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Struggles Pre-Service Teachers Experience When Taking a Pre-Symbolic Algebra Content Course

Siobahn Suppa
Stockton University

Charles Hohensee
University of Delaware

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Pre-symbolic algebra has been advocated for as a mathematics topic elementary students should experience to better prepare them for middle and high school algebra. However, most elementary pre-service teachers have little to no experience with pre-symbolic algebra. The study reported here analysed the struggles that ten elementary pre-service teachers experienced when learning about pre-symbolic algebra in a mathematics content course. Three types of struggles emerged, struggles with changes in the artefacts of algebra, the objects of algebra, and pre-service teachers' role while doing algebra. This study could inform efforts to better support elementary PSTs in preparing to teach pre-symbolic algebra.

Keywords · pre-symbolic algebra · pre-service teachers · apprenticeship of observation · consequential transitions

Introduction

Because pre-service teachers (PSTs) have been exposed to years of schooling while they were students, they have been encultured into an "apprenticeship of observation" (Lortie, 1975, p. 61). This refers to the idea that PSTs have served in a kind of apprenticeship for nearly all their lives as students, providing them with certain experiences learning mathematics. Because of these experiences, PSTs may want to continue to learn in a certain way. Research has shown that if they encounter an approach that differs from these experiences, PSTs may struggle (Ball, 1990; Brown et al., 1990; Buchmann, 1989).

Based on their apprenticeship of observation in elementary mathematics, PSTs will have preconceived notions about what should and should not be taught in elementary mathematics. Thus, when faced with the situation of learning a topic they believe does not belong in the elementary mathematics curriculum, they may struggle. They may also struggle if the topic is addressed in a way that is not familiar to them. One such topic is *pre-symbolic algebra*.

Reformers and scholars have advocated for pre-symbolic algebra to be taught in elementary school. However, most PSTs have never experienced pre-symbolic algebra, which may not align with their apprenticeship of observation. Specifically, PSTs might be surprised that algebra is being taught in elementary school and pre-symbolic algebra may seem very different from the algebra they learned in high school. Therefore, they may struggle in a mathematics content course focused on pre-symbolic algebra.

This study investigates the ways in which PSTs struggled while learning about pre-symbolic algebra in a mathematics content course for elementary PSTs. Learning about the struggles PSTs face may help inform the field about how mathematics teacher educators can teach more effectively.

Pre-Symbolic Algebra

Pre-symbolic algebra is not a new idea (Cai & Knuth, 2011; Kieran et al., 2016; National Council of Teachers of Mathematics, 2000) but it has yet to find its way into most elementary classrooms. Consequently, PSTs have rarely experienced pre-symbolic algebra. By *pre-symbolic algebra*, we mean "activities [that] can be engaged in without using the letter-symbolic, and that … can be further elaborated at any time so as to encompass the letter-symbolic" (Kieran, 2004, p. 148). Instead of exploring algebra using traditional algebraic symbols, pre-symbolic algebra uses quantitative reasoning (Ellis, 2011; Smith & Thompson, 2007), pictures and diagrams (Abrahamson, 2015; Cai et al., 2011; Carraher et al., 2008; Dickinson & Eade, 2004; Van Amerom, 2002), and story problems (Koedinger & Nathan, 2004; Russell et al., 2011). Mathematics education researchers have argued that formal algebraic symbols are less accessible to elementary students than informal representations of algebra (Kieran 1992; Van Reeuwijk, 1995). Therefore, pre-symbolic algebra represents a form of algebra that can be taught to elementary students.

For instance, consider the example of a pre-symbolic algebra task focused on functional reasoning given to a third-grade class from Carraher et al. (2008). A teacher fills two solid boxes with the same number of candies, without showing the students how many candies are in the boxes. Then, the teacher adds three pieces of candy to the top of one of the boxes, showing students that exactly three pieces were added. Then, the teacher asks students to describe how the two quantities of candies are related. This type of task treats unknowns as a tangible object or something that can be described in everyday words, as opposed to the symbolic representation of an unknown with a letter. As such, this kind of problem is an accessible way to introduce elementary students to functional relationships.

Indeed, in this example, while about two thirds of students drew pictures specifying exact quantities in the two boxes (i.e., they did not treat the quantities as unknowns), about one third drew pictures or wrote descriptions of how one box held three more candies than the other (i.e., by refraining from assigning specific numbers to the quantities in each box, they treated the quantities as unknowns). This example illustrates it is possible to teach algebraic reasoning and concepts (i.e., unknowns in this example) without using formal symbols such as letters and equations, hence the term *pre-symbolic algebra*. These types of activities can help PSTs experience pre-symbolic algebra from an elementary student's perspective.

Many researchers have been calling for the inclusion of pre-symbolic algebra throughout the elementary curriculum and a number of curriculum reform efforts from around the world have supported this recommendation, such as the following: (a) Mathematics Years 1-10 (Australia; https://www.qcaa.qld.edu.au), (b) the Ontario Curriculum (Canada; https://www.dcp.edu.gov.on.ca/en/curriculum/), (c) the Numeracy Project (New Zealand; https://nzmaths.co.nz/numeracy-project-teaching-resources), (d) the Mathematics Enhancement Program (United Kingdom; https://www.cimt.org.uk/projects/mep/), and (d) the Common Core State Standards (USA;

http://www.corestandards.org). Indeed, "scholars increasingly agree that it is the avenue through which young children can become mathematically successful in later grades" (Blanton & Kaput, 2005, p. 35). However, most elementary teachers are not familiar with the kinds of algebraic thinking and reasoning these scholars are recommending for elementary students. Moreover, research has shown that PSTs tend to view algebra as requiring symbols and symbol manipulation (Stephens, 2008). This is likely due to the inclination for in-service teachers and PSTs to teach the way they were taught (Ball, 1990; Barlow & Reddish, 2006; Brown et al., 1990; Conner et al., 2011; Ebby, 2000; Philipp et al., 2007; Wilson et al., 2005) based on their apprenticeship of observation (Lortie, 1975). For this reason, the move to include pre-symbolic algebra in the elementary curriculum has yet to gain substantial ground (Blanton et al., 2018).

Because of the powerful impact pre-symbolic algebra can have in preparing elementary students for middle and high school algebra (Blanton & Kaput, 2011; Carpenter et al., 2000), effective approaches must be found to help them embrace pre-symbolic algebra and prepare them to teach pre-symbolic algebra. This study represents a step towards understanding PSTs' experiences with learning pre-symbolic algebra.

Struggles Elementary Pre-Service Teachers Experience

Although there are many types of struggles elementary PSTs potentially face during their preparation to becoming a teacher (e.g., Frykholm, 1996; Reisman et al., 2019, Thomson et al., 2020), four types of struggles related to learning to teach mathematics identified by research appear most relevant to our study: elementary PSTs struggle to (a) accept that they lack content knowledge of elementary mathematics, (b) see mathematics as a field of connected abstract ideas that make sense, (c) realise they will need to teach in different ways from how they were taught, and (d) view mathematics problems as having more than one possible correct solution. We explain each type of struggle.

First, research in mathematics education has shown that elementary PSTs, "struggle to understand why they need to relearn the mathematics that they think they already know" (Thanheiser, 2018, p. 39). For instance, Nicol (1997) observed that many elementary PSTs struggled when "[becoming] aware of their limited understanding of [a mathematics] problem" (p. 100). This is important because if elementary PSTs struggle to understand why they need to study elementary mathematics, they may not "realize the value of opportunities to learn important mathematics" (Thanheiser, 2018, p. 48). This type of struggle has relevance to our study because pre-symbolic algebra is likely unfamiliar to PSTs. For this reason, they may not believe pre-symbolic algebra is or should be taught in elementary school, or that they need to learn about it.

Second, PSTs may struggle because they view mathematics as a set of disconnected procedural rules that must be memorised for the purposes of solving mathematical problems (e.g., Ball, 1988; Borko et al., 1992, Schoenfeld, 2016). To them, the purpose of learning mathematics is to master procedural skills, execute computations, and solve problems quickly. For example, Borko et al. (1992) made the following empirical observation:

> Ms. Daniels did not understand what it might mean to know mathematics, at least the mathematics of the division-of fractions algorithm, any differently than she did. Her own success in K-12 and early university mathematics appears to have been the result of her success in rote learning of fairly

> complex mathematical procedures and her ability to apply these procedures in a variety of problem situations. (p. 218)

The type of struggle illustrated in this excerpt has relevance to our study because the point of pre-symbolic algebra is to foster algebraic reasoning early on in a child's education, not focus on procedural aspects of algebra. This puts PSTs in a position where they will not be able to rely on the algebraic procedures they learned in high school.

Third, PSTs may struggle with the notion that they will need to learn to teach mathematics differently from how they were taught (Ball, 1990; Holt-Reynolds, 1992; Smith, 1996; Wilson, 1990). For example, Holt-Reynolds found that PSTs, who were most familiar with learning math via lecture, did not regard progressive instructional approaches "as appropriate substitute formats for traditional teacher-as-teller, lecture formats" (p. 330). Furthermore, Wilson (1990) observed that PSTs struggled with the idea of providing students opportunities to examine incorrect solutions, mistakes, and false starts. For example, "Fifteen minutes of students arguing" (p. 206) was viewed by PSTs as allowing the students to "go on too long" (p. 206) without seeing the correct answer. Since PSTs will be expected to provide elementary students with opportunities to debate and justify their reasoning (NGACBP & National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), PSTs struggle to move away from their view of teaching as telling (Holt-Reynolds, 1992; Wilson, 1990). Thus, pre-symbolic algebra represents a way to learn about beginning levels of algebra that diverges from how they were taught algebra.

Finally, PSTs struggle with the idea that multiple solution strategies and correct solutions can exist for the same problem (e.g., Ball, 1988; Crespo, 2000; Ma, 1999; Nicol, 1997; Thanheiser, 2009). For example, Thanheiser found that PSTs often struggle with how to explain that a solution strategy that does not employ standard algorithms or algebraic manipulation is mathematically valid. Also, Nicol (1997) found that, for an open-ended mathematical task which could be solved in several ways, "many [PSTs] find it difficult to accept that there could be more than one acceptable way to solve it" (p. 100). This type of struggle is relevant to our study because pre-symbolic algebra is more open ended and does not involve standard algorithms used in middle and high school algebra, and PSTs will repeatedly encounter multiple solutions and solution paths.

PSTs learning about pre-symbolic algebra may experience a compounding effect by experiencing all four types of struggles described above. It was our observations of PSTs experiencing significant struggles while learning pre-symbolic algebra that motivated us to study exactly what kinds of struggles PSTs experience in this context. Next, we explain the theoretical framework that guided our study.

Theoretical Framework

Our theoretical framework is grounded in King Beach's notion of an *encompassing consequential transition*. Beach (1999) conceptualised learning to participate in new ways (i.e., relearning) as *making a transition* from one way of participating to another and that this transition likely involves a "conscious reflective struggle" (p. 130). This kind of situation is possible whenever teacher preparation engages PSTs in learning experiences that diverge from their apprenticeship of observation. Since pre-symbolic algebra differs significantly from high school algebra, PSTs who learn about pre-symbolic algebra during teacher preparation likely need to learn to participate in algebra in new ways. Thus, we came to view PSTs' experiences with pre-symbolic algebra as a process of making a transition, one which required them to "relearn" algebra.

Beach (1999) defined *consequential transitions* as "a developmental change in the relation between an individual and one or more social activities" (p. 112). Beach further defined *encompassing consequential transitions* as "persons moving within the boundaries of a single activity" (p. 114) with the "social activity that is itself changing" (p. 117).[1] In our case, PSTs were engaging in algebra that was itself changing, from symbolic to non-symbolic form. Our initial observations of PSTs engaging with pre-symbolic algebra reminded us of an example of an encompassing consequential transition that American machinists faced when industry rapidly changed from mechanical to computer-controlled machining (i.e., relearning to participate in machining). Some machinists found this transition "sufficiently profound…that they left computerised machining and returned to work with mechanical machines" (Beach, 1999, p. 123), which illustrates that these transitions typically involve struggle. Similarly, we noticed that our PSTs experienced a variety of struggles when learning about pre-symbolic algebra. Therefore, we conceptualised PSTs' experiences with pre-symbolic algebra as a case of relearning to participate in algebra and thus, as an *encompassing consequential transition*.

Beach (1999) further elaborated on ways machinists struggled during encompassing consequential transitions. Namely, machinists struggled with changes in the "*artifacts*, *object*, and machinist's *role* [italics added]" (p. 122) of a changing activity. "A change in any one or two of the three components constitutes transformation, the creation of a new relation between machinist and the activity of machining" (p. 122).

Inspired by Beach (1999), we used these three components as our lens to investigate PSTs' struggles. Specifically, following Beach's theoretical framework, we investigated PSTs' struggles with pre-symbolic algebra in relation to changes in the artefacts of algebra, the object of algebra, and PSTs' role when doing algebra. Consequently, the following research question guided our study: *In what ways do PSTs struggle with changes in their relationships with the artefacts of algebra, the object of algebra, and their role while doing algebra when participating in pre-symbolic algebra activities?* Our goal was to develop nuanced understandings about PSTs' range of experiences with pre-symbolic algebra.

Methods

Setting

This exploratory study was conducted at a large university in the northeast of the United States with a well-established elementary teacher-education program. Each elementary PST in the program must complete three mathematics content courses specially developed for the program. Course 1 focuses on number and operations, Course 2 focuses on rational numbers and proportional reasoning, and Course 3 focuses on pre-symbolic algebra for the first half of the course and measurement and geometry for the second half of the course. This study focused on

[1] Beach also classifies three other kinds of consequential transitions: lateral, collateral, and mediational.

the first half of Course 3 and the second author served as the instructor for two sections[2] of Course 3.

For the pre-symbolic algebra portion of Course 3, 15 lessons were used and three themes were focused on: (a) *generalised arithmetic* (Lessons 1-5), (b) *functional relationships* (Lessons 6-10), and (c) *meaning of the equal sign* (Lessons 11-15). For a detailed breakdown of themes, see Hohensee (2017).

An inquiry-oriented approach was employed for Course 3 (class met twice weekly). Specifically, PSTs worked on pre-symbolic algebra activities in small groups for 20-40 minutes. Then, the instructor facilitated a whole-class discussion, during which PSTs projected their written work using a document camera, presented their progress on the activities, and explained how they were thinking. Most of the mathematical thinking originated with the PSTs, followed by the instructor engaging PSTs in unpacking, connecting, refining, and building upon ideas that emerged. One to two activities were explored during a typical 75-minute class period. This approach was similar to the approaches in Courses 1 and 2.

Participants

After securing Institutional Review Board ethics approval, PSTs from both the second author's sections of Course 3 were invited to participate in the study. The first author recruited participants so the second author/instructor could remain blind to their identity until the semester had ended and final grades had been submitted. Five PSTs—four 2nd year students and one 3rd year student—from each section volunteered to participate (N = 10). PSTs received course points (2% of their final grade) for participating. All participants were taking Course 3 for the first time and had already passed Courses 1 and 2.

Pre-Symbolic Algebra Lessons

As stated above, the following three themes formed the basis for the pre-symbolic algebra lessons: *generalised arithmetic*, *functional relationships,* and *the meaning of the equal sign*. Each theme was addressed in five 75-minute lessons (i.e., 3 themes x 5 lessons = 15 total pre-symbolic algebra lessons). For the generalised arithmetic lessons, PSTs explored how to use informal diagrams, rather than letter symbols, to represent, reason about, and to apply arithmetic to unknowns. For example, in diagrams like those in Figure 1, PSTs were asked to describe the unknown quantities that were represented (i.e., the unknown quantity of stars is at least 4 stars, the unknown area is less than 1 area unit, the unknown length is greater than 2 length units, the #2 unknown quantity of eggs is 2/3 times as much as the #1 unknown quantity of eggs). PSTs were also asked to create diagrams for unknowns like those in Figure 1.

[2] In the US, a *section* typically refers to one of several offerings of the same course in a given semester. Students taking the course enroll in one section, usually the one that best fits their schedule.

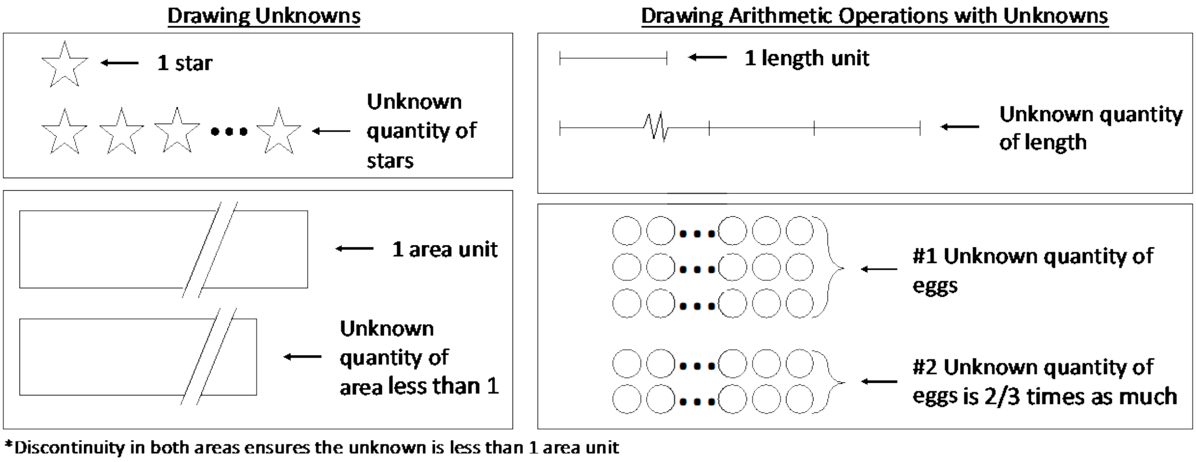

Figure 1. Informal Diagrams for Unknowns and Arithmetic Operations with Unknowns

For the functional relationships lessons, PSTs explored how to use dual number lines, rather than equations, to model and reason about variables and functions. For example, in dual number line diagrams (see Figure 2), PSTs were asked to identify the rates of change and the relationships between variables (e.g., the rate of change for the second dual number line diagram is $1.10 per bagel [or 1/1.10 bagels per dollar] and the relationship is such that the total cost including cream cheese is $2.15 plus $1.10 per bagel times the number of bagels). PSTs were also asked to create their own dual number line diagrams, like in Figure 2, for different real-world functional relationship scenarios.

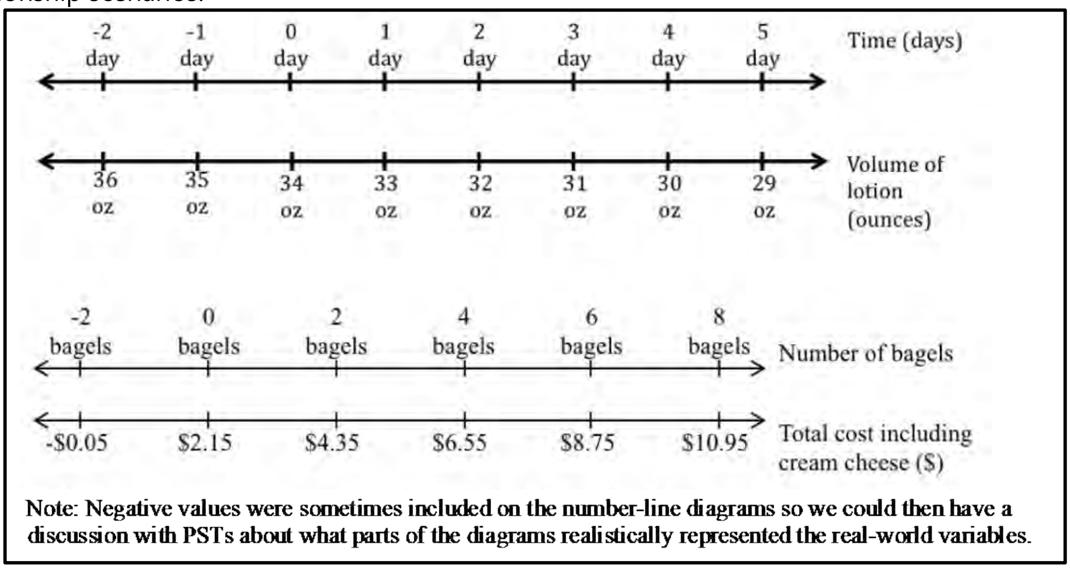

Figure 2. Modeling Variables in Relationships on Dual Number Line Diagrams

For the equal sign lessons, PSTs explored the meaning of the equal sign and how to use dual number lines and area-model diagrams, rather than equations, to solve word problems. For example, in diagrams like those in Figure 3, PSTs were asked to reason with the diagrams to solve for something (e.g., in the dual number line diagram, PSTs were asked to solve by identifying where on the top number line −1/3 hours was represented, determining the corresponding location on the other number line, and interpreting that as the solution). PSTs were also asked to create diagrams, like those in Figure 3, for different problem scenarios, and then use those diagrams to solve problems.

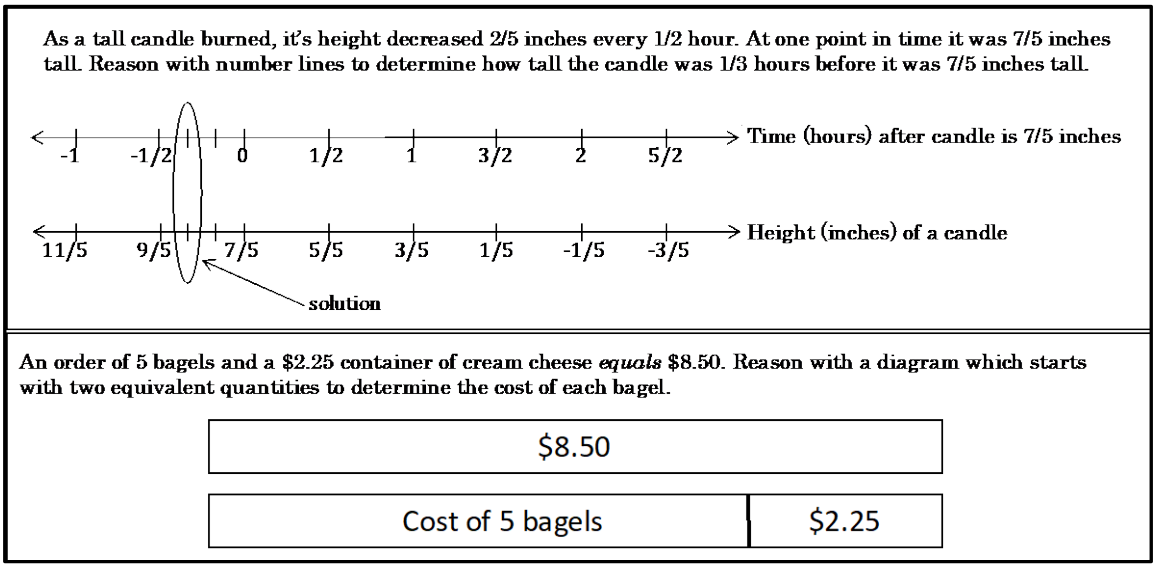

Figure 3. Solving for Unknowns by Reasoning About Number Lines and Equal Amounts of Quantities

An additional contextual feature of Course 3 was that PSTs' academic performance in the course was consistent with their academic performance in the prior two courses. Specifically, at the end of the semester, students in the two sections of Course 3 earned an average final grade of 88.6%, which was similar to the average final grade for students in Course 2 (e.g., in the three sections of Course 2 taught by the second author, PSTs earned an average grade of 86%). Additionally, the instructor for this study reported that he did not observe PSTs engaging in similar struggles when he taught Course 2. We interpreted this observation as an indication there was something unique about PSTs' struggles with pre-symbolic algebra, that was distinct from the struggles PSTs regularly experience when they are not academically successful, and that could not solely be attributed to the instructor.

As such, our study did not focus on *how* the instructor or learning context influenced PSTs' experiences. Instead, we focused on *what* PSTs struggled with while experiencing pre-symbolic algebra, according to their own reflections while learning pre-symbolic algebra in Course 3.

Data Collection

During data collection, each PST recruited from one section of Course 3 was randomly paired with one PST recruited from the other section of Course 3 (both sections taught by the second author) for a total of five pairs. We created pairs across sections because we hoped to hear participants compare their experiences across sections to create richer discussions during the interviews. Each pair participated in three interviews distributed over the 15 lessons, approximately one interview every five lessons. Thus, our data set consisted of 15 video-recorded interviews (i.e., 3 interviews × 5 pairs of PSTs). The first author conducted all interviews.

Each interview was clinical and semi-structured (Bernard, 1988; Ginsburg, 1997), lasted 45-55 minutes and had a similar three-part structure. Part 1 focused on the relation between PSTs and pre-symbolic algebra class activities; Part 2 focused on the relation between PSTs and algebra more generally; and Part 3 focused on the relation between PSTs, teaching mathematics, and becoming an elementary teacher. To align our interview protocol with Beach's theoretical lens on consequential transitions, Part I was designed to help PSTs articulate their views of the *artefacts* of algebra and the interviewer explicitly showed PSTs a worksheet of a task they recently experienced during class; Part II was designed to elicit PSTs' perspectives on the *object* of algebra by focusing on what algebra means to them, what algebra is used for, and why we need algebra; and Part III was designed to help PSTs consider their *role* while engaging in pre-symbolic algebra activities. Of course, PSTs' responses during any part of the interview might include their perspective on the artefacts or object of algebra as well as their role while doing pre-symbolic algebra. PSTs were asked the same main questions for each interview (follow-up questions depended on PSTs' responses) so that we could track changing relations over the course of the interviews.[3]

Data Analysis

We began by transcribing the 15 interviews. Next, we systematically developed a coding scheme grounded in the data (Strauss & Corbin, 1994) that addressed our research question. In particular, the second author rewatched the interviews to immerse himself in and build sensitivity to the data (Corbin & Strauss, 2008). During these viewings, the second author wrote memos that reflected themes emerging from the data (one memo per theme), with an orientation toward instances of PSTs experiencing a struggle of some kind. The first author used the memos to categorise themes and develop a first draft of codes. We then worked together to refine the code names and create well-articulated descriptions for each code.

Once we had a working coding scheme, we coded the same interview individually and compared our codes. During this process, we employed the *constant comparative method* to further refine codes based on evidence that did and did not fit (Corbin & Strauss, 2008). Then, we each chose a different interview to code individually and then compared our efforts. Through comparisons and competitive argumentation (Vanlehn et al., 1984), we came to a final agreement on how we interpreted the codes.

After the codes stabilised, we engaged in axial coding (i.e., looking for associations between codes; Strauss, 1987). Finally, we used Beach's interpretive framework to organise PSTs' struggles into the *artefacts* of algebra, *objects* of algebra, and PSTs' *roles* while doing algebra, to establish a framework for the entire coding scheme (see Table 1).

[3] To see the interview protocol go to the following link: https://docs.google.com/document/d/1yWOTJJqDZkxO12SW-lHDMXyB032WWp0eACBQWf5Sbnk/edit?usp=sharing

Table 1
Coding Scheme of PSTs' Struggles

- Struggles with changes in the artefacts of algebra
- Struggles with changes in the object of algebra
  - Struggles with changes in the object of solving algebra problems
  - Struggles with changes in the object of instructional algebra activities
  - Struggles with changes in the object of algebra homework
- Struggles with changes in PSTs' role when doing algebra

After the coding scheme was finalised, we divided the remaining interviews between us and each recoded our half. The results reflect our classifications about PSTs' struggles with changes in the artefacts and the object of algebra, and PSTs' role when doing algebra.

To further address our research question about ways in which PSTs struggled, we also coded the data on three levels of struggle: major, moderate, and minor. The second author developed these levels based on how many times PSTs mentioned a particular struggle, what PSTs explicitly said about how impactful that struggle was, and the implicit clues embedded in PSTs' talk about that struggle (e.g., was the description vehement, was it light-hearted, etc.). When presented with the categorisations, the first author agreed with how the struggles had been coded. Finally, we looked for patterns in how the levels of struggle changed over the three interviews for each pair of PSTs.

Results

As a reminder, our research question was: In what ways do PSTs struggle with changes in their relationships with the artefacts of algebra, the object of algebra, and their role while doing algebra when participating in pre-symbolic algebra activities? We will present three types of struggles. Our presentation of each struggle includes a description and illustration of the type of struggle, a characterisation of how that struggle manifested itself according to the three levels of struggle, and how the PSTs' levels of struggle changed over time.

Struggles with Changes in the Artefacts of Algebra

PSTs indicated they experienced struggles with changes in the *artefacts* of algebra. Recall Beach's (1999) example of machinists struggling with changes in the artefacts of machining, such as moving from mechanical levers and dials to computers. For us, the formal symbolic representations, such as letters and equations, qualify as artefacts of algebra, just as levers and dials qualify as artefacts of machining. Likewise, story problems and informal diagrams appropriate for elementary students (like those presented in Figures 1–3), qualify as artefacts as well. We consider all of these artefacts of algebra because they contribute to understanding algebraic ideas and ultimately to *doing* algebra, the same way that levers and dials contribute to *doing* machining.

PSTs indicated that they struggled with changes in the artefacts of algebra, from formal letter symbols to informal diagrams and stories. For example, consider Jess and Melanie's statements:

> M: I feel like I could write an equation for them if I was able to use variables . . . If I wrote 2 plus x equals 4 [or] equals question mark, I could see that 2 plus a equals b, but I don't know how you would in a model, or in a diagram what they're asking us to do here.
>
> J: We're so used to using the variables, in the past, that this is kinda difficult to grasp.

Our interpretation here was that because Jess and Melanie already knew how to represent unknowns with symbols ("I feel like I could write an equation"), they struggled with changes in the artefacts of algebra to more child-accessible ways to represent unknowns ("this is kinda difficult to grasp").

In a second example, Liz said:

> L: I remember being confused because I wanted to use variables in the problems…I wanted to do Susan's age was x or something and…Susan's sister was $x + 2$…but then we realised that we just had to use real numbers as kind of like examples even though you didn't really know their real ages…I thought we had to use variables and that's how my whole group was thinking. We were definitely getting a little bit frustrated about it.

In this example, Liz indicated she struggled ("a little bit frustrated") with the shift from formal symbols ("I wanted to do Susan's age was *x*") to informal algebraic reasoning ("we just had to use real numbers as kind of like examples").

We noticed differences over time in the levels of struggle PSTs experienced with changes in the artefacts of algebra. To capture these nuances, we coded their struggles according to three levels, major, moderate, and minor, which we explain next.

Levels of Struggles and Changes in Levels of Struggle with Artefacts of Algebra Over Time

We identified three levels of struggle PSTs exhibited with changes in the artefacts of algebra (see Table 2).

Table 2
Characteristics of Levels of Struggles with Changes in the Artefacts of Algebra

Levels of Struggles	Characteristics of Each Level of Struggles
Major Struggles	(a) Mentioned changes in the artefacts of algebra; (b) struggled with trying to not use formal algebraic symbols; and (c) rated pre-symbolic algebra activities and/or instructor negatively or disputed the instructor's mathematical interpretations.
Moderate Struggles	(a) Mentioned changes in the artefacts of algebra; (b) voiced frustration with not using formal algebraic symbols; and (c) the new ways of reasoning took time to make sense to them.
Minor Struggles	(a) Mentioned changes in the artefacts of algebra; (b) marginally struggled trying not to use formal algebraic symbols; and (c) noticed that the changes in perspectives of the artefacts of algebra led to new insight(s) about algebra.

We rated each pair of PSTs on the level of struggle that best characterised them during each interview (see Table 3). We decided on our ratings by comparing the struggles exhibited within the three interviews for any given interview pair, as well as the struggles exhibited across interview pairs.

Table 3
Summary of Varying Levels of PSTs' Struggles with Changes in the Artefacts of Algebra

	Interview	Ann & Nick	Britany & Katie	Dawn & Patty	Jess & Melanie	Christine & Liz
Changes in artefacts of algebra	1st	–	major	major	moderate	–
	2nd	minor	moderate	minor	–	moderate
	3rd	–	–	–	–	–

Note. Blank entries indicate PSTs did not talk about changes in artefacts during that interview.

We make two main observations about Table 3. First, PSTs experienced different levels of struggle with changes in the artefacts of algebra. Second, no pair of PSTs experienced struggles with changes in the artefacts in the final interview. Thus, it appears that while PSTs experienced a range of struggles with changes in the artefacts of algebra, their struggles appear to fully resolve by the end of the pre-symbolic algebra lessons. As we show next, PSTs exhibited a similar pattern of struggles with changes in the *object* of algebra, except that those struggles did not fully resolve.

Struggles with Changes in the Object of Algebra

Besides struggling with changes in the artefacts of algebra, PSTs also struggled with changes in the *object of algebra.* In particular, they experienced pre-symbolic algebra as a significant shift from the algebra they were familiar with and furthermore, this shift seemed to conflict with their notions of what algebra was and what it means to do algebra. The following statements from Christine, Katie, Ann, Melanie, Britany, and Patty illustrate this struggle with changes in the object of algebra:

> C: So, it's a very ambiguous algebra [pre-symbolic algebra].
>
> K: I think that because that's [pre-symbolic algebra] more of a new kind of way of doing it, it's harder for me.
>
> A: Algebra is just weird for me right now because of this [pre-symbolic algebra] class.
>
> M: When I took [high school] algebra or whatever, they had an equation or like they had a word problem and you would write an equation. But I don't think I was ever modeling that equation, if that makes sense.
>
> B: I think I would have liked this [pre-symbolic algebra] more if we had related it back to stuff that we already know, even though that's not the point.
>
> P: I feel like each time we're starting a new lesson or a new topic is brought up, that we're almost starting from scratch. It's another thing that we're like, I don't get what we're supposed to do.

We interpreted these statements as indications PSTs were experiencing struggle with changes in the object of algebra or what algebra is ("it's a very ambiguous algebra," "Algebra is just weird for me right now") and what it means to do algebra ("I don't think I was ever modeling that equation"). We viewed these struggles with changes in the object of algebra as similar to how machinists in Beach (1999) struggled with a change in the object of machining, when going from machining parts themselves to producing the computer programs that did the machining.

We broke down PSTs' struggles with the object of algebra into three sub-categories: changes in the object of *solving algebra problems*, changes in the object of *instructional algebra activities*, and changes in the object of *algebra homework*.

Struggles with Changes in the Object of Solving Algebra Problems

One sub-category of changes in the object of algebra was a change in the *object of solving problems*. For PSTs, the object of solving was mainly about determining a single correct solution. However, there can sometimes be multiple correct solutions when solving at the pre-symbolic level. One reason for this is because diagrams are used to represent unknowns, functional relationships, and algebraic expressions as a pre-symbolic substitute for algebraic symbols (e.g., see Figure 1). A consequence of using diagrams instead of algebraic symbols is that informal diagrams are not governed by definitive or fixed rules, as algebra symbols are. Thus, PSTs could draw diagrams in different ways and still be correct.

The idea that the object of solving algebra problems was not necessarily to find a single correct solution was something PSTs often struggled with. For example, Patty explained: "What I found most confusing with this lesson was that there was not really a right or wrong answer, which math is usually known for having." Patty's belief aligned with other students' beliefs that mathematics problems typically have a single correct answer (e.g., Lampert, 1990; Nicol, 1997). Ann also found multiple solutions confusing: "I was really confused at first, as was everyone in the class I think…There's no one exact answer anymore. As long as you explain it, it could be right." Christine echoed a similar sentiment:

> There's multiple ways of doing it but in [Course 1 and 2], they were so strict so it was easy. Even though it was more structured, I thought it was easier because I knew what was expected of me. I don't like how [Course 3] is so lenient, if that makes sense.

These statements suggested that PSTs struggled with multiple solutions for the same problem being seen as correct. Each group mentioned struggling with this change in at least one interview, and most groups mentioned it in multiple interviews.

Struggles with Changes in the Object of Instructional Algebra Activities

A second sub-category of changes in the object of algebra involved the *object of instructional algebra activities*. For PSTs, the primary object of instructional activities was to completely address (i.e., wrap up) one or more mathematical ideas. Yet, because pre-symbolic algebra ideas were so different from the algebra PSTs encountered in high school, it sometimes took more than one lesson about the same idea before they felt their questions had been fully resolved. We observed that PSTs struggled when activities concluded but unresolved questions remained. This may have been because this object of algebra activities diverged from PSTs' apprenticeships of observation.

For example, Melanie explained, "I found this a little bit more confusing than I remember [Course 1 and 2] being because we didn't at the end of class [in Course 3] come to something—like everything we had done in that class just automatically made sense now." Similarly, Anne said "I hate leaving class not knowing how to do something.

These statements suggested that PSTs struggled (i.e., "I found it a little bit more confusing," "I hate leaving class not knowing") when the pre-symbolic algebra activities did not fully resolve (i.e., "we didn't at the end of class come to something"). We interpreted this evidence as a change in the object of instructional activities, from completely addressing one or more algebra ideas to addressing algebra in ways that might leave questions unresolved. This struggle reminded us of the cultural belief that good teaching involves telling students the correct answer and resolving confusion immediately (Schoenfeld, 2016; Wilson, 1990). Each pair of PSTs provided evidence of this struggle multiple times during the interviews.

Struggles with Changes in the Object of Algebra Homework

A third sub-category of changes in the object of algebra was about the *object of algebra homework*. For PSTs, the object of algebra homework was primarily to practice what was done in class. However, as explained above, the pre-symbolic algebra ideas sometimes took more than one lesson for PSTs to fully understand. Therefore, PSTs were often assigned homework for which the object was to continue the exploration of the ideas introduced in class.

We observed that PSTs struggled with treating the homework as an opportunity to continue exploration. The following examples from Ann, Britany, Christine, and Patty illustrate PSTs' struggle with changes in the object of algebra homework:

> A: I hate leaving class not knowing how to do something, because what [Nick] said, when I'm doing the homework, it's pointless because I don't even know if I'm doing it right.

B: We spent the entire class going over the homework and he gave out the next lesson and the next homework and was like there ya go! Bring it back Tuesday!

C: He didn't clear that up before the homework so we couldn't do half the homework.

P: I felt like what was expected and what we were supposed to do and all of that, like all the directions for these and the homework and stuff. It's all just very vague and it's all very like figure it out on your own.

These statements suggested that PSTs struggled with feeling unprepared for homework assignments. Each group mentioned struggling with this change in at least one interview.

According to our interpretation, PSTs' struggles to adjust to changes in the object of algebra were indicative of a "conscious reflective *struggle to reconstruct knowledge*" (Beach, 1999, p. 130; italics added). Similar to the struggles with changes with artefacts of algebra, we also noticed differences over time in the levels of struggle PSTs experienced with changes in the objects of algebra.

Levels of Struggles and Changes in Levels of Struggle with the Object of Algebra Over Time

We identified three levels of struggle with changes in the object of algebra during the pre-symbolic algebra lessons (see Table 4).

Table 4
Characteristics of Levels of Struggles with Changes in the Object of Algebra

Levels of Struggles	Characteristics of Each Level of Struggles
Major Struggles	(a) Exhibited low confidence that struggles would eventually be resolved or provided no evidence of a belief that struggles would be resolved; (b) voiced frustration about the changes in the object of algebra and/or in their own inabilities to understand the changes; and (c) rated pre-symbolic algebra activities and/or instructor negatively or disputed the instructor's mathematical interpretations.
Moderate Struggles	(a) Eventually resolved struggles that extended beyond the class period (or expressed confidence that struggles would eventually be resolved); (b) voiced some frustration with changes in the object of algebra; and (c) interpreted struggles as beneficial to their learning.
Minor Struggles	(a) Resolved struggles within a given class period; (b) did not voice significant frustration; and/or (c) rated the pre-symbolic algebra activities positively.

We then rated each pair on the level of struggle that best characterised those PSTs during each interview (see Table 5).

Table 5
Summary of Varying Levels of PSTs' Struggles with Changes in the Object of Algebra

	Interview	Ann & Nick	Britany & Katie	Dawn & Patty	Jess & Melanie	Christine & Liz
Changes in the object of algebra	1st	moderate	major	major	moderate	minor
	2nd	major	major	moderate	minor	minor
	3rd	major	moderate	moderate	minor	minor

Two observations about Table 5 seem particularly informative. First, each pair exhibited some degree of struggle with changes in the object of algebra in all three interviews. Second, for most pairs, struggles with changes in the object of algebra seemed to be more major at the beginning and more minor by the end. In other words, although PSTs experienced a range of struggles with changes in the object of algebra, their struggles persisted over all of the pre-symbolic algebra lessons, but trended from more to less major.

Struggles with Changes in PSTs' Role When Doing Algebra

Finally, PSTs also indicated they struggled with changes in their role when doing algebra. In particular, PSTs felt they already knew how to solve pre-symbolic algebra problems using their prior knowledge from high school algebra or from other courses. In those instances, they tended to struggle to solve the problem the way a child might solve. We interpreted these instances as struggles with changes in their role when doing algebra.

Consider the comments from Britany and Katie when talking about solving a problem involving functional relationships:

> B: Like you already know how to do it one way, so the new way doesn't make sense because it's so different from your old way.
>
> K: And like I can't erase the concept of knowing numbers in my brain, like it's hard.
>
> B: Until you relate it back because they're totally different.

In this example, Britany and Katie indicated they already knew a way to solve the problem and struggled to think about the problem from a child's perspective ("I can't erase the concept of knowing numbers in my brain, like it's hard"). Here's a second example from Nick:

> N: Like I understand how it relates algebraically to the word problem, but why can't we just do that instead? Like if we're doing algebra, then let's do algebra. And I guess kids can't learn that at such a young age. So you have to do it in the form of word problems that kind of means the same thing, it's just easier for a little kid to understand. But like for—I don't know it's stupid. I just don't like it at all.

With this statement, Nick indicated he would rather solve the problem his way ("let's do algebra"), than the way a child might ("it's just easier for a little kid to understand").

Similar to the other struggles described above, we noticed differences over time in the levels of struggle PSTs experienced with changes in their role when doing algebra. We again coded their struggles according to levels, which we explain next.

Levels of Struggles and Changes in Levels of Struggle with PSTs' Role When Doing Algebra Over Time

We identified three levels of PSTs' struggle with changes in their role while doing algebra during the pre-symbolic algebra lessons (see Table 6).

Table 6
Characteristics of Levels of Struggles with Changes in PSTs' Roles While Doing Algebra

Levels of Struggles	Characteristics of Each Level of Struggles
Major Struggles	(a) Mentioned changes in their role while doing algebra; (b) voiced frustration about their role while doing algebra and/or in their own inabilities to understand their role; and (c) rated pre-symbolic algebra activities and/or instructor negatively or disputed the instructor's mathematical interpretations.
Moderate Struggles	(a) Mentioned changes in their role while doing algebra; (b) voiced some frustration with changes in their role while doing algebra; and (c) the new ways of reasoning took some time to make sense to them.
Minor Struggles	(a) Mentioned changes in their role while doing algebra; (b) did not voice significant frustration regarding their role while doing algebra; and/or (c) noticed that the changes in their role led them to new insight(s) about algebra.

We rated each interviewed pair on the level of struggle that best characterised those PSTs during each interview (see Table 7).

Table 7
Summary of Varying Levels of Struggles with Changes in PSTs' Roles When Doing Algebra

	Interview	Ann & Nick	Britany & Katie	Dawn & Patty	Jess & Melanie	Christine & Liz
Changes in PSTs' role	1st	–	–	major	moderate	–
	2nd	–	–	–	–	–
	3rd	major	major	–	–	minor

Note. Blank entries indicate PSTs did not talk about changes in their role during that interview.

Notice in Table 7 that PSTs' struggles with changes in their roles while doing algebra were less consistent compared to the other two struggles. Although each pair struggled with this change in one interview, some pairs only struggled at the beginning, while other pairs only struggled at the end. In addition, more than half of the pairs experienced major struggles with changes in their role while doing algebra at some point in the semester. However, overall, it appears that PSTs struggled with this change the least of the three kinds of change.

Discussion

While engaging in pre-symbolic algebra activities, the PSTs in our study struggled with the changes in the artefacts of algebra, changes in the object of algebra, and changes in PSTs' role while doing algebra. Beach (1999) explains that the process of creating new knowledge (by generalising one's knowledge) may involve a change in any one or more of these components. The PSTs in our study experienced struggles in all three components, to varying degrees, and at different times throughout the course.

A take-away from our results is that the frequency with which PSTs experienced struggles differed for the three main types of changes. For example, PSTs in general struggled more often (i.e., during more interviews) with changes in the object of algebra than with changes in the artefacts or their role. A second take-away is that the levels of struggle PSTs experienced over time differed for the three types of changes. For changes in the object of algebra, most PSTs' struggles lessened over time but never fully resolved. For changes in the artefacts of algebra, most PSTs' struggles lessened and appear eventually to fully resolve, because no pair struggled with this change by the third interview. In contrast, for their role when doing algebra, PSTs either went from not struggling to struggling or vice versa, and only struggled with this change in a single interview. A third take-away is that different pairs of PSTs exhibited different trends over the three interviews. For instance, Ann and Nick experienced increasing levels of struggle with changes in the object of algebra, while the other pairs experienced decreasing or constant levels of struggle with that change. These take-aways imply that learning pre-symbolic algebra after already experiencing symbolic algebra is a dynamic process that presents significant challenges and manifests itself in a range of ways.

Through these learning experiences, PSTs confronted breaking with their apprenticeship of observation and shifting their perceptions of algebra. Making the break was met with struggles. PSTs struggled with changes in the artefacts of algebra, the object of algebra, and their role in doing algebra, which likely will impact the degree to which they engage their future students in pre-symbolic algebra. Thus, their experiences help explain why pre-symbolic algebra has yet to be embedded in the elementary mathematics curriculum.

Significance

The findings from this study are significant for several reasons. First, this study connects Lortie's (1975) and Ball's (1990) claim that PSTs must break from their apprenticeship of observation, and Beach's (1999) conceptualisation of consequential transitions. Specifically, this study suggests that to break with one's apprenticeship of observation is a consequential transition. This connection is significant because it likens the experiences of elementary PSTs breaking with their apprenticeship of observations to individuals facing consequential transitions in other professions, and thus our study may foster deeper understandings of PSTs' experiences as part of a broader class of experiences.

Second, the findings show that PSTs breaking with their apprenticeship of observation when learning about pre-symbolic algebra within a teacher education program can be accompanied by struggles that are non-trivial. This is particularly salient when one considers Beach's claim about changes in the artefacts, object, and roles of social activities, that "a change in any one or two of the three components constitutes transformation" (Beach, 1999, p. 122). In other words, a change need only occur in one of the three components of social activity to result in a consequential transition. The PSTs in our study indicated they were experiencing changes in all three. For this reason, it should be strongly noted that, although pre-symbolic algebra is arguably an important topic for elementary PSTs to learn for more of their future students to be successful at algebra, pre-symbolic algebra could put significant demands on PSTs.

Third, some struggles PSTs exhibited while learning about pre-symbolic algebra seem unique to the context of learning about pre-symbolic algebra, while others may extend more broadly to when PSTs learn about elementary mathematical content in general, or even to learning about elementary mathematics teaching methods. Examples of the former was when PSTs had to generalise their knowledge of algebra to include artefacts of pre-symbolic algebra (a struggle with changes in the artefacts of algebra), and when PSTs had to break away from preestablished notions that doing algebra involves representing variables and unknowns with letters and solving problems with equations (struggles with changes in their role while doing algebra).

On the other hand, an example of a struggle that extends more broadly to when PSTs learn about elementary mathematical content in general, or even to learning about elementary mathematics teaching methods, was when PSTs struggled with the idea that more than one solution could be considered correct (a struggle with changes in the object of algebra). A similar struggle has been documented in other mathematics content and methods courses (e.g., Nicol, 1997; Thanheiser, 2009). In particular, evidence has shown that when PSTs learn to teach mathematics for conceptual understanding, the object of mathematics in general may change for them. Because PSTs in our study were engaged in activities designed to help them teach for conceptual understanding, it made sense they would struggle with a change in the object of algebra. Together, the general struggles associated with learning to teach for conceptual understanding compounded with the struggles that were specific to learning about pre-symbolic algebra created a complex learning environment for PSTs to navigate.

Fourth, the findings are significant, not only because they show *that* PSTs struggled, but also *what* they struggled with. Identifying what PSTs struggled with when learning about pre-symbolic algebra is significant because it could inform efforts to develop targeted interventions that better support elementary PSTs through the struggles. Based on this study, future research could work on developing and testing such interventions. As explained next, research efforts to develop interventions have already begun.

Implications

Based on this study, the implication arises that targeted supports are needed to help PSTs navigate struggles with pre-symbolic algebra. In fact, each aspect of pre-symbolic algebra PSTs struggle with could be specifically targeted to better support PSTs. For example, the struggle with more than one solution being considered correct could be explicitly targeted by making sure PSTs also have a discussion about an incorrect solution or solutions, so PSTs have contrasting cases (Marton, 2006) to which they can compare the multiple correct solutions.

Another approach would be to develop holistic supports for PSTs experiencing struggles with pre-symbolic algebra. One such holistic support has already been developed and tested with preliminary findings showing promising results. Specifically, McKenney (2020) and the second author developed a daily 5-minute intervention for elementary PSTs that was based on three principles: that meaningful learning (a) takes time, (b) involves productive struggle, and (c) requires a growth mindset. The primary goal of the intervention was to foster patience in PSTs when they struggled because, as shown in our study, for most PSTs, the struggle lessened as the course went on. McKenney and the second author reasoned that by fostering patience, PSTs would be less inclined to give up on the course before their struggles could become more manageable. This kind of support might also be generalisable to other contexts in which PSTs might struggle with breaking with the apprenticeship of observation (e.g., when they are asked to teach using small groups, inquiry-based learning, or open-ended problems, etc.).

A second implication is that PSTs may require more *convincing* about the merits of pre-symbolic algebra. Convincing them that learning about pre-symbolic algebra is vitally important, may provide motivation to help them navigate those struggles. Furthermore, PSTs may also require more convincing that children can actually engage with pre-symbolic algebra, given that they themselves struggle to learn these concepts (Hohensee, 2017). To test this idea, the second author has incorporated into several of the pre-symbolic algebra lessons videos of children learning about pre-symbolic algebra, and has created two activities in which PSTs interview children about pre-symbolic algebra. The video and interview activities are designed to help convince PSTs that pre-symbolic algebra is important for and accessible to children. Preliminary observations indicate these strategies have been helpful.

A final implication is that PSTs may need more time to make the transition to the ideas in pre-symbolic algebra. One short unit on pre-symbolic algebra may not suffice for them to break with years of apprenticeship of observation and make this important consequential transition. Instead, similar to individuals making consequential transitions in other professions, making such drastic changes may take PSTs more time and exposure to the ideas. Without sustained exposure to pre-symbolic algebra, PSTs may not be able to make it past their own struggles.

Conclusion

Our study shows that when learning pre-symbolic algebra, PSTs may experience a consequential transition that involves breaking with their apprenticeship of observation. When learning about pre-symbolic algebra, PSTs may struggle with changes in the artefacts of algebra, changes in the object of algebra, and changes in their role while doing algebra. Because consequential transitions are a form of generalisation of knowledge, one way to think about PSTs' struggle with pre-symbolic algebra is as a struggle with generalizing their knowledge of algebra to include pre-symbolic algebra.

PSTs in teacher preparation programs learn innovative ways to teach mathematics, which may not align with their own apprenticeship of observation for how to teach mathematics. Furthermore, it may come as a surprise to them to learn they will be expected to teach familiar content in innovative ways, such as learning that pre-symbolic algebra is expected to be taught in elementary school. Our study offers an explanation for why pre-symbolic algebra has yet to be infused into elementary mathematics curricula and informs the field about ways to better support PSTs so they are more likely to engage their future students in pre-symbolic algebra.

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Authors

Siobahn Suppa
School of General Studies
Stockton University
Galloway, NJ, USA
email: siobahn.suppa@stockton.edu

Charles Hohensee
University of Delaware
Newark, DE, USA
email: hohensee@udel.edu