem involving two decimals and the division problem also involving two decimals, as opposed to the multiplication and the division problems with a whole number (e.g., Baroody & Coslick, 1998; Hiebert & Wearne, 1985; Joung, Lin, & Kim, 2021; Lortie-Forgues & Siegler, 2015). All tests were administered at the beginning of the first semester.

### **Coding and Data Analysis**

An interpretive framework was used to analyze PTs' responses of DKT and to identify their *procedure-based* and *concept-based* errors of decimal multiplication and decimal operations. This framework has been largely used in decimal knowledge research (Bell, Swan & Taylor, 1981; Hiebert & Wearne, 1985) and we chose this to analyze categories of decimal error patterns in multiplication and division. To do this, all PTs were required to show more than one different way when solving decimal multiplication and division operations. A spreadsheet was used to record all responses, including incorrect responses. Frequency counts were then used to tally correct responses and common incorrect responses. The PTs responses were coded as "correct", "incorrect", and "no response." Two mathematics educators independently used open codes, identifying the characteristics of each error. Each error may receive more than one code (e.g., place value with arithmetic errors). Similarly, the *concept-* and *procedure-based* errors were identified and categorized by the two mathematics educators, focusing on the nature of errors, that is, whether they result from a lack of PTs' conceptual or procedural knowledge. Inter-rater reliability was calculated as 95%. Discrepancies between the error codes were discussed and resolved.

## Results

In the results of the PTs' total responses in relation to correct responses, incorrect responses, and no responses on decimal multiplication and division operations, the frequency of total responses (220) were as follows: 87 multiplication (decimal  $\times$  whole number), 52 multiplication (decimal  $\times$  decimal), 43 division (decimal  $\div$  whole number), and 38 division (decimal  $\div$  decimal). Among them, number of errors were found in order as follows: 1) multiplication involving two decimals (i.e., 17 errors on  $0.05 \times 0.8$ ), 2) division with a whole number (i.e., 16 errors on  $0.25 \div 5$ ), 3) division involving two decimals (i.e., 9 errors on  $0.96 \div 0.8$ ), and 4) multiplication with a whole number (i.e., 5 errors on  $2.5 \times 8$ ). Also, divisions involving two decimals showed the largest "no response (18.42%)" percentages. The results showed that division involving two decimals appeared to be most difficult for the PTs because their total responses were very minimal (38 out of 220) and the percentages of "no response" was the highest among the 4 operations. These results are in line with those of previous studies that found students have difficulty with decimal division operations, in particular, decimal division involving two decimals (Graeber & Tiros, 1990; Hiebert & Wearne, 1985; Lortie-Forgues & Siegler, 2015).

### PTs' Common Error Patterns for Decimal Multiplications and Divisions

To respond to research question 1, several major types of error patterns related to decimal multiplication were listed as shown in Table 2. In decimal multiplication, the three frequent types of error patterns (EP) could be detected: Misunderstanding of the place value (EP1), Misunderstanding of mathematical equivalence (EP2), and Arithmetic errors (EP3).

Table 2. Decimal Multiplication Error Pattern Coding

Operation	Error Patterns	Characteristics
Multiplication	<b>Concept-based errors:</b> misunderstanding of the place value (EP1)	Using whole number thinking, PTs misplace the decimal point: PTs multiply two decimal numbers, the same as they would with whole numbers, and then place a decimal point in the product directly below the decimal points in the factors.
	<b>Concept-based errors:</b> Misunderstanding of mathematical equivalence (EP2)	PTs fail to make mathematical equivalence (e.g., after multiplying 100, then they need to divide the result by 100)
	<b>Procedure-based errors:</b> Arithmetic errors (EP3)	PTs give the correct location of the decimal points but show minor arithmetic errors while multiplying.

*Misunderstanding of the Place Value (EP1)*

The main pattern of responses was identified from the PTs who incorrectly computed these multiplications. The majority of the PTs (76%) demonstrated EP1 in multiplication involving two decimals (i.e.,  $2.5 \times 8$  &  $0.05 \times 0.8$ ). As shown in Figure 1, most PTs in this EP1 misplaced the decimal point when multiplying two decimal numbers. They multiplied two decimal numbers just as with whole numbers, and then aligned the decimal point in the factors just as in the addends or subtrahends for the decimal addition or subtraction respectively. The PTs in Figure 1 lined up the rightmost digit of the decimals by adding 0 after 0.8 or 8 respectively. After multiplying two decimal numbers just as with whole numbers, they aligned the decimal point in the product with decimal points in the factors. This indicates that the PTs may not have a well-developed concept of the place value.

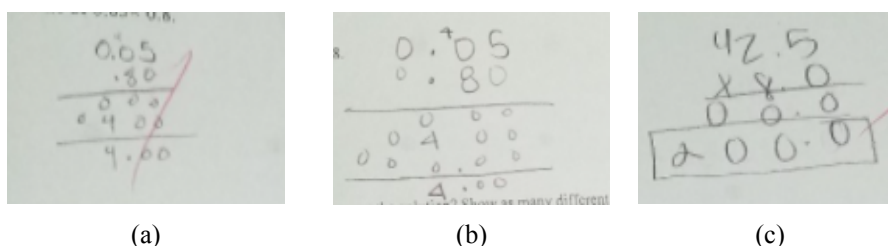


Figure 1. Examples of EP1 in Multiplying Decimals

*Misunderstanding of Mathematical Equivalence (EP2)*

PTs (24%) showed the most common EP 2 in multiplication involving two decimals. For instance, a PT solved the problem  $.05 \times .8$  incorrectly as shown below:

$$.05 \times .8 = .05 \times .80 (\times 100) = 5 \times 80 = 400 (\div 100) = 4$$

The correct answer should be 0.04; however, several PTs multiplied 100 on both decimals but failed to divide 100 on both decimals due to the lack of knowledge of the definition of mathematical equivalence. This problem should be solved as below:

$$\begin{aligned} 0.05 \times 0.8 &= (0.05 \times 100)/100 \times (0.8 \times 100)/100 = \\ &= (5 \times 80) / 10000 = 400/10000 = 0.04 \end{aligned}$$

This problem could be also solved based on the definition of decimals and decimal fractions as below.

$$\begin{aligned} 0.05 \times 0.8 &= (5 \times 0.01) \times (8 \times 0.1) = (5 \times 8) \times (0.01 \times 0.1) = 40 \times 0.001 \\ &= 40 \times 1/1000 = 40/1000 = 4/100 = 0.04 \end{aligned}$$

A mathematical equivalence applies based on the definition of decimals and decimal fractions. Decimal fractions (finite decimals) are defined as fractions with denominators of 10, 100, or 1,000 and can be expressed in base-ten notation (Van de Walle, Karp, & Bay-Williams, 2019). The example shown in Figure 2 (a) illustrates that the PT multiplied 100 on both decimals but did not divide the product by 10,000 ( $100 \times 100$ ) after the multiplication. The PT in Figure 2 (b) multiplied by 100 on both decimals but moved the decimal point to the left only two places, not four places (i.e., divided the product by 100, not 10,000). This may demonstrate that



the PT may have a weak knowledge of mathematical equivalence. No EP2 were shown in the decimal multiplication with a whole number and division involving two decimals.

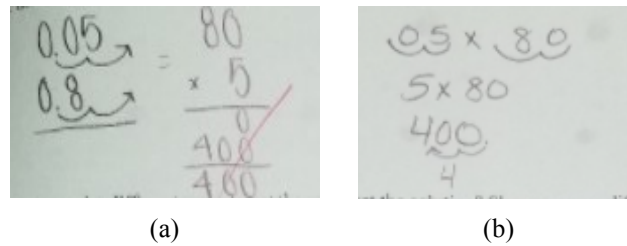


Figure 2. Example of EP2 in Multiplication involving Two Decimals

*Arithmetic Errors (EP3)*

Although the PTs generally used the correct location of the decimal point, they showed incorrect numerals. The PTs in our study presented the most common EP3 when multiplying with a whole number (60%). As shown in Figure 3(a) and (b), PTs used the correct location of the decimal point with incorrect numerals (i.e., in (a),  $16 + 4 = 20$ , not 23). These incorrect responses demonstrated a lack of PTs’ procedural knowledge.

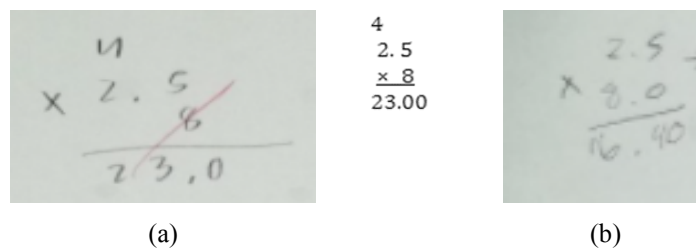


Figure 3. Example of EP3 in Multiplication with a Whole Number

In operating decimal divisions, the most common four types of error patterns were detected as follows: Misunderstanding of the place value (EP1), Misunderstanding of mathematical equivalence (EP2), Arithmetic errors (EP3), and Reversing the positions of the divisor and dividend (EP4). Table 3 presents the major types of error patterns determined in dividing decimals.

Table 3. Decimal Division Error Pattern Coding

Error Patterns	Characteristics
<b>Concept-based errors:</b> Misunderstanding of the place value (EP1)	Using whole number thinking, PTs misplace the decimal point: PTs divide two decimal numbers, the same as they would with whole numbers, and then place a decimal point in the quotient directly above the decimal point in the dividend.

Error Patterns	Characteristics
<b>Concept-based errors:</b> Misunderstanding of mathematical equivalence (EP2)	PTs fail to use equivalent fractions (e.g., if multiplying by 100 both the divisor and dividend, the fraction will not change)
<b>Procedure-based errors:</b> Arithmetic errors (EP3)	PTs give the correct location of the decimal points but show minor arithmetic errors while dividing.
<b>Concept-based errors:</b> Misunderstanding of division operation - reversing the positions of divisor and dividend (EP4)	PTs think the dividend should be bigger than the divisor, heavily relying on whole number thinking.

*Misunderstanding of the Place Value (EP1)*

One quarter of the PTs demonstrated EP1 in dividing decimals with a whole number (i.e.,  $0.25 \div 5$ ), while about half of the PTs (45%) with incorrect responses computed the division without multiplying the divisors and dividends by 10 correctly when dividing decimals involving two decimals (i.e.,  $0.96 \div 0.8$ ). For example, in Figure 4(a), the PT divided two decimal numbers the same as they would with whole numbers, and then aligned the decimal point in the quotient (0.12) with the decimal point in the dividend (0.96). In Figure 4(b), we obviously notice that the PT just divided 25 by 5 and placed a decimal point directly above the decimal point in the dividend (0.25). This indicates the PTs with those incorrect responses may have limited knowledge of decimal place value and conceptual understanding of place value, which is affected by PTs' whole number thinking.

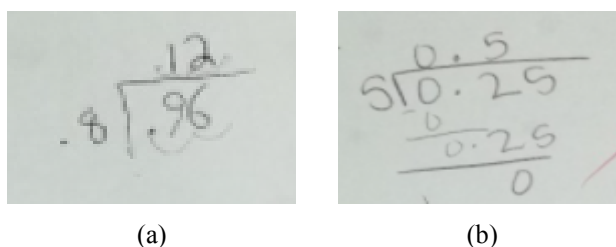


Figure 4. Examples of EP1 in Decimal Divisions

*Misunderstanding of Mathematical Equivalence (EP2)*

In operating decimal divisions, EP2 wasn't shown much. Only 19% of PTs showed EP2 on the division with a whole number problem (i.e.,  $0.25 \div 5$ ). In Figure 5, we notice that the PT tried to multiply 100 on both divisor and dividend, but didn't make a correct answer, failing to use equivalent fractions. The PT also showed EP3, presenting miscalculation of the quotient (i.e.,  $25 \div 500 = 0.05$ , not 0.005)

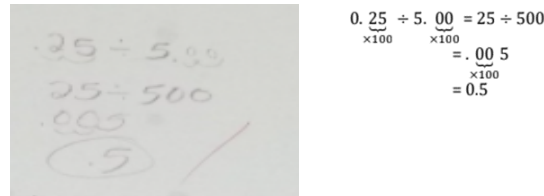


Figure 5. Examples of EP2 in Decimal Divisions

The correct answer should be 0.05; however, several PTs multiplied 100 on both the dividend and divisor for the division, and multiplied 100 on the miscalculated quotient (i.e., 0.005) due to the lack of knowledge on mathematical equivalence. Using the definition of equivalent fractions and fractions as division, this problem should be solved as below:

$$0.25 \div 5 = \frac{0.25}{5} = \frac{0.25 \times 100}{5 \times 100} = \frac{25}{500} = 0.05$$

This problem could also be solved based on the definition of decimals and decimal fractions as below.

$$0.25 \div 5 = \frac{25}{100} \div 5 = \frac{25}{100} \times \frac{1}{5} = \frac{25}{5 \times 100} = \frac{25}{500} = 0.05$$

*Arithmetic Errors (EP3)*

Similar to the multiplications, the PTs used the correct location of the decimal points with incorrect numerals (see Figure 6). The PTs in our study presented the most common EP3 when dividing decimals with two decimals (33%) and division with a whole number (12%). These incorrect responses demonstrated a lack of PTs' procedural knowledge. For example, in Figure 6(b), the PT showed evidence of knowledge on the definition of decimals (by converting decimal numbers to decimal fractions) but made a simple mistake (2/10 is not 0.5 but 0.2).

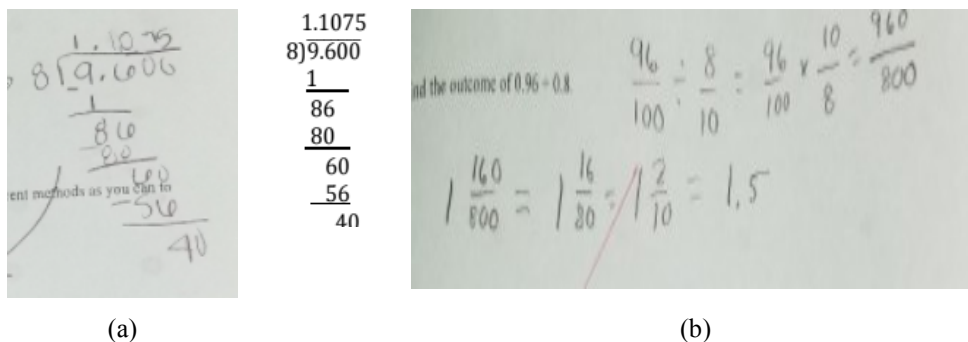


Figure 6. Examples of EP 3 in Dividing Decimals

*Misunderstanding of Division Operation - Reversing the Positions of the Divisor and Dividend (EP4)*

EP4 was shown only for dividing decimals: division with a whole number (54%) and division involving two decimals (22%). Interestingly, more than half of PTs switched the divisor and dividend, making the dividend the larger number when dividing decimal with a whole number (i.e., 0.25 ÷ 5). We assume that the PTs who made

EP4 believed that the dividend should be bigger than the divisor. The results showed a lack of conceptual understanding of division that is also heavily affected by PTs' whole number thinking.

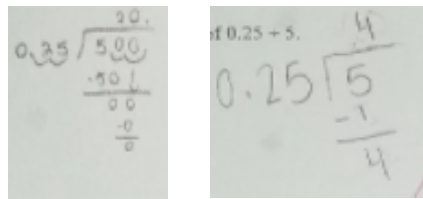


Figure 7. Examples of EP 4 in Dividing Decimals

**Characteristics of *Concept-* and *Procedure-based* Errors**

To respond to research question 2, we summarized to what extent PTs reveal common errors in relation to their conceptual and procedural knowledge as shown in Table 4. The PTs showed two types of errors when multiplying and dividing decimals. Most of the PTs showed *concept-based* errors except with multiplication involving a whole number. However, since we have five errors (three *concept-based* errors & two *procedure-based* errors) in multiplying with a whole number, it is difficult to simply conclude that the PTs may have been likely to have *procedure-based* errors.

Table 4. PTs' Identification of *Concept-* and *Procedure-based* Errors

Type of Errors	$2.5 \times 8$	$0.05 \times 0.8$	$0.25 \div 5$	$0.96 \div 0.8$
<i>Concept-based</i> errors	2 (40%)	17 (100%)	13 (88%)	7 (78%)
<i>Procedure-based</i> errors	3 (60%)	0 (0%)	3 (12%)	2 (22%)
Total errors	5	17	16	9

In multiplication involving two decimals, all of the errors appeared to be *concept-based* errors, and it is probably because of the PTs' lack of understanding of place value. Most of the PTs overgeneralized the rule for adding and subtracting decimals by applying the rule to moving the decimal point. Also, it seems that PTs do not check on the reasonableness of the answer.

For division with a whole number, the results showed that 88% of incorrect answers derive from the *concept-based* errors. The PTs showed a lack of understanding of place value, meaning that they incorrectly placed the decimal point in the answer. They often reversed the positions of the divisor and dividend. This showed PTs' limited knowledge on the definition of division and the definition of decimals (i.e., decimal fractions).

Lastly, as for the division involving two decimals, 78% of incorrect answers stemmed from *concept-based* errors. The PTs failed to multiply by 10 or 100 to make a whole number divisor and divide decimals as is, aligning the decimal point in the dividend. This error seemed to have been derived from ignoring the concept of

division of the whole number. They did not understand the concept that  $0.96 \div 0.8$  could be considered  $96 \div 80$  by multiplying 100. This also showed that they may have a limited knowledge of the meaning of equivalent fraction of decimal division  $0.96/0.8 = 96/80$ . Interestingly, 22% of *procedure-based* errors showed that the PTs successfully converted decimals into fractions but showed arithmetic errors while simplifying.

## Discussion

The overall aim of this study is to provide insights on how to support PTs' growth in knowledge of decimals through exploring PTs' major error patterns in multiplying and dividing decimals and identifying the nature of their errors, that is, whether the errors are *concept-* or *procedural-based*. We now summarize the major findings of this study as explicit partial answers to the two research questions.

To respond to the first research question, what error patterns emerge when PTs solve decimal multiplications and divisions, the three major types of error patterns emerged in decimal multiplications: Misunderstanding of the place value (EP1), Misunderstanding of mathematical equivalence (EP2), and Arithmetic errors (EP3). Similarly, four types of errors are emerged in decimal divisions: Misunderstanding of the place value (EP1), Misunderstanding of mathematical equivalence (EP2), Arithmetic errors (EP3), and Reversing the positions of the divisor and dividend (EP4).

As for EP1 for decimal multiplications, we found that 76% of PTs revealed place value errors in multiplication involving two decimals. They misplaced the decimal point after multiplying two decimal numbers, as when doing whole number multiplications. They placed a decimal point in the product directly below the decimal points in the factors. Whole number thinking might affect the PTs' place value errors in solving decimal multiplications. This finding was also reported by Lai and Murray's research (2014). This means that the PTs' understanding of whole numbers might not support their understanding of decimal place value. Thus, we assume that the PTs in this study might not have applied the conceptual changes.

In terms of EP2 for decimal multiplications, a quarter of the PTs failed to make mathematical equivalence in multiplication involving two decimals. The PTs multiplied 100 on both two decimals but did not divide the product by 10,000 ( $100 \times 100$ ) after the multiplication. This indicates that the PTs showing EP2 might have a limited knowledge on the definition of decimals and lack conceptual understanding of the mathematical equivalence concept. Several studies (Asghari, 2009; Baratta, 2011) emphasize that understanding equivalence can build students' foundation for success in algebra, geometry, and other higher-level mathematics. McNeil (2014) points out that children face difficulties with mathematical equivalence because it isn't interconnected with their early experiences with arithmetic. Similarly, our PTs showed their difficulties with applying mathematical equivalence that emerged as one of their major error patterns (EP2).

For EP3 of the decimal multiplications, more than half of the PTs (60%) showed arithmetic computation errors when multiplying with a whole number. The majority of PTs for this study applied standard algorithms but did not correctly apply them. These results seem to be consistent with other research (Fuson & Beckmann, 2012;

Joung et al., 2021; Ortiz-Laso & Diego-Mantecón, 2020) which found that the standard algorithm is still the dominant strategy used when teaching decimal operations in school mathematics. Also, the majority of PTs for this study did not apply the standard algorithm correctly due to a lack of procedural knowledge, ultimately leading to a lack of conceptual knowledge.

As for the EP1 in decimal divisions, a quarter of PTs showed EP1 errors in dividing decimals with a whole number, and 45% of PTs presented EP1 errors when dividing decimals involving two decimals. In general, the PTs misplaced the decimal point in the quotients: dividing two decimal numbers just as solving whole number divisions, and then placing a decimal point in the quotient directly above the decimal point in the dividend. This implies that PTs' whole number thinking might be used when solving decimal division operations as in decimal multiplications. These results seem to be consistent with other research (Lai & Murray, 2014; Lortie-Forgues et al., 2015) which found the most common misconception for decimal operations in young children is "whole number thinking." The whole number concept may not always be supportive of learning rational numbers for both children and PTs alike. PTs' whole number thinking may interfere with their knowledge of decimals, resulting in misunderstanding of place value and division operations. Thus, conceptual changes might not be completely achieved by many PTs in the process of decimal multiplication and division involving two decimals, as confirmed by this study. These results corroborate the ideas of previous researchers (Vamvakoussi & Vosniadou, 2004; Vosniadou, 1994) in that students were facing difficulties in learning decimal division operations due to the incompatibility between new knowledge (e.g., decimal operations) to be learned and what is already known (e.g., whole number operations).

In terms of the EP2 of decimal divisions, one-fifth of PTs showed mathematical equivalence errors on the division with a whole number problem. The PTs who showed the EP2 errors multiplied by 100 on both decimals, but failed to use equivalent fractions. This implies that they have weak knowledge of the definition of mathematical equivalence in decimal divisions as shown in decimal multiplications.

For the EP 3 in decimal divisions, the PTs gave the correct location of the decimal points; however, 33 % of PTs presented arithmetic computation errors when dividing decimals with two decimals, and 12% of them showed those errors in division with a whole number. Like decimal multiplications, PTs tried to solve decimal divisions using standard algorithms, but the majority of them didn't give a response to decimal divisions. This implies that the PTs might have a weak procedural knowledge on decimal divisions, particularly, involving two decimals. In this regard, Ortiz-Laso and Diego-Mantecón, (2020) point out that one of the main reasons for PTs to frequently forget the steps in solving multi-digit division problems is that there is too much reliance on calculators at schools. We agree that this could be a significant problem. Thus, mathematics educators should rethink students' immense use of calculators at school.

Lastly, for the EP 4 in decimal divisions, more than half of PTs switched the divisor and dividend, making the dividend the larger number when dividing the decimal with a whole number. This implies that those PTs might think the dividend should be bigger than the divisor, as the students participating in this study were confused (Fischbein, Deri, Nello, & Marino, 1985; Graber et al., 1989). This indicates that the PTs' whole number

thinking might heavily affect misunderstanding of division operations, in particular, reversing on the positions of divisor and dividend in decimal divisions. The errors may also result from overgeneralizing the commutative property or from mixing up the order of the divisor and dividend in  $a \div b$  form, and  $a / b$  form (Graber et al., 1989) due to a lack of conceptual understanding of fraction as division.

In the 1980s and 1990s, many research studies explored students' common misconceptions of decimal divisions (e.g., Bell, 1982; Brown, 1981; Fischbein et al, 1985; Graber et al., 1989). For example, students struggled with understanding difference between  $a \div b$  with  $b \div a$  (Bell, 1982; Brown, 1981; Graber et al., 1989). Students were also confused that the divisor must be smaller than the dividend and that the divisor must be a whole number (Fischbein et al, 1985; Graber et al., 1989). Surprisingly, those errors, reversing the divisor and dividend, have been still evident over the past decades, as confirmed by the PTs in this study. It is therefore likely that the PTs might have limited access to timely educational interventions that emphasize the knowledge of conceptual understanding of decimal divisions.

In response to the second research question, to what extent do PTs reveal common errors in relation to their conceptual and procedural knowledge, we found that the majority of the PTs showed *concept-based* errors, in particular, misunderstanding of place value. An understanding of place value plays a crucial role in enhancing students' mathematics understanding (Hansen, 2017). The lack of their understanding may create difficulties that might lead to long-standing errors. Place value concepts could be very complicated and abstract for children to understand (Cooper & Tomayko, 2011; Hansen, 2017; Luneta & Makonye, 2013). However, unfortunately, students tend to learn place value concepts algorithmically without deeper understanding (MacDonald, Westenskow, Moyer- Pakenham & Child, 2018). If children have not properly developed decimal knowledge during an appropriate instructional period, it would be very difficult for them to correct it later. The PTs showing *concept-based* errors might not develop their conceptual knowledge of decimals in their younger school years. As a result, teachers' early intervention with young children on conceptual understanding and procedural knowledge may be an effective way to reduce the occurrence of place value errors of students as advanced as PTs.

In this study, we found that the conceptual change process may not seem to apply to our PTs successfully. When the PTs encountered rational numbers (e.g., decimals), the whole number thinking still interferes with decimal multiplication and division operations and therefore they generate *concept-based* errors. This incompatibility of new information with their existing conceptual structure in learning decimal multiplication and division may be the source of students' difficulties in understanding mathematical concepts such as algebra and rational numbers (Durkin & Rittle-Johnson, 2014; Hartnett & Gelman, 1998; Kieran, 1992; Merenluoto & Lehtinen, 2002).

## Conclusion

Conceptual change framework can be used in mathematics as an approach to identify concepts that students have difficulty with and as a medium to predict and explain the systematic errors and misunderstandings that students have. The major error patterns in multiplying and dividing decimals and their characteristics found by

this study indicate that almost all of the participants demonstrated *concept-based* errors that might be caused by misunderstanding of decimal place value, decimal division operation (reversing the positions of the divisor and dividend), and mathematical equivalence. In particular, the PTs' misunderstanding of decimal place value and division operation might result from their whole number thinking that it may interfere with their knowledge development on decimals and decimal operations. As a result, it implies that the PTs in this study might fail to achieve successful conceptual change.

However, the major error patterns and their characteristics found by this study can provide PTs with a window into their own learning progress and a clue for successful conceptual change. PT's insufficient conceptual knowledge can inhibit their future students from developing a deep understanding of decimal multiplications and divisions. As PTs become more aware of their errors, they put more effort on avoiding making errors in problem-solving and finding ways to prevent their future students from those errors.

More specifically, using an improved understanding of those common error patterns, mathematics educators can be more effective in remediating PTs who have limited knowledge of conceptual understanding. In particular, Teacher Education programs should develop a math content and methods course that includes class activities to explore major error patterns (e.g., misunderstanding of place value) revealed by PTs and discuss what caused the errors (e.g., whole number thinking) and how to prevent them for a successful conceptual change (e.g., knowledge development on rational numbers). The course work must also include strategies to teach the conceptual basis of decimal operations that are useful and helpful for PTs to achieve a successful conceptual change, and ultimately to support the successful conceptual change of their future students.

Most of all, an early intervention for students who struggle with decimals is important to provide the students with continuous opportunities for success in math education. Additionally, teacher education programs should ensure that PTs are given appropriate instructional time to discuss their own possible error patterns while taking mathematics content and methods courses. PTs would have a strong conceptual knowledge of mathematics if they focus more on explaining why and how they generate errors on specific mathematical concepts, realizing why and how their future students might make errors. To promote students' understanding, the PTs will be able to deliver meaningful instruction to individual students, including a diverse population of students through identifying PTs' own error patterns and student errors.

## **Limitations and Recommendations**

There are several limitations and recommendations to this study:

- First, we analyzed PTs' error patterns on decimal multiplication by a whole number, involving two decimal numbers, decimal division by a whole number, and involving two decimal numbers. Other decimal numbers may produce different results. Thus, it is important for future research to consider other variables of the problems such as the semantic, number size, and exact division.
- Second, in our sample, most of the PTs are Caucasian (88%). The results may not have been the same if more diverse PTs were available.



- Third, we conducted qualitative research using only PT's written work without cognitive interviews. All PTs were required to *show their work* in different ways, and it allowed us to figure out what and why they did wrong, without having to guess, even if we didn't conduct cognitive interviews. However, we recommend a mixed approach using cognitive interviews to get insights from PTs and understand their responses in depth in order to identify the causes of common errors.
- Fourth, EP3, Arithmetic errors, were frequently shown in decimal multiplications and divisions. Careless errors might be accidental, many errors in arithmetical computation can be learned and can also be developed habitually. Thus, we recommend that studies need to further examine where those errors came from.
- Lastly, only 37 PTs were used for this study. Although generalization is not the aim of this study, a larger sample size would be more representative of the populations.

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