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COUNTEREXAMPLES: CHALLENGES FACED BY ELEMENTARY STUDENTS WHEN TESTING A CONJECTURE ABOUT THE RELATIONSHIP BETWEEN PERIMETER AND AREA

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Abstract

One pedagogical approach to challenge a persistent misconception is to get students to test a conjecture whereby they are confronted with the misconception. A common misconception about a 'direct linear relationship' between area and perimeter is well-documented. In this study, Year 4-6 students were presented with a conjecture that a rectangle with a larger perimeter will always have a larger area. Eighty-two (82) students' written responses from three elementary schools in Victoria, Australia were analyzed. The findings revealed that Year 4-6 students could find multiple examples to support the conjecture but they struggled to find counterexamples to refute the conjecture. The findings underscored the importance of developing elementary school students' capacity to construct counterexamples and recognize that it is sufficient to offer one counterexample in refuting a conjecture about all cases. Implications for teaching practice to support investigating and testing a conjecture are discussed.

Keywords: Counterexamples, Conjectures, Perimeter, Area, Elementary Students, Justifying

Abstrak

Salah satu pendekatan pedagogis untuk menantang miskonsepsi yang terus-menerus adalah membuat siswa menguji dugaan yang mana mereka dihadapkan pada suatu miskonsepsi. Kesalahpahaman umum tentang 'hubungan linier secara langsung' antara luas dan keliling didokumentasikan dengan baik. Dalam penelitian ini, siswa Kelas 4-6 disajikan dengan dugaan bahwa persegi panjang dengan keliling yang lebih besar akan selalu memiliki luas yang lebih besar. Delapan puluh dua (82) tanggapan tertulis siswa dari tiga sekolah dasar di Victoria, Australia dianalisis. Temuan mengungkapkan bahwa siswa Kelas 4-6 dapat menemukan banyak contoh untuk mendukung dugaan tersebut, namun mereka berjuang untuk menemukan contoh tandingan untuk membantah dugaan tersebut. Mengembangkan kapasitas siswa sekolah dasar untuk membangun contoh tandingan dan menyadari bahwa cukup menawarkan satu contoh tandingan untuk menolak dugaan tentang semua kasus menjadi perhatian utama dalam penelitian ini. Penelitian ini juga membahas terkait implikasinya pada praktik pengajaran untuk mendukung penyelidikan dan pengujian dugaan.

Kata kunci: Contoh Pembanding, Dugaan, Keliling, Luas, Siswa Sekolah Dasar, Pembenaran

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Generating counterexamples is challenging for students (Zaslavsky & Ron, 1998; Zazkis & Chernoff, 2008) and its role to refute a conjecture might not be recognized. The majority of studies investigating the use of counterexamples and examples in refuting a conjecture involved secondary or university students (Yopp, 2013; Zazkis & Chernoff, 2008). A small number of studies have focused on elementary students' capacity to work with examples and counterexamples (Knuth, Zaslavsky, & Ellis, 2019; Komatsu, 2010; Markovits, Brisson, de Chantal & St-Onge, 2016). Mathematical reasoning is one of the proficiencies in the *Australian Curriculum Mathematics* (ACARA, nd) and in addition to analysing and generalising, students are expected to learn to justify that is, "to prove that something is true or false."

Watson and Mason (2005) assert that having students search for and construct counterexamples deliberately to explore the limitations of a relationship might lead to a better understanding and a deeper appreciation of conjectures and properties. Creating a cognitive conflict by presenting a situation where students are confronted with a common misconception is recognized as a pedagogical strategy to help learners recognize and rectify their misconception (Limon, 2001; Tirosh & Graeber, 1990; Watson, 2007).

A common misconception about a 'direct proportional relationship' between area and perimeter was reported among students of different ages (Cavanagh, 2007; De Bock, Verschaffel, & Janssens, 1998; Tan Sisman, & Aksu, 2016; Tirosh & Stavy, 1999). Tirosh and Stavy (1999) reported that a high proportion of students assumed that a linear relationship exists between area and perimeter and envisaged when the area of a figure decreases or increases, the perimeter will also decrease or increase. They linked this phenomenon to student use of intuitive rule 'more A – more B'. Similarly, De Bock, Verschaffel, and Janssens (1998) observed this phenomenon among lower grades of secondary students and referred to this as 'the illusion of linearity'. Fernández, De Bock, Verschaffel, and Van Dooren (2014) extended earlier studies by De Bock and colleagues (e.g., De Bock, Verschaffel, & Janssens, 1998; Van Dooren, De Bock, Janssens, & Verschafell, 2008) by making a distinction between dimensionality and "directionality".

This study aims to examine upper elementary school students' capacity to generate examples and counterexamples to test the conjecture of a linear relationship between perimeter and area of a rectangle. The following research questions were addressed:

a. What understanding do elementary school students have of the roles of examples and counterexamples in the process of testing a conjecture?
b. How do elementary school students use examples and counterexamples to test a conjecture?
c. What levels of justifying are evident when testing a conjecture?
d. How does the use of a task to test a conjecture reveal elementary students' understanding of the relationship between perimeter and area?

Use of Counterexamples and Examples in Refuting A Conjecture

Earlier studies (Goldenberg & Mason, 2008; Pedemonte & Buchbinder, 2011; Watson & Mason, 2005) ascertain different roles and uses of counterexamples and examples in mathematics learning. Effective construction and use of counterexamples and examples require strategic thinking beyond algorithmic or procedural thinking. It is vital for students to learn about the limitation of the scope of examples in proving. That is, examples could not be counted as proof because it violates the intellectual-honesty principle of proof (Buchbinder & Zaslavsky, 2019; Stylianides, 2007). Pedemonte and Buchbinder (2011) recognized different levels of efficacy in example use. They argue it is necessary to have a cognitive unity and structural unity between the argumentation leading to a conjecture and its subsequent proof in order for examples to be productive in proving.

There is prevalent use of examples as a form of justification in elementary school (e.g., Carpenter et al., 2003; Goldenberg & Mason, 2008; Martin & Harel, 1989; Mason, 2019). An explicit attention to build students' capacity in using and choosing examples and counterexamples is key in order to lay a solid foundation for a more formal mathematics, particularly in relation to proof (Martin & Harel, 1989, Mason, 2019; Stylianides, 2007). Martin and Harel (1989) stated "If [elementary] teachers lead their students to believe that a few well-chosen examples constitute a proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students" (pp. 41-42). Mason (1982) and Ellis et al. (2019) discuss ways that teachers might engage students in exploring examples and counterexamples.

Mason (1982) argued that conjecturing involves a cyclical process that requires verifying a conjecture, checking if the conjecture encompasses all identified cases and examples, and testing the conjecture by trying to refute it using a counterexample. Ellis et al. (2019) examined the use of examples in exploring conjectures and developing appropriate justifications and distinguished two different ways to view examples connected to different mathematical reasoning processes. In exploring conjectures, students might use examples to explore and make sense of the conjecture or use counterexamples to refute a conjecture. Secondly, examples might be used to form a new conjecture. In the justifying process, examples might be used to "convey the claim of the conjecture is true (or false), or to convey a general argument" (p. 269).

A counterexample is a mathematical concept that is used to test the limitation of a relationship between mathematical concepts or to contest a conjecture (Komatsu, 2016; Watson & Mason, 2005; Yopp, 2013; Zazkis & Chernoff, 2008). Counterexamples play a critical role to "delineate the example space… and to understand and appreciate conjectures more deeply" (Watson & Mason, 2005, p. 60). However, the efficacy of counterexamples relies upon a learner having a personal history of constructing counterexamples (Watson & Mason, 2005; Zazkis & Chernoff, 2008). Zazkis and Chernoff (2008) stated "Different counterexamples, while serving the same mathematical purpose of rejecting a conjecture, may not be equally effective in serving a pedagogical purpose of helping a learner recognize the faulty conjecture." (p. 206).

Research on secondary student difficulties with counterexamples revealed that students had trouble in accepting the logic that a counterexample refutes a rule (Stylianides & Al-Murani, 2010; Peled & Zaslavsky, 1997; Zaslavsky & Ron, 1998). Widjaja et al. (2021) previously reported Year 3 and 4 Australian and Canadian students' capacity to search for examples and counterexamples when testing a conjecture that was true for a task called "Magic V" (NRICH, 2018). They found that some students argued that because they could not find counterexamples then the conjecture that a Magic V using the numbers 1 to 5 could not have an even number in the vertex was true. These students used the absence of counterexamples, rather than a logical argument, to accept the conjecture. They did however believe that if they could find a counterexample they would be able to refute the conjecture. Zazkis and Chernoff (2008) attributed the challenges faced by pre-service elementary teachers in realizing the significance of a counterexample in refuting a conjecture to an assumption that students would follow a proof scheme similar to the expert's proof scheme. Furthermore, they argued that some students might not grasp the significance of counterexamples and dismiss them as an exception. Similarly, Stylianides and Al-Murani (2010) reported that some secondary students maintained that a true mathematical statement and a counterexample could co-exist together.

Factors Contributing to Misconceptions about Perimeter and Area

Several researchers (Grant & Kline, 2003; Kamii & Clark, 1997; Moyer, 2001) argued that the pedagogical approach in teaching measurement place too much prominence on the measurement procedures of 'how to measure' and not enough emphasis on the key attributes and ideas of measurement in order for students to attach meaning to the concept of area and perimeter. Lack of understanding of length and area and a hasty introduction of the formulas were attributed as possible reasons for students to overgeneralize the relationship between perimeter and area. This common misconception was also noted in The National Council of Teachers of Mathematics documents (NCTM, 1989), "Most students in grades 5–8 incorrectly believe that if the sides of a figure are doubled to produce a similar figure, the area and volume also will be doubled" (NCTM, 1989, pp. 114–115). Other researchers (Livy, Muir, & Maher, 2012; Yeo, 2008) observed a similar misconception among pre-service teachers and reported a strong reliance on procedural knowledge. This suggests that the confusion about the relationship between perimeter and area is persistent.

The *Australian Curriculum Mathematics* (ACARA, nd) for teaching measurement suggests a sequence that in Foundation year and Year 2 students estimate which one is bigger i.e., form a conjecture, and then use direct or indirect comparison to verify or refute the conjecture about which is bigger or to order of the size of objects. In Year 4 students are expected to explore the areas of different rectangles using concrete representations of metric units. They are expected to develop understanding of relationship between area and length and width of rectangle; not recognizing that a square is a rectangle may hinder development of this relationship. In Year 5 the focus is on formalizing formulae for calculating area and perimeter and in Year 6 they are expected to "solve problems involving the comparison of lengths and areas using appropriate units" (ACARA, nd). Whilst the curriculum does expect that students will begin to explore the relationship between area and perimeter in Year 4, the focus on understanding this relationship is not explicit in the curriculum statements for Years 5 and 6.

Mathematical Reasoning

Jeannotte and Kieran (2017) proposed two conceptual frameworks for mathematical reasoning - a process framework for reasoning in addition to a structural framework for reasoning. They distinguished mathematical reasoning processes into two broad categories of searching for similarities and differences, and validating. In their view, conjecturing fits under a reasoning process related to searching for similarities and differences whilst justifying is considered as a mathematical reasoning process related to validating. Jeannotte and Kieran (2017) argued that the process of justifying is associated with two epistemic paths and elaborated the distinction between the two as follows:

> The first is related to the justification of a conjecture that arises from the process of conjecturing. This passage allows for changing the epistemic value from likely to more likely… The second type of epistemic passage is related to a validation that changes the epistemic value from likely to true or false, without being considered necessarily as constituting the process of proving. (p. 12)

In their framework, Jeannotte & Kieren framework (2017) emphasise the importance of focusing on the processes aspect of reasoning and interrogating the connections between different reasoning processes of searching for similarities and differences, conjecturing and validating. Vale et al. (2017) introduced a framework called 'Mathematical Reasoning Actions and Levels (MRAL)'. This framework drew on earlier work (e.g., Carpenter et al., 2003; Ellis, 2007; Lobato, Hohensee, & Rhodehamel, 2013). It elaborated and extended the three 'reasoning actions': comparing and contrasting, generalising, and justifying by theorizing 'levels of reasoning using a generalising task of "What else belongs?" (Small, 2011). The MRAL framework was then used to map Year 3-4 and Year 4-5 students' reasoning when testing a conjecture that arose when exploring examples for the "Magic V" task (NRICH, 2018). Analysis of students' arguments led to revision of the levels of justification in the MRAL framework (Widjaja et al., 2021).

In the larger study, in order to support teachers to teach and assess elementary students' mathematical reasoning, we developed a generic assessment rubric (see Table 1) to assist teachers in developing awareness of students' reasoning actions and to assess their levels of reasoning (Loong et al., 2018). The rubric built on previous studies and was developed through an iterative design-based research process of design, testing with elementary school students, and getting feedback from teachers to refine the rubric. In this version of the rubric 'comparing and contrasting' was relabeled as 'analysing,' 'forming conjectures' was added to 'generalising', and 'logical argument' was included in the heading for 'justifying'. These terms were included to support teachers as they aligned with terms used in the *Australian Curriculum Mathematics* (ACARA, nd) to describe mathematical reasoning.

Table 1. Levels of Mathematical Reasoning (Source: Loong et al., 2018)

	Analysing	Generalising	Justifying and Logical argument
Not evident	• Does not notice numerical or spatial structure of examples or cases. • Attends to non-mathematical aspects of the examples or cases.	• Does not communicate a common property or rule for pattern. • Non-systematic recording of cases or pattern. • Random facts about cases, relationships or patterns.	• Does not justify. • Appeals to teacher or others.

Beginning	• Notices similarities across examples • Recalls random known facts related to the examples. • Recalls and repeats patterns displayed visually or through use of materials. • Attempts to sort cases based on a common property.	• Uses body language, drawing, counting and oral language to draw attention to and communicate: o a single common property o repeated components in patterns. • Adds to patterns displayed verbally and/or visually using diagrams or through use of materials.	• Describes what they did and why it may or may not be correct. • Recognises what is correct or incorrect using materials, objects, or words. • Makes judgements based on simple criteria such as known facts. • The argument may not be coherent or include all steps in the reasoning process.
Developing	• Notices a common numerical or spatial property. • Recalls, repeats and extends patterns using numerical structure or spatial structure. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Describes the case or pattern by labelling the category or sequence.	• Communicates a rule about a: o *property* using words, diagrams or number sentences. o *pattern* using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • Explains the meaning of the rule using one example.	• Verifies truth of statements by using a common property, rule or known facts that confirms each case. May also use materials and informal methods. • Refutes a claim by using a counter example. • Starting statements in a logical argument are correct and accepted by the classroom. • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.
Consolidating	• Notices more than one common property by systematically generating further cases and/or listing and considering a range of known facts or properties. • Repeats and extends patterns using both the numerical and spatial structure. • Makes a prediction about other cases: • with the same property • included in the pattern.	• Identifies the boundary or limits for the rule (generalisation) about a common property. • Explains the rule for finding one term in the pattern using a number sentence • Extends the number of cases or pattern using another example to explain how the rule works.	• Uses a correct logical argument that has a complete chain of reasoning to it and uses words such as 'because', 'if…then…', 'therefore', 'and so', 'that leads to' ... • Extends the generalisation using logical argument.
Extending	• Notices and explores relationships between: o common properties o numerical structures of patterns. • Generates examples: o using tools, technology and modelling o to form a conjecture.	• Communicates the rule for any case using words or symbols, including algebraic symbols. • Applies the rule to find further examples or cases. • Generalises properties by forming a statement about the relationship between common properties. • Compares different symbolic expressions used to define the same pattern.	• Uses a watertight logical argument that is mathematically sound and leaves nothing unexplained. • Verifies that the statement is true or the generalisation holds for *all* cases using logical argument.

METHOD

The study reported here is part of a larger research project to develop teaching resources to support elementary teachers' understanding, teaching, and assessing of mathematical reasoning. In the current study, we used a task for which a counterexample could be used to refute the conjecture, but given the documented misconception, discussed above, students are likely to initially believe the conjecture.

The task was presented to students as follows:

Nathan said: "When you increase the perimeter of a rectangle, the area always increases".

Explain why or why not Nathan might be correct? Is this statement true for all cases?

The aim of the lesson was to engage students in testing a conjecture and to confront a potential misconception that a larger perimeter will always result in a larger area. In particular, students will learn that it is sufficient to offer one counterexample to refute a conjecture. In supporting and challenging students to test and justify a conjecture, students were expected to use their understanding of area and perimeter to select examples or counterexamples (analysing) and to use the results of their trials to refute the conjecture (justifying).

While all the schools and the teachers chose the same task for their students, they set out the task differently with their students. In School A, the teachers presented the conjecture and discussed/presented the rules for finding perimeter and area (see Figure 1a). The students worked in pairs to respond to the task using a blank sheet of paper. In school B, students recorded their exploration of examples and counterexamples in a blank sheet of paper and they mainly worked individually with some exception of a group of students worked in a small group of three. In School C, the teachers differed in their introduction of the task. One of the teachers introduced the task and then asked students to work in pairs to explore different rectangles using a geoboard on their iPad and then record their examples in a table (see Figure 1b). Another teacher from School C introduced the task and provided her students with concrete materials (tiles) to generate different rectangles and record their rectangles in a table.

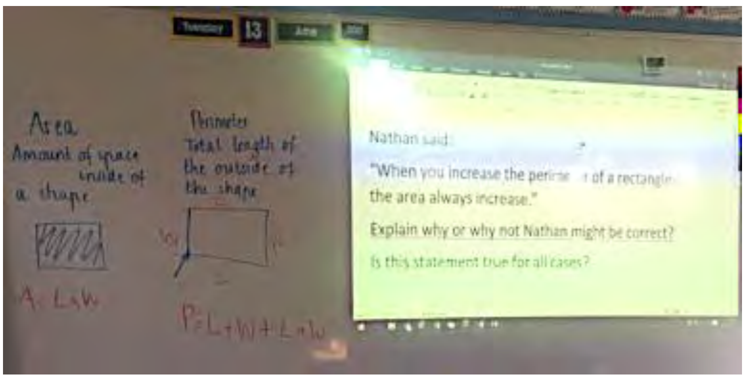

Figure 1a. A presentation of the task in School A

Figure 1b. A pair of students working on the task in School C

Participants

In total, 119 elementary school students from three elementary schools in Victoria participated in the project (see Table 2). Students typically worked in pairs on the task in Schools A and C and individually at School B.

Table 2. List of participating schools and year levels

School	Year level	Teachers	Students
A	4	2	28 (14 pairs)
B	5 & 6	2	51 (47 individuals, 2 pairs, 1 group of 3)
C	5	2	40 (2 individuals, 19 pairs)

Methods of Data Collection

Data were collected from three elementary schools in Victoria, Australia. The teachers participated in the professional development session delivered by the research team and were given resources to support the teaching of the task including suggested prompts to elicit, support and challenge reasoning, along with anticipated student solutions with examples of how to use the reasoning rubric to assess students' mathematical reasoning. Following the lesson, each pair of teachers participated in a discussion with the research team to examine samples of student written work, identified different levels of student reasoning actions based on their work samples and informed by classroom observations.

The task was taught by the classroom teacher in each school as a one-off mathematics lesson and not as part of a sequence of lessons on area and perimeter due to a logistical limitation. The lesson was observed by another teacher who taught the same year level from each participating school, and two members of the research team. In the lesson materials, there was a clear expectation for students to communicate their reasoning to one another and to the teacher. However, as the schools and the participating teachers were relatively new to mathematical reasoning, it was unclear if there was an established classroom culture that expected students to communicate their reasoning in their regular classroom practice. The teachers interacted with students during the lesson using prompts provided in the materials. The nature and content of these interactions was not a focus of this study. Rather, evidence of students' reasoning, both their analysing and justifying actions, was gathered from their written work as well as their verbal and non-verbal communication captured in the videos of paired and whole class discussions.

Data Analysis

Using the levels of reasoning framework (Loong et al., 2018), the levels of justifying of 82 work samples were classified (Table 2). Some work samples were from pairs of students and some were individual student's work. They were analysed for the reasoning processes of justifying in line with the focus of the task. Evidence of students' reasoning actions of analysing and justifying were gathered from their written work and were analysed using the rubric presented in Table 1. The first author used the levels of mathematical reasoning to classify the levels of justifying based on the written work collected which was then checked and verified by the second author using multiple rounds of coding and checking the coding (Corbin & Strauss, 2008). Categories of example usage (random, ordered, systematic) were generated in line with the levels of reasoning for analysing of levels of reasoning framework in Table 2. Selected written work from the three participating schools will be presented and discussed to elucidate elementary school students' use of examples and counterexamples to test Nathan's conjecture.

RESULTS AND DISCUSSION

Different levels of justifying were identified among students' written responses to the task and the analysis of these work samples also revealed the analysing processes students used to test the conjecture. We found that there were students at each school whose reasoning was classified at each of the justification levels, except School A, where no Year 4 students demonstrated reasoning at a level higher than developing. In examining the developmental aspect of learning, we analysed evidence of students' justifying levels based on their written work and cross-tabulated it with the year levels (See Table 3). Some examples of students' responses to a conjecture and the levels of justifying are included in Table 4. More than half of the work samples indicated that students could not justify or they did not provide a coherent argument in responding to Nathan's conjecture. There were variations in students' analysing the conjecture that is, the process of exploring examples and counterexamples. Some used random approaches in their search for examples and counterexamples. Other students were more systematic when generating examples, for example increasing perimeter by changing the length and/or width, or keeping the perimeter or area the same to explore the area and perimeter of other rectangles. Some students were prompted by the teacher to use a table for a more systematic recording of their examples and counterexamples.

Table 3. Frequency of justifying levels for work samples grouped based on schools and year levels (n=82)

Schools	**Year levels**	**Levels of justifying**				
		Not evident	Beginning	Developing	Consolidating	Extending
A	4	3	10	1	0	0
B	5	13	12	9	5	0
	6	3	3	1	0	1
C	5	5	4	10	1	1
Total		24	29	21	6	2

Table 4. Frequency and examples of justifying levels for work samples (n=82)

Levels	Frequency (%)	Example of responses to Nathan's conjecture
Not evident	24 (29%)	Nathan is incorrect because [blank]
		Nathan is not correct because Matt proved that he is wrong by showing us a rectangle 50 by 1 so not for all cases. (Year 4, School A)
		Nathan is correct. Ms Bree showed me the answer and I wouldn't know the right answer (Year 5, School B)
		Making a graph helped me to understand how Nathan was wrong. (Year 5, School C)
Beginning	29 (35%)	We believe Nathan is correct and incorrect because different examples are different (Year 4, School A)
		Yes, I think Nathan is correct if the rectangle increases, so will the area and perimeter because the rectangle is getting bigger so will the number. (Year 5, School B)
		Nathan is correct because it always increases. It is not always the case because sometimes the perimeter can be small. (Year 5, School C)
Developing	21 (26%)	We believe Nathan is incorrect because we found an example that the area stays the same but the perimeter got larger. (Year 4, School A).
		I think Nathan is correct because if you increase the outside of a rectangle, it will always increase the inside of a rectangle. The rectangle will always be bigger, so perimeter and area will both get bigger (if might be different for a square. I don't know). (Year 6, School B)
		I learnt what area and perimeter and that the area is not always bigger than the perimeter. I also learnt what Geoboard was. (Year 5, School C)
Consolidating	6 (7%)	Nathan is incorrect because the area can stay the same even if the perimeter increases. The statement is true in some cases but not all. (Year 5, School B)
		With 12 blocks, a block of 6 by 2, the perimeter is 16cm and the area is 12cm^2. With the same amount of blocks, a block of 3 by 4, the perimeter is 14 cm. (Year 5, School B)
		I learnt that if you get a rectangle and increase both sides the area will increase, but if you decrease one side and increase the other side, the area will not increase. (Year 5, School C)
Extending	2 (3%)	I believe this statement is false. However, this is not true in every case. In A (a 6 cm × 4cm rectangle), the perimeter is 20 cm and the area is 24 cm^2. In B (a 23cm × 1cm rectangle), I increased the perimeter to 48 cm but the area decreased (23 cm^2). (Year 6, School B)
		It [the conjecture] does not work unless you add more cubes to the rectangle. We have learnt how to disprove mathematical hypothesis by testing its area and perimeter. (Year 5, School C)

The work samples are classified as *Not evident* level of justifying when students did not offer justification or appealed to authority or others in their justification. As an example, one student provided a justification stating: "Nathan is incorrect because Mathew proved he was wrong" (Table 3, row 1). This response indicated that these students wrote their justification after the presentation by other students at the end of the lesson. The distinguishing features between *Beginning* and *Developing* levels of justifying are characterized by the fact that the work at the beginning level seemed to focus mainly on extending both sides of the rectangles as they explored the example space (see Table 4, row 2). This led them to arrive at the conclusion to support Nathan's conjecture. Work samples categorized as beginning failed to recognize other ways of increasing perimeter to generate cases where a counterexample could be found. The work samples that were classified as *Developing* showed evidence that they could generate examples to verify Nathan's conjecture and they could also find counterexamples to refute Nathan's conjecture (see Table 4, row 3 and Figure 2). However, they did not arrive at the logical conclusion that a counterexample refutes Nathan's conjecture.

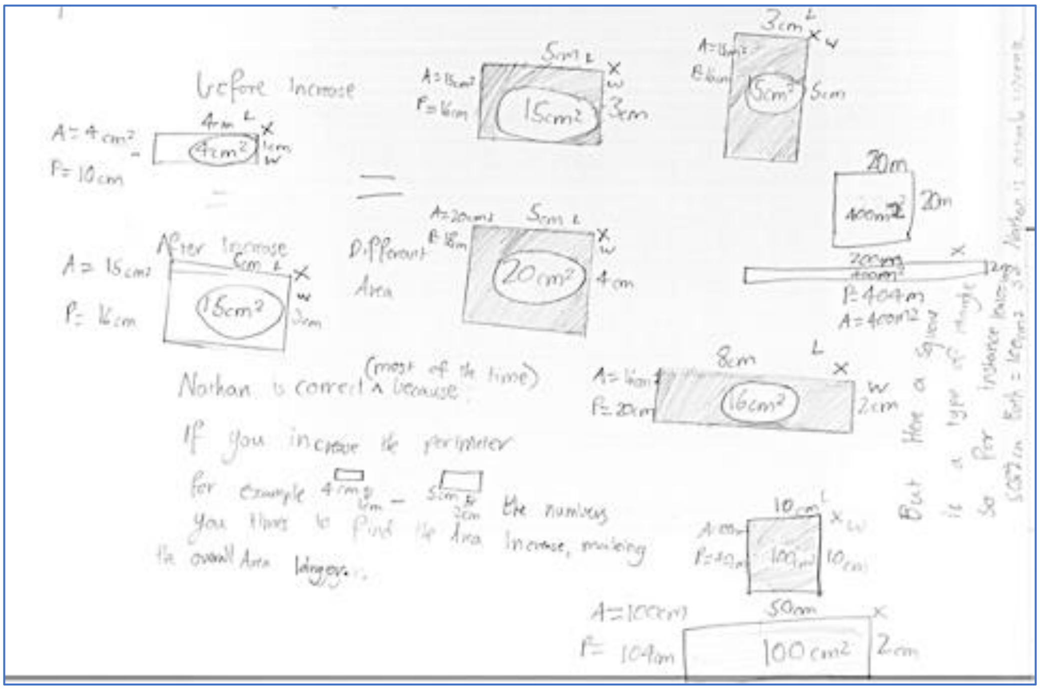

Figure 2. An illustrative work sample exploring area and perimeter of rectangles and squares (Yung Qi, Donald, &, Heather, Year 5, School B)

For instance, in Figure 2 the pair of students generated 10 rectangles. They initially compared 4 cm × 1 cm, 5 cm × 3 cm and 5 cm × 4 cm rectangles to show that Nathan was correct: "Nathan is correct because if you increase the perimeter for example [rectangle] 4 cm & 1 cm [and rectangle] 5 cm & 3 cm, the numbers you times to find the area increase, making the overall area bigger" (Yung Qi, Donald, &, Heather, Year 5, School B). They continued to search for examples to include an 8 cm × 2 cm rectangle with a perimeter of 20 cm and an area of 16 cm^2 but did not yet realize that this provided a counterexample when compared with the 5 cm × 4 cm rectangle. However, they continued their exploration and found a few counterexamples. They compared a 200 cm × 2 cm rectangle with a 20 cm × 20 cm square and a 10 cm × 10 cm square with a 50 cm × 4 cm rectangle. They changed their conclusion about the conjecture by including a qualification: "Nathan is correct (most of the time) because…" (Yung Qi, Donald, &, Heather, Year 5, School B). The fact that they found more than one counterexample might indicate that they did not realize that one counterexample would be sufficient to refute Nathan's conjecture.

As the previous work sample shows, while some students could find counterexamples, it was quite challenging for them upon finding counterexamples to refute a conjecture. This was evident in a work sample that was classified as *Developing* (Figure 3) where a systematic search for examples started by finding another rectangle with a longer length but a shorter width compared to the first rectangle. The third rectangle had the same area as the second rectangle with a larger perimeter. The students were able to identify this as a counterexample as evidenced by the asterisk. They went a bit further by finding another counterexample (marked by an asterisk), the fourth rectangle with a smaller area but a larger perimeter compared to the second rectangle. However, the reflection of what they have learnt did not show a coherent argument in response to Nathan's conjecture:

> That the area is not always bigger then [than] the perimeter… In some mathamaitals [mathematical] minet [minds] says it is bigger but in some ways it is not bigger…I in my way think that the area can be bigger but sometimes not. (Sarah & Lily, Year 5, School C)

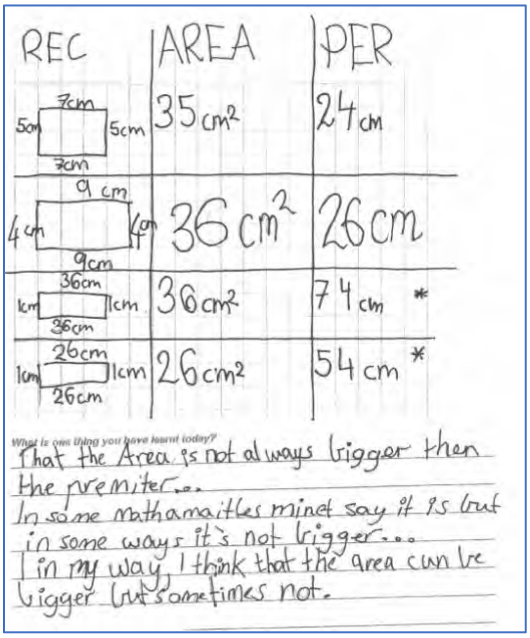

Figure 3. An illustrative work sample of developing level of justifying (Sarah & Lily, Year 5, School C)

The *Consolidating* level is characterized by evidence of a correct logical argument in refuting Nathan's conjecture. A work sample at the Consolidating level of justifying (Figure 4) showed evidence of a systematic search for examples and a clearer explanation about the process to reach a conclusion as recorded in their justification. "I learnt that if you get a rectangle and increase both sides the area will increase but if you decrease one side and increase the other side, the area will not increase". While this argument is logical and correct, it does not meet the requirement of a sound logical argument as decreases of either the width or length whilst increasing the other side do not always result in smaller area, as is claimed.

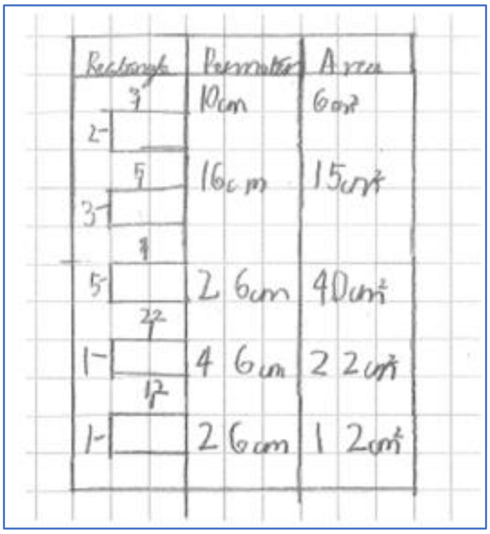

Figure 4. An illustrative work sample of consolidating level of justifying (Cody, Jay, & Neo, Year 5, School C)

Lastly, the *Extending* level is characterized by evidence of a sound logical argument in refuting Nathan's conjecture. A work sample of Extending level of justifying (Figure 5) showed evidence of a systematic search whereby students kept the area of the rectangles constant but altered the dimensions of the rectangles. As a result, they found different perimeters. Their justification "It does not work unless you add more cubes to the rectangles" (Archie & Scott, Year 5, School C) suggested that the students realized the power of counterexamples to refute a conjecture and a different outcome if they did not keep the area constant. They have identified that they have learnt "How to disprove mathematical hypothesis by testing its area and perimeter" (Archie & Scott, Year 5, School C). Hence, they have demonstrated evidence of an argument at the Extending level.

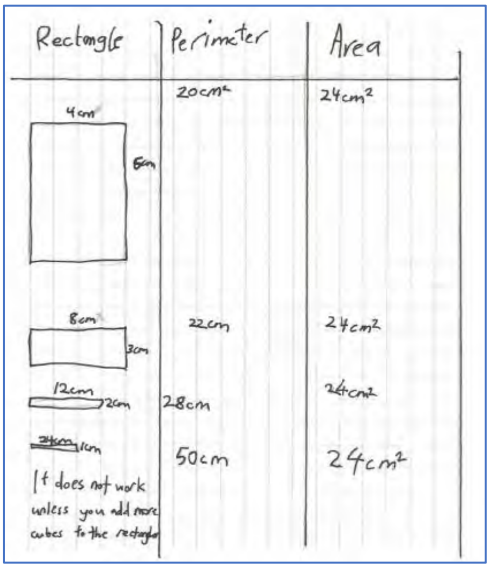

Figure 5. An illustrative work sample of Extending level of justifying (Archie & Scott, Year 5, School C)

The analysis of 82 work samples from Year 4-6 students revealed that refuting a conjecture using a counterexample was challenging for students at all year levels and that it requires a higher level of justifying compared to verifying truth of statements by using a common property, rule or known facts that confirms each case. In the reasoning framework (Loong et al., 2018), refuting a conjecture using a counterexample was classified as *Developing*. Findings from this study contend that it is necessary to revise the framework and include different levels for the way in which students use counterexamples as an argument to test and refute conjectures. The framework was subsequently changed so that "refutes a claim using a counter example" was included at the consolidating level in the Assessing Mathematical Reasoning Rubric (AAS, 2020) as shown in Table 5.

Our argument to reconsider the level of justifying in relation to refuting a conjecture using a counterexample is based on the perspective that the justifying process should be perceived not only as a *disciplinary practice* (Davis & Hersch, 1981; Lakatos, 1976) but it is important to also emphasize justifying as a *learning practice* (Cohen & Ball, 2001; Staples et al., 2012; Staples, 2014). Furthermore, Staples (2014) argued that "What "counts" as a justification is locally defined, and the nature of justification activity is locally constituted in the classroom through engagement of the members of the community. Hence we argue that 'refuting a conjecture using a counterexample' should be classified at the higher level of 'Consolidating' instead of 'Developing'. We posit the challenges evident in students' work samples relates to complexities related to aspects of proof, that needs to be a focus in the classroom when students communicate their argument (Stylianides & Ball, 2008).

Table 5. Assessing Mathematical Reasoning Rubric: Developing and Consolidating levels (Source: AAS, 2020; Loong et al., 2018)

	Analysing	Generalising	Justifying
Developing	• Notices a common property, or sorts and orders cases, or repeats and extends patterns • Describes the property or pattern.	• Generalises: communicates a rule (conjecture) using mathematical terms, and records other cases or examples.	• Attempts to verify by testing cases, and detects and corrects errors or inconsistencies. • Starting statements in a
Consolidating	• Systematically searches for examples, extends patterns, or analyses structures, to form a conjecture. • Makes predictions about	• Generalises: communicates a rule (conjecture) using mathematical symbols and explains what the rule means or explains how the rule	• Verifies truth of statements by confirming all cases or **refutes a claim by using a counter example.** • Uses a correct logical

Year 4-6 elementary students involved in this study were not accustomed to testing conjectures that were not true or realising that they only need to generate one counterexample to disprove a mathematical statement such as for Nathan's conjecture. Justification recorded in students' work samples at the 'Developing' level suggested that these students did not realize that a mathematics statement cannot be both true and false at the same time (Stylianides & Al-Murani, 2010; Zaslavsky & Ron, 1998; Zazkis & Chernoff, 2008). The analysis of the written work samples suggested that most students used a random strategy to generate examples that supported Nathan's conjecture. One approach to address the challenge of a widespread overreliance on examples as justification is "to help students to understand the limitations of examples (Sowder & Harel, 1998; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2011).

One of the teachers in School C provided their students with tiles and asked them to work with the tiles to generate different rectangles after realizing that many of their students would benefit from exploring the conjecture using concrete manipulatives. This teaching action is consistent with findings from earlier studies (Chen & Herbst, 2013; Komatsu, 2010; Lin & Tsai, 2016; Schifter, 2009) about the importance of using appropriate modes of argumentation and choice of representations such as concrete manipulatives in elementary school particularly to reason and communicate justification effectively.

The findings from this study does show that directly challenging students' misconceptions of the relationship between area and perimeter provided students with the opportunity to develop some understanding of the relationship. Whilst only a few students rejected the conjecture based on counter examples, about a third of students in this study did find counterexamples and realized that the conjecture did not work for all cases. The different approaches used by teachers to introduce the problem did influence students' strategies for generating the examples and counter examples and comparing and contrasting. For example, the use of concrete materials such as tiles and digital technology such as *Geoboard* and *Show me* supported students to investigate and justify their reasoning. The lesson materials included examples of prompts for teachers to use in their interactions with students such as "Is it always true or just sometimes true?" and enabling prompts such as "Have you searched for examples that show Nathan is not correct" (AAS, 2020)? However, it is evident that teachers were not experienced in providing prompts to challenge the students' misconceptions about the relationship between area and perimeter that persisted in the approach that students took to generate and compare and contrast their examples.

In examining elementary school students' use of examples and counterexamples in testing a conjecture using only written work, we realized the limitation of only using written arguments to classify their justifying levels. Campbell, King, and Zelkowski (2020) compared written and oral arguments of 47 Year 8 students who worked in groups to solve proving tasks. They found that students' oral arguments often were at the higher level than their written arguments. Their finding concurred with Soto-Johnson and Fueller (2012) who recommended the potential benefit of student audio-recording their oral reasoning to improve the quality of their written reasoning. We acknowledge this as a limitation of this study as we did not capture recording of students' oral reasoning. As described when discussing the response from the student who said that they had been convinced by another student that the conjecture was not true, it is therefore possible that their justifying may have been more sophisticated as they orally argued about the significance of counterexamples.

CONCLUSION

Testing a conjecture about the relationship between the perimeter and area of a rectangle was challenging for most students in this study. The majority of students confirmed the misconception that a rectangle with a larger perimeter will also have a larger area using examples as evidence, even though many of these students did find counterexamples. About a third of the students did make a qualifying argument about this conjecture but only a few students were able to provide a logical argument using a counterexample to reject the conjecture. Rubrics for assessing reasoning need to include this trajectory of understanding the use of counterexamples when justifying. It needs to include identifying counterexamples, qualifying conjectures using counterexamples and refuting conjectures using counterexamples.

Prompts to support systematic exploration of examples and to challenge students to search for counterexamples were provided for teachers but it is not clear that all the teachers were prepared to use prompts to address students' misconceptions about the relationship between perimeter and area as their student generated examples. This means that teachers need to understand the relationship between area and perimeter and use of a counterexample to refute a conjecture, as particular to mathematical argumentation, if students' misconceptions are to challenged and argumentation developed. Many of the opportunities for developing reasoning proficiency in the elementary mathematics curriculum are focused on making arguments about common properties or relationships (generalizations). More opportunities are needed for students to identify what is different, and to appreciate that in mathematics using a counterexample is an acceptable means of disagreement. The teachers in the three schools in this study presented the task and supported their students differently. Further research of the role of teacher knowledge and teacher actions when introducing the task, supporting students to explore and to encourage argumentation during orchestrated whole class discussion is needed to develop coherent approaches for developing students' use of counterexamples.

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