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Dyscalculia in Algebra: A Case Study

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Algebra is a gatekeeper. For the 6% of students with dyscalculia (i.e., mathematical learning disabilities), an inability to pass algebra may significantly limit academic and career opportunities. Unfortunately, prior research on dyscalculia has focused almost exclusively on elementary-aged students' deficits in speed and accuracy in arithmetic calculation. This case study expands our understanding of dyscalculia by documenting how one college student with dyscalculia understood algebra during a one-on-one design experiment. A detailed case study of 19 video recorded sessions revealed that she relied upon unconventional understandings of algebraic quantities and notation, which led to persistent difficulties. The design experiment involved designing alternative tools to enable the student to reason about algebra, but the unconventional understandings persisted. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon, particularly in understudied mathematical domains, like algebra.

Keywords: **Case Report, Mathematics Learning Disability, Algebra, Dyscalculia**

Introduction

Although many students may have difficulties with mathematics, the 6% of students with dyscalculia (Shalev, 2007) have a neurological difference in how their brains process quantity (Butterworth, 2010). Research on dyscalculia has identified that students have difficulty processing both symbolic (e.g., 5) and pictorial (e.g., *****) representations of quantity (Butterworth, 2010). This neurological difference in number processing may render standard math-

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ematical tools, like symbols or representations, less accessible for students with dyscalculia (Lewis, 2014; 2017). Currently, research on dyscalculia has predominantly focused on elementary-aged students engaged in basic arithmetic (Lewis & Fisher, 2016). It remains largely unknown what kinds of difficulties students may experience when encountering more complex mathematics, like algebra. This is a critical omission because algebraic reasoning is qualitatively different than arithmetic (e.g., Carraher & Schliemann, 2007; Kaput, 2008; Kaput et al., 2008; Kieran, 1992; Stephens et al., 2013), quantities are represented abstractly in a variety of forms (Kaput et al., 2008; Kieran, 1992), and failure to pass algebra can limit students' academic and career opportunities (Adelman, 2006).

Large-scale studies of students with dyscalculia in algebra are not currently feasible because of difficulties in accurately identifying students with dyscalculia. Researchers emphasize the importance of differentiating between students with dyscalculia and students who have mathematical *difficulties* that are due to environmental, language, instructional, or affective factors (Lewis & Fisher, 2016; Mazzocco, 2007; Mazzocco & Myers, 2003). Researchers also argue that it is essential to differentiate students with dyscalculia from other disabilities (e.g., dyslexia) who may struggle with math, because these students have different cognitive profiles (Lyon et al., 2003) and conflating these groups of learners may mask unique characteristics of each (Mazzocco & Myers, 2003). To establish whether students' low mathematics achievement is due to cognitive or noncognitive factors, researchers often use longitudinal designs (e.g., Geary et al., 2012; Mazzocco & Myers, 2003; Mazzocco et al., 2013) or work with adult learners (e.g., Lewis, 2014; Lewis & Lynn, 2018). For example, in the context of fractions, longitudinal research has found that the difficulties experienced by students with dyscalculia are qualitatively different than low achieving students (Mazzocco et al., 2008), and that these difficulties have been found to persist over years (Mazzocco et al., 2013). Detailed analyses of adults with dyscalculia have demonstrated that these difficulties may be due to persistent, unconventional understanding and use of standard mathematical tools, which suggests that all mathematical tools are not equally accessible for students with dyscalculia (Lewis, 2014; 2016; 2017; Lewis et al., 2020). Although both studies of adults with dyscalculia and those with a longitudinal design have identified characteristic patterns of reasoning that students with dyscalculia demonstrate in fractions (Lewis, 2016; Lewis et al., 2021; Mazzocco et al., 2013), no similar studies have been conducted in algebra.

To extend work on dyscalculia into algebra, we conducted a detailed analysis of an adult learner with dyscalculia ("Melissa") as she engaged in a weekly videorecorded one-on-one design experiment focused on algebra. We adopt an anti-deficit theoretical orientation to disability (Vygotsky 1929/1993), and we identify the understandings she relied upon rather than interpreting her data through a deficit frame. A detailed analysis of 19 weekly hour-long videorecorded sessions suggests that the student relied upon unconventional understandings of algebraic symbols. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the unconventional understandings that students with dyscalculia persistently rely upon.

In this section we provide the rationale for a detailed case study of one adult student with dyscalculia. We review research on algebra teaching and learning, which has established both the common misconceptions experienced by all students when learning algebra, as well as instructional approaches intended to address these issues. We then present our theoretical framework – grounded in an anti-deficit Vygotskian framing of disability. We conclude by considering how this framing influenced the design decisions for our one-on-one learning environment.

Strength of Case Reports of Learning

Although case study reports rarely appear in special education journals, there is considerable benefit in this type of research (Grünke et al., 2021) – particularly for disabilities that are not well understood or are hard to accurately identify, like dyscalculia. Historically, detailed case studies have been essential in identifying the defining characteristics of other disability categories, including attention deficit hyperactive disorder (Lange et al., 2010), autism (Wolff, 2004; Verhoeff, 2013) and dyslexia (Duane, 1979). Early clinical identification of *extreme* cases often led to defining characteristics of the disability that were used to identify and further refine the definition (e.g., Verhoeff, 2013). For dyscalculia, an extreme case could be an adult with a long history of significant and pervasive issues with math, who continued to struggle with basic mathematics despite sufficient educational opportunities. Detailed analyses of these kinds of extreme cases can allow researchers to identify characteristic patterns of understandings evident in individuals with dyscalculia. Indeed, detailed case studies of adults with dyscalculia have begun to identify the characteristics of this disability in fractions (Lewis, 2014; 2017), which have later been identified in younger students with dyscalculia (Lewis et al., 2022). Through these kinds of detailed analyses of adult cases, we can begin to further delineate the characteristics of dyscalculia across a range of mathematical topics.

Prior research on algebra

In this study, we aimed to extend research on dyscalculia to the mathematical topic of algebra. Algebra is a particularly appropriate content area to explore dyscalculia because algebra is representationally and conceptually far more complex and abstract than arithmetic (Kaput, 2008). Kaput (2008) defines algebraic reasoning as generalizations within a conventional symbol system and syntactically guided action on those symbols. Because students with dyscalculia have difficulty both using symbols to represent quantities and manipulating those quantities in arithmetic (Piazza et al., 2010) – it is critical that we begin to explore how these difficulties emerge in algebra when symbol use and manipulation is core to the mathematical activity.

Fortunately, research with *nondisabled* students offers considerable insight into the nature of common student difficulties and a wealth of instructional approaches for addressing these difficulties (e.g., Carraher & Schlieman, 2007). Research in mathematics education has demonstrated the difficulties that students experience when transitioning from arithmetic to algebra (see Kieran, 2007 for a review). For example, research has shown that students may have difficulty understanding the relational meaning of the equal sign and students may treat the equal sign as a unidirectional operator which yields an answer (Booth, 1984; Kieran, 1981; 1992). Similarly, students may also have difficulty using letter symbols to represent unknown or variable quantities (Booth, 1984; Kieran, 2006) and consequently have difficulty manipulating terms in an equation in valid ways when solving for an unknown (Filloy & Rojano, 1989). To address these kinds of difficulties, physical or dynamic two-sided scales (i.e., pan balances) have been used in instruction. These two-sided scales have been shown to be effective in supporting students' reasoning about equations, including helping students eliminate like terms from both sides of the equation and solve for unknown on both sides of the equation (Vlassis, 2002). Two-sided scale models have been widely recommended to support students' understanding of equality and the equal sign (e.g., Common Core State Standards, National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Van de Walle et al., 2016).

Although common difficulties and effective instructional approaches have been identified for nondisabled students, it is unclear what kinds of unique difficulties students with dyscalculia may experience, as well as which mathematical representations and tools may be inaccessible. To explore the unique difficulties that students with dyscalculia experience when learning algebra, we draw upon a sociocultural framing of disability and understand mathematics as a mediated activity.

Theoretical Perspective – Reconceptualizing Dyscalculia as Difference

Although dyscalculia is typically conceptualized in terms of cognitive *deficits* (e.g., Geary, 2010), we argue that it is more productive to conceptualize dyscalculia in terms of cognitive *difference.* Our perspective is derived from a Vygotskian perspective of disability (Vygotsky, 1929/1993). Vygotsky argued that mediational signs and tools (e.g., language, symbols), which developed over the course of human history, were often incompatible with the biological development of children with disabilities (Vygotsky, 1929/1993). For example, the mediational tool of *spoken* language is not accessible to a Deaf child, and therefore does not serve the same role in supporting the child's development of language as it would for a hearing child. In the case of students with dyscalculia, it is possible that standard mathematical mediational tools (e.g., numerals, graphs, equations), which support the mathematical development of most students, may be incompatible with how a student with dyscalculia cognitively processes numerical information (e.g., Piazza et al., 2010). Students may have difficulties accessing and using these standard tools and may understand representations and symbols in unconventional ways. Although all students may use standard tools in unconventional ways as they are first learning a topic, we propose that students with dyscalculia may experience *persistent* incommensurability because of the inaccessibility of these mathematical mediators. To address this inaccessibility, alternative mediators must be designed (re-mediations) which provide the student with access to the mathematics. Therefore, in this study, we identify unconventional use or understanding of standard mathematical tools that persist across problems and contexts – we term these *persistent understandings*. We explore how these persistent understandings interact with our attempts to provide alternative mediational tools (re-mediations).

Disability Through the Lens of a Design Experiment

Aligned with our anti-deficit theoretical perspective, we argue that dyscalculia must be understood by examining the student in the *process* of doing mathematics. Too often research on learning disabilities focuses on outcome measures (e.g., written performance on an assessment) and makes inferences about the characteristics of the disability from these outcome measures. In this study, we captured the student's attempts to learn during a design experiment (Cobb et al., 2003). A design experiment involves engineering learning environments and systematically studying the forms of learning (Cobb et al., 2003). In this design experiment not only did we capture the student's unconventional understandings as she was engaged in attempts to learn, but we attempted to design instructional approaches to address her difficulties. It is through the iterative cycles of design, enactment, and analysis that we can understand both the student's unconventional understandings and what instructional approaches were accessible for the student. The outcome of this design experiment is not a recommendation for a particular sequence of instructional activities or tools. Instead, the design experiment serves as the context through which we are able to better understand the kinds of inaccessibility this student with dyscalculia experienced in mathematics and what kinds of tools were more accessible. In this research we aimed to identify the kinds of unconventional understandings of mathematical mediators that the student persistently relied upon during the design experiment (persistent understandings). We were also interested in the ways in which these persistent understandings interacted with alternative mediational tools, intended to provide the student with access.

Methods

Case Study Participant History

Melissa was a 31-year-old woman, native English speaker, who identified as half Black and half White. Although she never received a disability diagnosis, she reported that, despite having a private tutor, she had persistent mathematics difficulties throughout her schooling. She took pre-algebra in middle school, and algebra in high school. After leaving high school in 11th grade, she worked in child care and senior care for ten years. When she returned to college, she took a mathematics placement test and was placed into the lowest level mathematics class offered by the college, which covered arithmetic content. She did not pass the class the first time and repeated it. She reported that the mathematics requirement was the primary barrier for her. During the data collection she completed a course entitled "Foundations of Algebra" which covered: variables and equations, linear equations, graphing linear equations, exponents and roots, quadratic equations, and polynomials. She reported doing all her homework and practicing problems "over and over and over again," but she still struggled to understand the content. She did not pass this class. She explained, "how my mind processes it, is quite different than the average person. It seems easy for other people, but for me you have to explain it in a different way." She explained that she did well in all her other classes, "it's just math that gets me."

We recruited Melissa from a pre-college mathematics class at a community college. All students in the pre-college mathematics class were given a written fractions assessment (Lewis et al., 2022), which has been shown to identify students who demonstrate unconventional understandings, characteristic of dyscalculia (Lewis & Thompson, 2015; Lewis et al., 2022). Melissa demonstrated unconventional understandings on this assessment and was invited to participate in an interview, formal assessment, and design experiment focused on algebraic concepts. On the Woodcock-Johnson Test of Achievement IV (WJ-IV; Schrank et al., 2014), Melissa scored between the 13th and 29th percentile on the mathematics subtests (Applied Problems SS=83, PR=13, Calculation SS=92, PR=29, Math Facts Fluency SS=89, PR=24). Melissa's composite mathematics score was at the 19th percentile – which is below the 25th percentile – the most commonly used cutoff for determining dyscalculia eligibility (Lewis & Fisher, 2016).

Although Melissa expended considerable effort and reported that she had sufficient resources and instruction, she had a history of struggling and repeatedly failing mathematics classes. She also demonstrated unconventional understandings of fractions, characteristic of students with dyscalculia (Lewis et al., 2022). These data along with her mathematics achievement score (below the traditional cutoff), and the difficulties evident during our one-on-one sessions, suggest that she meets the criteria for dyscalculia (see Lewis et al., 2020 for more details).

One-on-One Design Experiment

We conducted 19 videotaped design experiment sessions with Melissa. This design experiment involved iterative microcycles of design, enactment, and analysis (Gravemeijer & Cobb, 2006). Each microcycle involved designing and enacting an individual, one-on-one instructional session and then analyzing a video of the session in order to design the subsequent session. The first and third authors participated in the design microcycles and the first author (Katie) was the tutor for all sessions. The goal of these sessions was to identify the ways in which Melissa used mathematical tools in unconventional and problematic ways and to provide Melissa with alternative mediational tools to support her understanding (for more details about this iterative approach to design see Lewis et al., 2020). In designing alternative mathematical mediators we (a) drew upon prior research on the teaching and learning of algebra with nondisabled students (e.g., Kieran, 2007), (b) leveraged instructional recommendations offered by an adult with dyscalculia who developed ways of compensating (Lewis & Lynn, 2018) and (c) built upon Melissa's intuitive notations about mathematics and what she reported was more or less effective for her. In our design we aimed to provide Melissa with mediators that would help support a conventional understanding of algebra, specifically solving for an unknown, which is a core algebraic concept.

Retrospective Analysis

After the conclusion of data collection, we began our retrospective analysis. We transcribed all video recordings and scanned all written artifacts. We parsed each transcript into individual problem instances, which began with a question and ended with a student answer. The first and second author iteratively reviewed videos of each of the sessions and generated and refined analytic categories that captured the nature of the student's understanding. For example, in session 1 we noted that the student explained that unknowns were equal to 1. We noted this unconventional understanding and continued to iteratively refine this preliminary coding category as we reviewed subsequent sessions and learned more about how she thought of unknowns. At the end of this initial review of data we had produced a small set of operational definitions (see Appendix), which specified inclusion criteria and identified prototypical examples of each. Five persistent understandings were related to Melissa's understanding of algebra (see Appendix and see also Lewis et al., 2020 for a description of persistent understandings associated with integer operations). Three coders (first, fourth, and fifth authors) systematically coded each problem instance for correctness and any persistent understandings (see Figure 1 for an illustration of this systematic coding process). Each problem instance was coded by at least 2 coders. Reliability for the coding of the 5 algebraic persistent understandings was 95.4%. Any discrepancies in coding were resolved during our weekly research team meetings by rewatching the video and discussing whether there was sufficient evidence to warrant the attribution of that operational definition (for a similar approach see Schoenfeld et al., 1993; Lewis, 2014).

Findings

The detailed analysis of video recordings revealed a collection of five persistent algebraic understandings that reoccurred, were unconventional, and led to difficulties (see Appendix for operational definitions of each of these persistent understandings). These persistent understandings were related to (1) the value of unknowns, (2) the equal sign, (3) coefficients, (4) the meaning of "x=", and (5) the value of zero. In this section, first we present a high-level overview of the analysis with a graphic illustrating her performance across the sessions. Second, we provide a detailed view of each persistent understanding. For each persistent understanding, we provide a description, illustrate the ways in which this persistent understanding led to unconventional answers, and report the frequency of this understanding throughout the sessions. We then illustrate how these persistent understandings often appeared in conjunction and disrupted her ability to make sense of algebraic content, like solving for an unknown. We end by considering how the alternative mediational tools designed for Melissa provided her with supports to ground her reasoning in physical quantities and increased her access, supporting her conventional use of mathematical mediators.

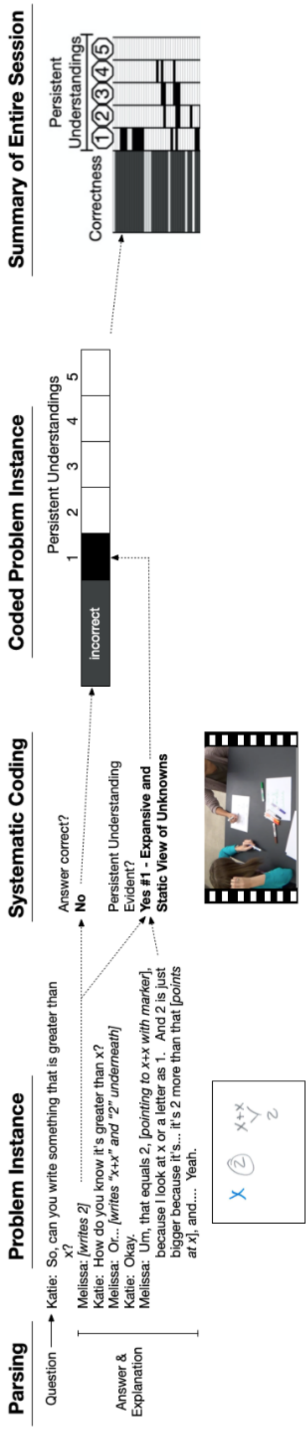
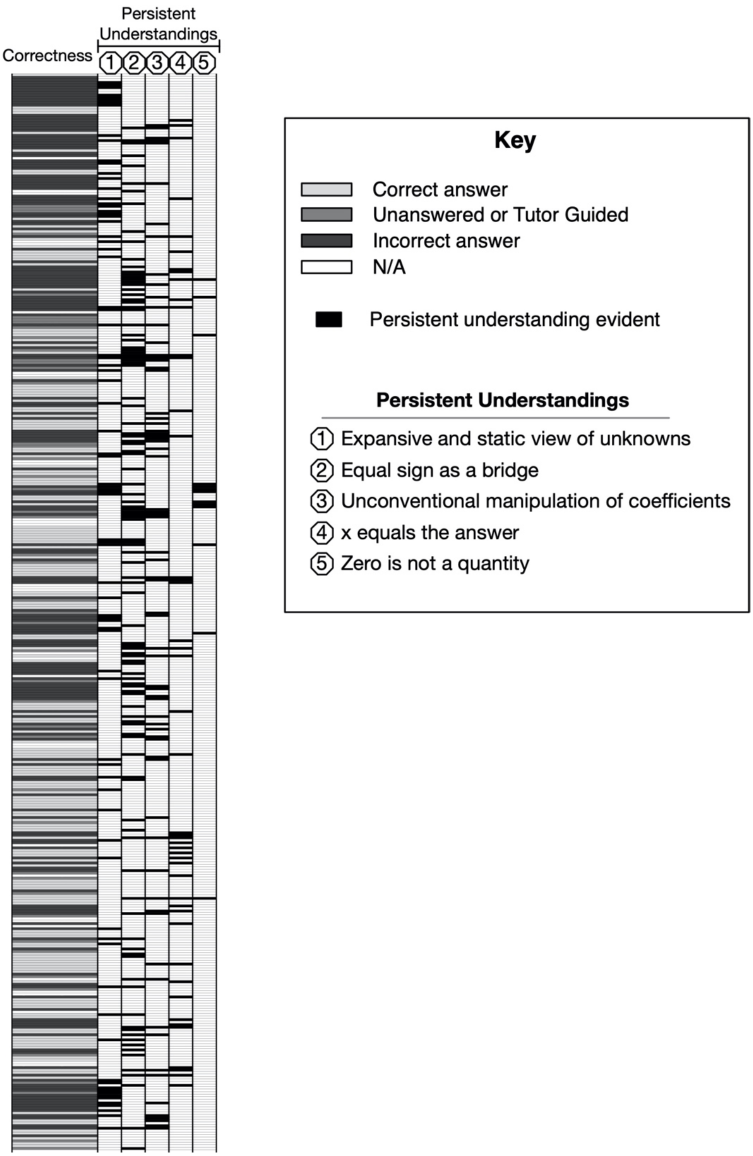


Figure 1. **Illustration of the parsing of problem instances, systematic coding, and representation of data.**



Note. Each horizonal segment represents a problem instance that starts with a question and ends with the student's answer.

***Figure 2.* Problem-by-problem coding of each problem instance involving algebraic content.**

To provide a high-level view of the 19 instructional sessions, the correctness of the student's answer and evidence of any algebraic persistent understandings is shown in Figure 2. Within these 19 individual sessions, there were 427 problem instances that involved algebra content (for analysis of integer problems please see Lewis et al., 2020). Each horizontal segment represents an individual problem instance, in chronological order. This view of the data provides a sense of the student's ongoing difficulties (as illustrated by the prevalence of incorrect answers) and reveals the persistence of each of the understandings identified. Of the 427 problem instances 186 were coded as incorrect (44%) and 90% of these incorrect answers were associated with either an unconventional integer persistent understanding (see Lewis et al., 2020) or one or more of the five algebraic persistent understandings. Only 19 incorrectly answered problems were *not* associated with one of these persistent understandings. These 19 problems were further analyzed. These incorrect answers involved calculation errors (n=5), miscounting (n=4), incorrect operation (n=3), or did not include an explanation (n=7) and were therefore lacking sufficient data to warrant classification. Therefore, we argue that these five persistent understandings (along with the integer persistent understandings, Lewis et al., 2020) provide a relatively comprehensive explanatory frame for the difficulties that the student experienced. We now present each of these persistent understandings and provide examples and excerpts to illustrate each in turn.

Persistent Understanding #1: Expansive and Static View of Unknowns

The first persistent understanding – *expansive and static view of unknowns* – involved an unconventional understanding of algebraic unknowns and their values, that was both overly expansive and overly static. Melissa was overly *expansive* in her definition of unknowns, in that she used the term "unknown" or "variable" to refer to any non-numeral mathematical symbol. She explained, "a variable is a… an unknown number," and that a variable could be an "addition or subtraction problem" and then identified a whole range of different mathematical symbols (e.g., +, π, [, x, m, ÷, <, =) as variables. She explained, "[The equal sign] is a variable, as well as an x is a variable, or a plus is a variable." In addition to treating all symbols as variables, she was also overly expansive about her understanding of unknowns in that she believed that an unknown (e.g., x) could be different values within the same problem (e.g., 2x+4=3x), and argued that even after solving for x, that that unknown could still be anything.

Although she was often overly expansive in her understanding of unknowns, she also demonstrated a *static* view of unknowns, and often asserted that unknowns were equal to 1. She explained, "The rule of x is 1, that's the most common unknown, in other words, for x to be 1." For example, during one session Katie asked her to write a value that was greater than x. She incorrectly determined that 2 was larger than x and explained, "because I look at x, or a letter, as 1. And 2 is just bigger."

Believing that unknowns were static values equal to 1 was sometimes problematic when she attempted to solve for x. For example, when asked to solve 12=x+5, she replaced the x with a 1 simplifying the equation to 12=6, then divided both sides by 6 to get an answer of 2.

> Katie: Okay, so if we were looking at this problem, [*writing "12 = x + 5"; see figure 3a*], how would you solve that?
>
> Melissa: Well... it depends. Because the way I look at this, I'm like, well, x could equal 1 [*writes "1" see Figure 3b].* So therefore, I'd want to distribute – well, kind of distribute. So it would be fi– I would just, be 5 [*writes "5"*] and 5 [*writes "5" see Figure 3c*]. Or no, it would be... 6. So plus [*writes "+"; see Figure 3c*] equals 6 [*writes lines and "6"; see Figure 3d*] , and then under here it would be 6 [*overwrites the 5 with "6"; see figure 3e].* [*writes "12/6 =" and writes "6" underneath 6, writes "= 2"; see figure 3f*]

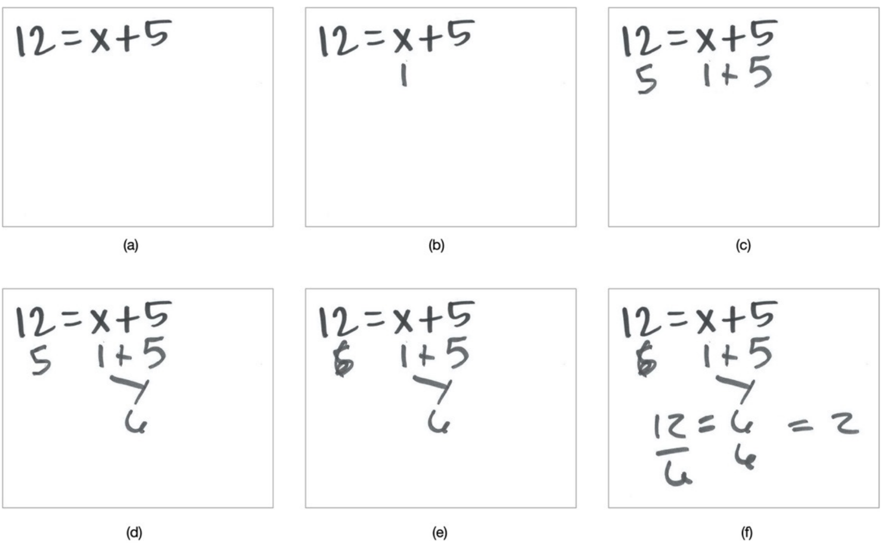


Figure 3. **Melissa's written work for the problem 12=x+5 digitally recreated to illustrate her process**

Melissa's static understanding of x, being equal to 1, was used in this example to create an invalid equation 12=6. She did not find this to be problematic and continued to procedurally manipulate the values, as if she was still solving for x, determining that the answer was 2.

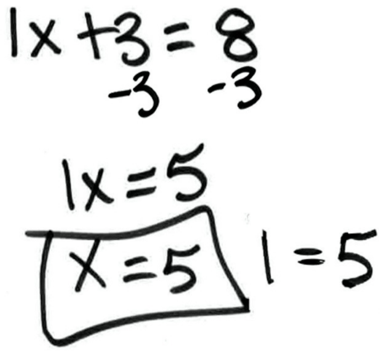


Figure 4. **Melissa's written work to solve the problem x+3=8**

In another example, when she solved the problem x+3=8, she correctly determined that x=5, but when Katie asked her what it meant that x=5 (a standard question), she explained that "one equaled five [*writes 1=5; see Figure 4*] because I see x as 1." In this instance, despite the fact that she had just determined that x was equal to 5, her understanding that x=1 emerged. This resulted in her creating an invalid equality 1=5, which she did not find problematic.

Both Melissa's understanding of x as a static value, equal to 1, and her overly expansive understanding of unknowns, which involved believing that x could be any value, even after determining the value of x, led to her unconventional use of unknowns and errors across the sessions. This persistent understanding was evident in 80 problem instances across the sessions, and 80% of the time was associated with an incorrect answer.

Persistent Understanding #2: Equal Sign as a Bridge

The second persistent understanding we identified was *equal sign as a bridge.* This persistent understanding was characterized by Melissa treating the equal sign as something which separates, but also allows movement of quantities, rather than a symbol that shows equality between two quantities. This resulted in Melissa often using an equal sign in between solution steps, having more than one equal sign in a given equation, or omitting the equal sign during her solution process. Because she did not use the equal sign to show equality between two quantities (or two sides of the equation), she did not see invalid equalities (e.g., -4=5) as problematic. She described an equal sign as a "bridge" that allowed quantities to move back and forth. Because of this she often performed invalid transformations, moving terms to the opposite side. For example, she rewrote 2x+3=3x as 2x+3x=3 – exchanging the location of the 3 and 3x. When asked about this kind of transformation, she said, "I'm sure it turns out the same. Well, it should. It should."

Because she did not think of the equal sign as showing equality, when solving for x she routinely performed different calculations to the left and right side of the equal sign. When I asked her what it meant to "solve for x" she explained that "x is going to be on one side, and a whole number is going to be on the other side." She focused on separating x from the rest of the equation, "I have to isolate, isolating x is necessary." When I asked her what it meant to "isolate x" in the context of a problem like x+3=9, she attempted to solve this problem by subtracting 3 on one side and dividing by 3 on the other. She explained, "Well I minused 3 on this side [*points to x+3*], for cross-canceling those [*crosses out the 3-3*], and I divided by 3 on this [*points to 9 ÷ 3*], equaling 3. I brought the x down, x equals 3." When I pointed out that she did different things to the sides of the equation, she did not find this problematic. For the left side she explained, "I wanted to minus it on this side, since it didn't have a numeral with this x [*points to x*]" and for the right side she explained, "When you have x on one side [*points to x+3*], you have to divide it on the other side [*points to right side of equation*]."

Melissa's unconventional understanding of the equal sign – treating it as a bridge in between intermediate steps of the problem or allowing quantities to move across it – occurred frequently in our sessions because of our focus on solving for unknowns. This understanding occurred 110 times across the sessions and was often (59%) associated with an incorrect answer.

Persistent Understanding #3: Unconventional Manipulation of Coefficients

A third persistent understanding we identified had to do with Melissa's *unconventional manipulation of coefficients.* Melissa often assumed an additive relationship of the coefficient and unknown, and would therefore inappropriately subtract the coefficient away from the unknown to solve for x. For example, when solving 15=5x, she subtracted 5 from both sides of the equation (15-5=5x-5), incorrectly simplifying the right side of the equation (i.e., 5x-5) to x, to incorrectly determine that x=10. When she did divide to simplify unknowns with coefficients, she often divided the sides of the equation by different amounts – commonly dividing the x term by the coefficient and x, rather than the coefficient alone. For example, to simplify 2x=12, she divided 2x on the left and 2 on the right (see Figure 5). She explained her process, "I brought the 2x underneath the 2x and did the same thing to both sides. I took 2x and 2x, and that canceled out, and then I brought the 2 over where 12 was, I brought the 2 underneath the 12 and divided 2 divided – or 12 divided by 2.

It equaled 6." In this problem she referred to the movement of the coefficient to the other side (*equal sign as a bridge)*, and although she said she "did the same thing to both sides" she divided 2x on one side of the equation and divided only by 2 on the other side (*unconventional manipulation of coefficients)*.

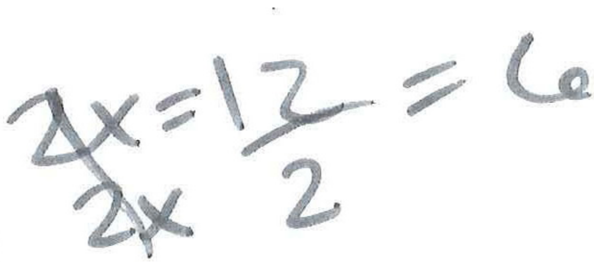


Figure 5. **Melissa's written work to solve the problem 2x=12**

Because Melissa often assumed that coefficients had an additive relationship with the unknown, she sometimes was confused by situations where x was represented with the coefficient of 1. For example, in the following problem (3x=2x+2; see Figure 6), Melissa subtracted 2x from both sides to determine that 1x=2. Although she had correctly solved the problem, she wanted to further simplify 1x, and so subtracted 1x from the left side and 1 from the right, determining (incorrectly) that the answer was x=1 (see Figure 6). This example illustrates both her difficulty in understanding that 1x=2 is the same as x=2, and her tendency to incorrectly subtract the coefficient away from the unknown. As evident in this example, when simplifying unknowns with coefficients she would often subtract or divide by the coefficient *and* the unknown on one side, and subtract or divide by the coefficient alone on the other side (see Figure 6).

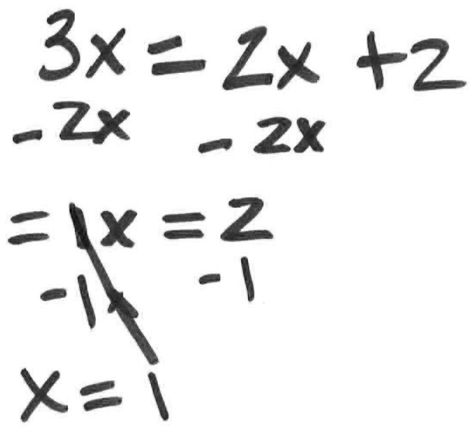


Figure 6. **Melissa's written work for solving the problem 3x=2x+2**

Melissa's unconventional understanding of coefficients led to difficulties. She often assumed an additive relationship between the coefficient and the unknown and she would often perform arithmetic calculations that were invalid, (e.g., subtracting the unknown away from the coefficient (2x-2=x) or subtracting or dividing by both the coefficient and unknown). This understanding occurred 70 times across the sessions and in 74% of cases was associated with an incorrect answer.

Persistent Understanding #4: x = the Answer

The fourth persistent understanding involved Melissa treating "x=" as a way of linguistically and symbolically demarking an answer, rather than defining the value of the unknown x. Melissa would often ignore the location of x in an equation and tack "x=" in front of the answer she calculated, regardless of whether it represented the value of the unknown. For example, in the following problem Melissa attempted to solve for x for the equation 2-x=-5 (see Figure 7). She added 2 to both sides of the equation, and incorrectly determined that the 2+2 would "cancel" resulting in: -x =-5+2. She then simplified the right side of the equation (-5+2) to -3. She then appended an x= to the front of the -3, without recognizing that the previous line of the equation established that -x was equal to that quantity.

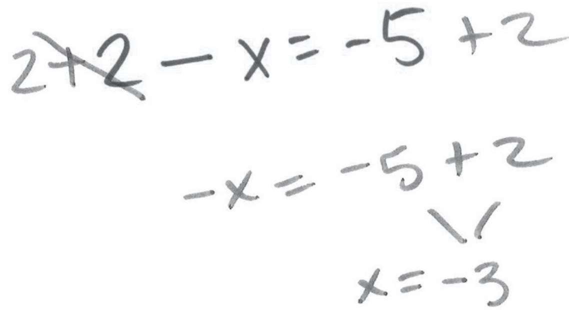


Figure 7. **Melissa's written work for the problem 2 - x = -5**

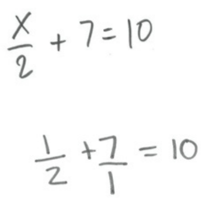
Perhaps because Melissa thought of "x=" as demarking and signifying the answer, she was confused by problems that asked her to interpret a quantity with a known x value. For example, when she was told that x=5 and asked to determine the value of 3x, she set up an equation, setting 3x equal to 5 (e.g., "3x=5"), and then attempted to solve for x. Because she thought of "x=" as signifying the answer, she created a situation where she could solve for x, when she was asked to produce an answer. This understanding occurred 46 times across the sessions and was often (74%) associated with an incorrect answer.

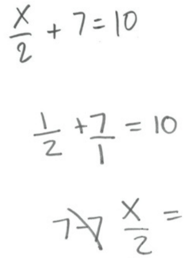
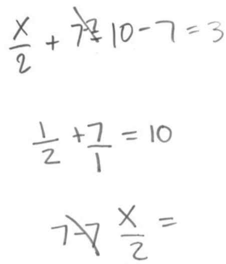
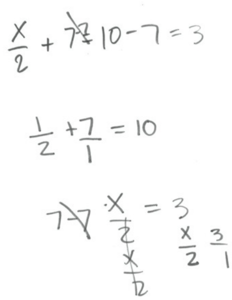
Persistent Understanding #5: Zero is Not a Quantity

The final persistent understanding identified was *zero is not a quantity*. Melissa did not think of zero as a valid quantity, was confused by situations in which x was equal to zero, and did not know how to simplify when zero was added or subtracted from unknown. For example, when attempting to solve the problem x+7=7, she subtracted 7 from both sides of the equation, to produce the equation x-0=0. She was unclear how to further simplify this problem, and so subtracted 0 from both sides of the equation – treating zero like any other constant. When she "cancelled" the zeros and determined that x=0, she rejected that answer. She explained, "x can equal one, but x can't equal zero." When Katie prompted her to say more about this assertion, she again argued that "x cannot equal zero" before clarifying that "x can equal zero, x can't be zero." When Katie asked her to explain, she rewrote the problem again, first subtracting 7 from both sides, and then subtracting zero from both sides, to determine that x=0, which she interpreted as "the answer is x," and added an arrow notation "x=0 → x". When Katie asked if a situation like x=4 feels different to her than a situation of x=0, she explained, "It is different, because zero has value when it's attached onto a 4 (writes a 4 in front of 0), it has to be attached to something for it to have value. Am I right? I suppose zero alone can't be value." Melissa did not understand the digit 0 alone as a value, and only saw 0 as a value if it was written with another digit (e.g., 40). Melissa had an unconventional understanding of 0, that involved zero not being a valid value, and resulted in difficulties in problems where x was equal to zero or when zero was added or subtracted to an unknown. Throughout the sessions, there were 13 instances of this understanding, and in 61% of these instances this understanding was associated with an incorrect answer.

Persistent Understandings Occurring in Tandem

To demonstrate how these persistent understandings often appeared together in the same problem, we illustrate how Melissa relied upon several persistent understandings as she solved the problem x/2+7=10. This episode was taken from the first instructional session as Katie tried to assess her existing strategies for solving for x. The prevalence of these persistent understandings in this first section suggests that Melissa came to this design experiment with these persistent understandings. In Figure 8 we present the transcript in the left column, the artifact (digitally modified to illustrate the relative state of student's work) in the center column, and the persistent understanding identified along with the rationale for that attribution in the right column. Figure 8 lays out how Melissa's attempts to solve this problem involved replacing the x symbol with a 1 *(static view of unknowns)*, using multiple equal signs in one equation *(equal sign as a bridge)*, moving quantities by performing different operations on each side of the equation *(equal sign as a bridge)*, dividing by both the coefficient and the x term *(unconventional manipulation of coefficients)*, and ignoring the x term in her calculations to determine "the answer" *(x is the answer)*. This excerpt illustrates how Melissa's persistent understandings often occurred together and resulted in difficulties

Transcript	Artifact	Persistent Understanding / Rationale
Katie: I'm going to throw a different one at you. [*writing "x/2 + 7 = 10"*] What if we had that? Melissa: Um, I would picture it as 1 over 2, plus 7 over 1, equals 10 [*as writing "1/2+7/1=10"*] Katie: Ok.		***Expansive and static view of unknowns*** Melissa replaced the variable x with the value 1 when she rewrote the equation.

Melissa: And – oh, no, I can't do that. Um, I would go 7 minus 7 [*writes "7-7"*], and cross-cancel that [*draws line through 7-7*], it would be the same thing over 2 [*writes "x/2"*], equals [*writes "="*]		None, but note omitted plus sign in her rewritten problem
Melissa: And that would equal 10 minus 7 [*writes "-7" on original problem after the 10*]. 7 minus 7 [*writes "-7" next to 7*], that cross-cancels [*crosses out 7-7]*, and that would equal 3 [*writes "=3"*]? Yeah, 3.		***Equal sign as bridge*** Melissa uses the equal sign to show the results of a calculation and has multiple equal signs within her equation.
Melissa: And then therefore I would go 3 [*writes "3" on bottom equation*], and then I would take x over 2 [*writes "x/2" below x/2 and then draws slash through both*], and then it would be x over 2 [*writes "x/2"*], 3 over so that would be 3 over 1 [*writes "3/1"*], and it would be... [*pointing back at written work*], x over 2, x over 2 [*pause*]... and then... um... minus 3, I confused myself. Okay, hold on.		***Unconventional manipulation of coefficients*** To simplify x/2=3, she divides both sides by the coefficient (1/3) and the x term.

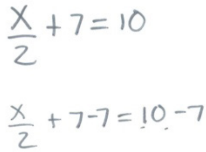
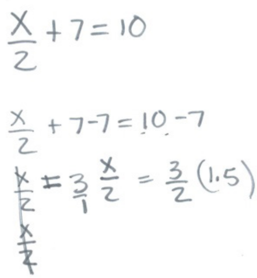
Katie: We can just get a new piece of paper if you want... Melissa: I'm going to turn this over here – [*flips over written work*], Katie: Let's keep the problem here. Melissa: Okay. x 2, plus 7 equals 10, [*writing "x/2 + 7 = 10"*] right? Katie: Mm hm [affirmative]. Melissa: Okay, so I would take – I would go x 2 plus 7 minus 7 equals [*writing "x/2 + 7-7 ="]* 10 minus 7 [*writing "10-7"*],		None. Note Melissa is referring to x/2 as x2.
Melissa: and I would go down here and go x over 2 uh, equals 3 [*writes "x/2 = 3"*]. And then I would go x over 2 [*writing "x/2"*], cross-cancel those [*draws line through x/2 and x/2*], and then I would go x over 2 [*writes "x/2"*], and 3 over 1 [*writes "/*1" below 3] equals 3 over 2 [*writes "3/2"*]. And then it would be a decimal because it would be 1.5 [*writes "1.5"*] if you were to divide it.		***Unconventional manipulation of coefficients*** To simplify x/2=3, she divides and multiplies by the coefficient and the x term. ***Equal sign as bridge*** Melissa divides by x/2 on the left and multiplies by x/2 on the right. She also uses the equal sign to show the results of a calculation resulting in multiple equal signs within one equation. ***x = the answer*** She ignored the x term when simplifying (3/1)(x/2) and treated the result of the calculation (1.5) as the answer.

Figure 8. **Illustration of the persistent understanding analysis of Melissa's solution to the problem x/2 + 7 = 10**

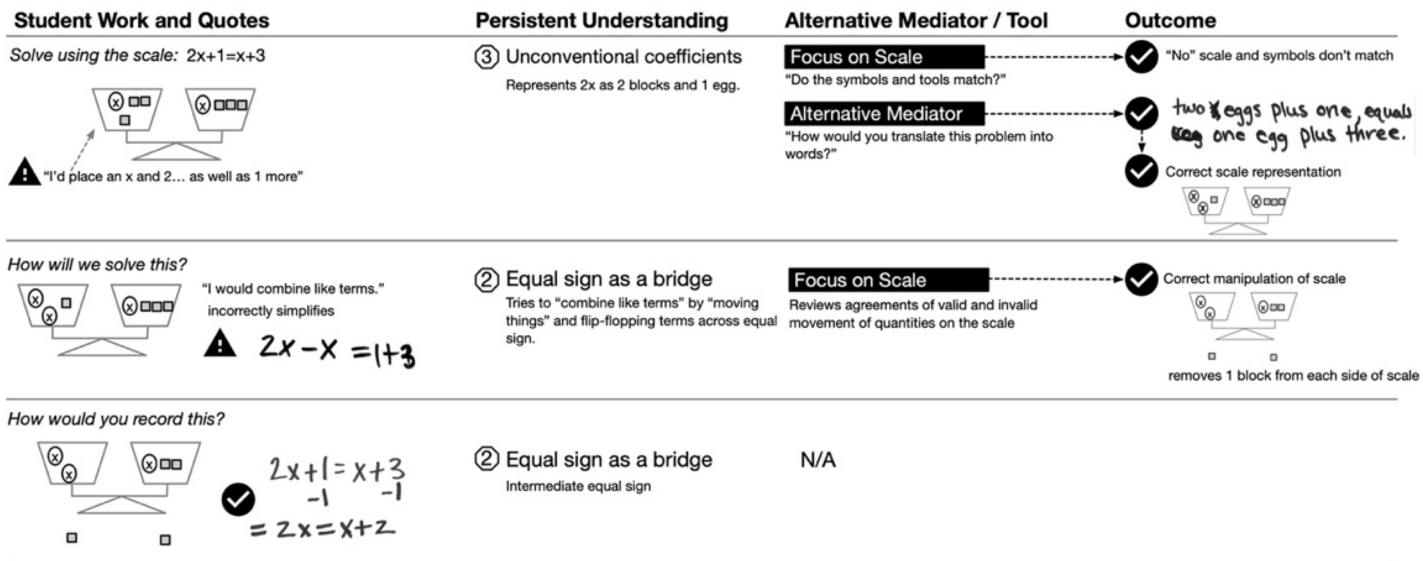
Instructional Design Decisions

To address the persistent understandings that were identified in the sessions the research team developed alternative mediational tools to support Melissa's understanding. After each session, the first author and a graduate research assistant (third author) reviewed the video recordings, made individual interpretations, and then met to discuss our interpretations, conjectures, and hypotheses and to make instructional decisions for the following session (Cobb et al., 2017). Although each of these persistent understandings was identified at different times, and we only fully defined and operationalized the persistent understandings in our retrospective analysis (after the sessions were completed), we attempted to design alternative mediational tools to address the unconventional understandings we identified in this preliminary analysis.

To address her tendency to think of any symbol as a variable and x as equal to 1 (*expansive and static view of unknowns*), we represented unknowns with plastic eggs labeled with "x", and used yellow base-ten block cubes to represent positive quantities, which aligned with how we had been using these blocks when solving integer operation problems. Katie explicitly told Melissa that for each problem all eggs (x) would have the same number of cubes inside them, but that between problems she would see Katie change the number in each egg, and Melissa's goal was to figure out how many blocks were in each egg. Our goal was for Melissa to begin thinking of unknowns or variables as a particular kind of mathematical symbol – one differentiated from other mathematical symbols – that stands in place of, or can physically hold an unknown value. We hoped that this physical representation of the unknown x as an egg would also help her reason about contexts in which x=0 (*zero is not a quantity*), and understand that "x=" indicates the quantity in the egg, rather than *x=the answer*. In later sessions we also began using contextualized word problems and jellybeans in place of blocks, to help Melissa focus on the number of jellybeans in each egg.

To address Melissa's belief that she can simply move terms to the other side of the equal sign (*equal sign as bridge*), we introduced the scale as a tool to help provide a physical experience with equivalence. An equation was represented on a two-sided balance scale, with the equal sign in the middle. This enabled us to discuss valid and invalid transformations to equations, and draw upon her intuitive understanding of balance.

To address Melissa's tendency to treat the coefficients as if they had an additive relationship (*unconventional manipulation of coefficients*), and her tendency to perform different operations to each side of the equation, we often had Melissa translate the symbolic equation into written words. 2x was therefore represented as "2 eggs" which could be physically represented on the scale. We hoped that this would help her differentiate between constants and coefficients.


Student Work and Quotes
Persistent Understanding
Alternative Mediator / Tool
Outcome
Solve using the scale: 2x+1=x+3
"I'd place an x and 2… as well as 1 more"
③ Unconventional coefficients
Represents 2x as 2 blocks and 1 egg.
Focus on Scale
"Do the symbols and tools match?"
"No" scale and symbols don't match
Alternative Mediator
"How would you translate this problem into words?"
two eggs plus one, equals one egg plus three.
Correct scale representation
How will we solve this?
"I would combine like terms."
incorrectly simplifies
2x − x = 1+3
② Equal sign as a bridge
Tries to "combine like terms" by "moving things" and flip-flopping terms across equal sign.
Focus on Scale
Reviews agreements of valid and invalid movement of quantities on the scale
Correct manipulation of scale
removes 1 block from each side of scale
How would you record this?
2x+1 = x+3
-1 -1
= 2x = x+2
② Equal sign as a bridge
Intermediate equal sign
N/A

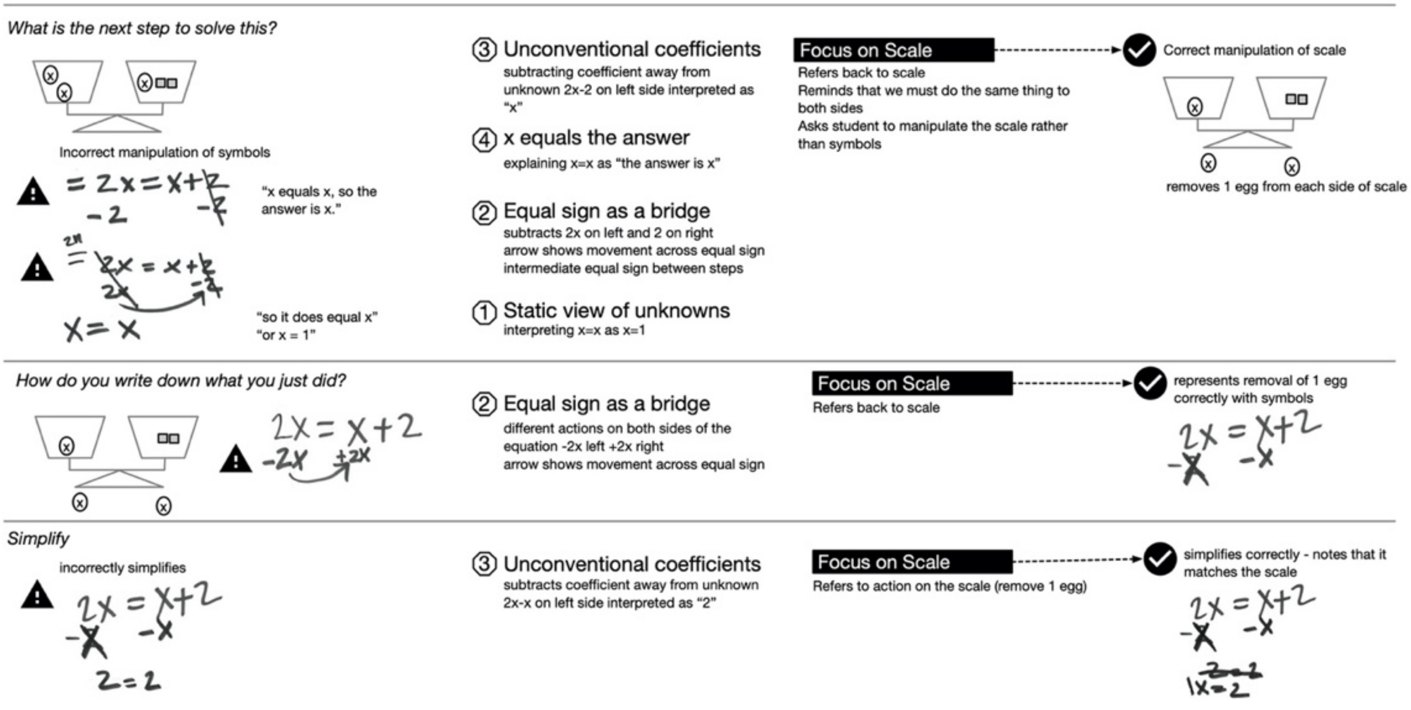


Figure 9. **Illustration of student's work and quotes, the persistent understandings that emerged, the tutors reorientation to alternative tools and mediators, and the outcome of this reorientation, while the student solved the problem 2x+1=x+3 with the scale.**

When each mediational tool was introduced, we spent some time connecting the physical manipulatives to the symbols. We generated written agreements about the tools and valid and invalid use of the mediators (e.g., each egg has same amount, you can add or remove a given number of blocks or eggs from each side of the scale, you cannot move a block or egg from one side of the scale to the other). Melissa's use of these alternative mediators provided us with greater insight into these persistent understandings, and also provided productive supports for her to address the ways in which these unconventional understandings were detrimental to her learning. To illustrate this, we present a focal episode from shortly after the introduction of the scale and the eggs.

Immediately before this focal episode, we had explored valid and invalid actions on the scale, and she was able to solve both 5+x=12 and 10=3+x+2 successfully using the scale and the eggs. In this focal episode we illustrate how Melissa's persistent understandings emerged when she was asked to solve the problem 2x+1=x+3 with the scale. Throughout this problem Melissa kept attempting to solve this problem algebraically with the symbols, each time resulting in her persistent understanding emerging, but repeated reorientation to the scale manipulatives and alternative tools (written words, scale, eggs) helped her correctly work through this problem. Figure 9 illustrates the problem that Melissa was solving, illustrations and quotes of her progress, the persistent understandings that emerged, Katie's reorientation to the alternative mediational tool, and the outcome of this reorientation. This episode illustrates both how Melissa's persistent understandings often reoccurred when working with standard algebraic notations, and also how working with the tools enabled her to access the problem, solve for the unknown, and represent her actions on the scale with standard algebraic notation.

Effectiveness of the Alternative Mediational Tools

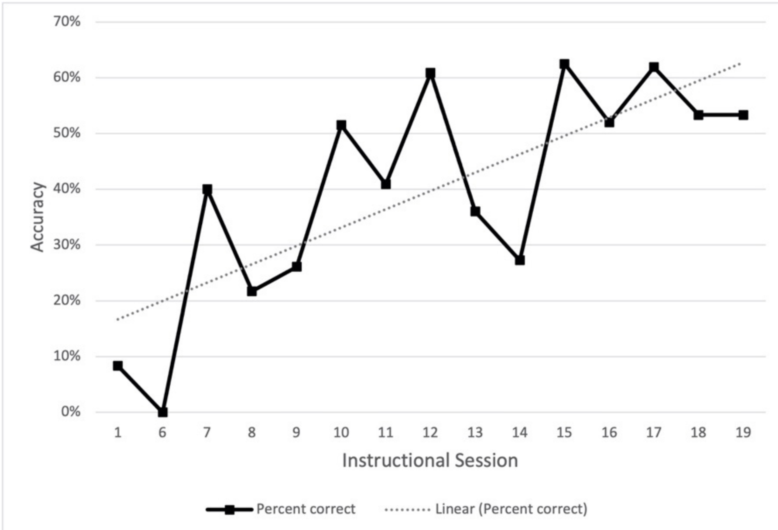


***Figure 10.* Melissa's accuracy in each session on problems where Melissa solved for x (with trendline to illustrate change over time).**

To illustrate the implementation of these instructional approaches over time and reflect upon the ways in which this supported her in making sense of contexts, we provide an illustration of the percent of problems she answered correctly that involved solving for x (see Figure 10). In the first two sessions focused on algebraic content, Melissa's attempts to solve for x were largely unsuccessful. In the next session (session 7) we introduced the eggs and cubes as a way of helping Melissa make sense of unknowns. In session 8 when she had difficulty interpreting what the coefficients meant in relationship to the eggs, the tutor asked her to translate the problem into words. Because her understanding of valid and invalid transformations of equations was an issue, in session 9 we introduced the scale as a model for solving for x. We introduced word problems (and switched from blocks to jelly beans) in session 15 after it became clear that Melissa was focusing on the total number of blocks on the scale rather than the amount in each egg. The trend line shows a general increase in her accuracy over time. It is important to note that our instructional goal was not to have her solve 100% of the problems accurately, but to provide her with tools which enabled her to make sense of and access the problems. When she was able to reason effectively the with the tools, we increased the difficulty of the problems to see if it would enable her to make sense of and solve increasingly complex problems. Providing Melissa with a tool to leverage her innate competencies (e.g., an intuitive understanding of balance), and continually reorienting to that tool, provided the student with support to address the persistent understandings when they emerged.

Discussion

This detailed case study of Melissa, an adult student with dyscalculia, found that she had persistent unconventional understandings of standard mathematical symbols (e.g., unknowns, the equal sign, coefficients, x=, and zero). Unlike the kinds of difficulties that all students experience when first learning a topic, these unconventional understandings were persistent. We argue that the persistence of these unconventional understandings suggests that these standard mathematical tools – used to represent quantities and relationships between quantities – were at least partially inaccessible to the student. Alternative mediational tools (re-mediations), which provided the student a way of drawing upon her intuitive understandings of physical quantity, weight, and balance enabled her to access and understand problems she had previously been unable to solve correctly. It is worth noting that the persistent understandings continued to emerge in the context of the alternative tools, but reorientation to the tools enabled Melissa to reason in more conventional ways.

In this section we first describe how these findings extend prior research on dyscalculia. Next, we argue that this kind of detailed case study work can provide important insights into dyscalculia, particularly in new mathematical content domains, like algebra. Finally, we consider the implications of this work for dyscalculia identification and intervention.

This detailed case study extends prior research on dyscalculia in several important ways. First, this research demonstrates how number processing difficulties found in younger students with dyscalculia (Landerl, 2013; Rousselle & Noël, 2007), occur in older students engaged in algebraic reasoning. Just as prior research has demonstrated that students with dyscalculia are slower and more error prone when asked to compare or manipulate *arithmetic* quantities (e.g., Desoete et al., 2012), Melissa often made errors (44% of algebra problems were incorrect) and she experienced persistent difficulties understanding, comparing, representing, and manipulating *algebraic* quantities. This study, therefore, extends findings that have been documented in students with dyscalculia, and begins to identify how these difficulties would emerge in an algebraic context. This kind of detailed case study can enable researchers to begin to explore mathematical topic domains beyond basic arithmetic, and provides much needed insight into dyscalculia across mathematical topic domains.

Second, this study offers an anti-deficit framing of dyscalculia by providing a detailed depiction of a student engaged in the process of learning and doing mathematics, rather than describing the student's performance from a deficit frame in terms of speed and accuracy. Unlike prior research on dyscalculia which infers learning difficulties based on patterns of errors on outcome measures (e.g., Bouck et al., 2016; Mazzocco et al., 2008), this study explored the student's reasoning underlying these errors. This study documented the ways in which Melissa was understanding, representing, and manipulating quantities, while engaged in problem solving. In this study, we analyzed the video data to determine the understandings she held and relied upon, rather than simply the skills that she lacked. This anti-deficit framing is critical for making progress in the field towards accurate identification of dyscalculia and offers new avenues to explore for re-mediation.

Implications for Identification

One benefit of this kind of detailed case study is the ways it enables researchers to make inroads into otherwise intractable methodological problems with respect to dyscalculia identification. Prior research has relied on low math achievement scores to identify students with dyscalculia (e.g., Geary et al., 2012). Unfortunately, this research has been hampered by an inability to accurately determine if a student's low score is due to dyscalculia or environmental factors (Lewis & Fisher, 2016; Mazzocco 2007). Detailed case studies, which identify the persistent understandings the student relies upon, enable researchers to begin to identify characteristics of the disability itself. The goal is to be able to more accurately define the disability by using behavioral characteristics. This definition can be used to develop screening measures to identify these characteristics, rather than relying upon the crude achievement measures that cannot differentiate dyscalculia and low achievement. Detailed case studies, like this, represent a first step towards more accurate behaviorally-based identification of dyscalculia. Future research is needed to determine whether these understandings identified in this study are unique to Melissa, or if they are typical of students with dyscalculia. A model for building from case study research to large scale studies that examine the prevalence of these characteristics in order to develop accurate screening measures has been demonstrated in the domain of fractions (Lewis et al., 2022) and could be extended to algebra.

Implications for Instruction

A second benefit of this kind of detailed case study research is that it fundamentally transforms the ways in which we think about instruction for students with dyscalculia. Rather than trying to identify and address supposed deficits within the student, we identify the ways in which the mathematical tools are inaccessible to the student and design alternative tools to increase their access. This is an important distinction because it changes the origin of intervention. Rather than trying to "fix" something internal to the student, it acknowledges the inaccessibility of mathematical tools and attempts to address that. The goal with the design of the alternative mediators was to provide a foundation, grounded in physical manipulation of quantities, and bridge to conventional representations of quantities in algebraic form. The goal therefore is to use these supports to ensure that the student has access to the canonical ways of representing and manipulating quantities. This kind of design experiment case study enables researchers to both identify these sites of inaccessibility and explore ways of meaningfully addressing them.

Limitations

Several limitations of the current study should be noted. An inherent limitation of case study research is that it is unknown if the characteristics identified for the student are unique to that individual, or whether they are typical of students with dyscalculia more broadly. Future research is needed to explore whether these persistent understandings occur in other students with dyscalculia. In addition, the focus of this design experiment was narrowly focused on the student's understanding of unknowns and solving for x. This represents just a small slice of the algebraic content area. Additional work needs to be done to explore how students with dyscalculia make sense of a range of algebraic content.

Conclusion

This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra. Findings suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon to design alternative tools to increase their access. Beginning to understand dyscalculia in algebra is critical, as algebra often acts as a gate keeper, like it did for Melissa, limiting students' academic and career opportunities.

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Appendix

Operational Definitions for the Persistent Understandings

#	Name	Operational Definition
1	Expansive and static view of unknowns	Problems were coded as indicative of an "expansive and static view of unknowns" if the student: (1) referred to any non-numeral mathematical symbol as a variable or unknown (e.g., π, =, >, []), (2) believed that *x* could be anything in all circumstances and that we could never know what an unknown is, even after solving for it, (3) did not understand that all unknowns (e.g., all *x*) in the same problem had the same value, (4) explained that a truly unknown value was equal to 1, or (5) referred to a coefficient and an unknown as just the coefficient (e.g., referring to 2x as "the 2").
2	Equal sign as a bridge	Problems were coded as indicative of an "equal sign as a bridge" understanding if the student (1) described her algebraic process as moving from one side to the other, particularly in instances where this was a non-valid transformation (e.g., take 1 cube away on 1 side and add it to the other side or simplifying 2x+3=3x to 2x+3x=3), (2) treated the equal sign as a boundary, separating the left from the right, rather than establishing equivalence (e.g., 1=4 was not treated as problematic), (3) performed different operations to each side of the equation to move terms to the other side of the equation (e.g., subtracting or dividing by different amounts), (4) used the equal sign to show that the calculations have yielded something (e.g., intermediate equal sign between equations), or (5) used a double equal ("==") as an assignment meaning (to set x equal to a value).

3	Unconventional manipulation of coefficients	Problems were coded as indicative of an "unconventional manipulation of coefficients" understanding if the student (1) assumed an additive relationship between the coefficient and the unknown rather than a multiplicative relationship (e.g., simplifying 2x=6 by subtracting 2 from both sides to produce x=4), (2) treated a constant as if it were a coefficient (e.g., 12=x+5, dividing both sides by 5 to solve for x), (3) divided by the coefficient and unknown (e.g., for 6x=12 dividing by 6x, rather than dividing both sides by 6), (4) tried to continue to "solve for x" when she determined the value of 1x, or (5) substituted the symbol x with a numerical digit, (e.g., if x=5 interpreting 2x as 25).
4	x = the answer	Problems were coded as indicative of an "x = the answer" understanding if the student (1) tacked on "x=" in front of whatever she calculated to be the answer, (2) ignored the location of x within the equation when solving for x, (3) believed that she solved for x by counting the number of cubes/jellybeans on the scale (e.g., 2 eggs=8 jellybeans, "the answer is 8", x=8, believes there is 8 in each egg), (4) totaled all the blocks in the problem (e.g., 5 cubes on scale plus 5 in the egg = 10) to determine "the answer", or (5) treated x with and without coefficients as if they were interchangeable, (e.g., treating 2x as an inseparable entity "the answer" or when given x=5 and asked to interpret 3x, setting 3x=5).

5	Zero is not a value	Problems were flagged as indicative of “zero is not a value” if the student (1) referred to calculations that equal 0 as “nothing” (e.g., “8-8 is nothing”) (2) argued that x cannot be equal to 0, (3) treated 0 as she would any other constant (e.g., x+0=8, subtracting 0 from both sides), or (4) explained that if x=0 then x equals any value (if x is nothing, it could be anything).