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CONCEPTIONS OF RATIO AND PROPORTIONS AMONG YEAR FIVE PUPILS: CASE STUDY

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ABSTRACT

Ratios and proportions are new topics in the primary curriculum. Previous research shows that Year Five pupils have difficulty in learning ratio and proportion. This study aims to identify the conceptions of ratio and proportion of seven year five pupils. It is based on radical constructivism using qualitative case study. Data was collected through a clinical interview technique. The data collected include the mental image of the ratio and proportion, representation of ratio and proportion, and the meaning of ratio and proportion. The cross-case analysis between the samples showed that the year five pupils could construct knowledge that make sense and gave the meaning of ratio as comparison of quantities, part to part comparisons, useful in daily activities, belongings, cooking, and parts of the body. Meanwhile, proportions are meaningful to them as part to whole comparisons and as two equal ratios that are related multiplicatively. This study gives implications to a few aspects mainly to the theory, teaching and learning practice, curriculum, textbook developers, and implication for future research.

Keywords: Clinical Interview, Conceptions, Radical Constructivism, Ratio and Proportion

INTRODUCTION

Ratio and proportion are important in daily mathematics. Research shows that ratio and proportion are important topics in the mathematics curriculum (Artut & Pelen, 2015; Andini & Jupri, 2017; Cramer, 2017; Diba & Prabawanto, 2019; Dole, 2008; Dougherty et al., 2016; Hulbert et al., 2017; Lobato et al.,2014: Tiflis et al.,2019; Petit et al.,2020). The knowledge of ratio and proportion is not limited to the comparison of quantities. In the Malaysian primary mathematics curriculum, proportion has been introduced in Year 4, ratio in Year 5, and the integration of ratio and proportion in year six (Curriculum Development Centre, 2016). The current curriculum KSSR syllabus shows that truly little focus is given to the development of the basic concepts and the sense-making of ratio and proportion. However, many primary pupils face difficulties in learning ratio and proportion (Andini & Jupri, 2017; Diba & Prabawanto, 2019; Dougherty et al.,2016; Monteiro, 2003; Tiflis et al., 2019).

Lin and Cooney (2011) stated that the primary goals of teaching mathematics are to enable students to make sense of mathematics. Primary pupils should know that comparing two similar measures are pure ratios (Lamon, 2012; Parker & Baldridge, 2015). If the comparison is between two different quantities,

then it will be a rate (Lamon, 2012; Petit et al., 2020). The knowledge of part-to-part and part-to-whole and whole-to-whole is important in acquiring the knowledge of ratio and proportion. Ratios are part-to-part comparisons between two different quantities or partitions. There is a discussion that ratio can be written as a fraction and this situation gives a confusion between ratio and fraction (Petit et al., 2020). Pupils should know that ratio can be part-to-part or part-to-whole comparison whereas fraction as part-to-whole comparison. Further, pupils must be able to differentiate between ratios and fractions and the interconnection between them (Lamon, 2020).

Upper Primary pupils know about multiplication and division, and they can use them as prior knowledge to create meaning to proportion as two equal ratios by using the building up strategies, repeated addition, and the scalar or functional relationships. Multiplication is sometimes conceptualized as repeated addition by primary pupils, and they use this knowledge to solve multiplication word problems as Larsson, Petterson & Andrews (2017) stated that primary pupils conceptualized multiplication as repeated addition or equal groups. Multiplication and division are the arithmetical knowledge useful and relevant in the iteration process (Singh, 2000; Hulbert et al.,2017).

Therefore, there is an urge to find out what are the pupils' concepts and ideas which can help them in acquiring the knowledge of ratio and proportion. The cognition factor that has a connection to the conceptions of ratio and proportion needs to be studied to build a strong foundation for a higher level of mathematics. According to von Glasersfeld (1982), conceptions can be defined as the representations we construct from our experience. Glasersfeld (1987) added that conception means a general idea or understanding of something. It can be referred to as the ability to form or understand mental concepts and abstractions. Meanwhile, Nik Azis (2014) defines conception as meaning, idea or opinion, and thoughts about something which is formed in an individual mind. It can be something conceived in the mind; a concept, plan, design, idea or thought.

This study is carried out by focusing on how children built their knowledge about ratio and proportion and their ability to transfer prior knowledge to new situations. Further, the variety of skills and conceptions to approach ratio and proportion flexibly will be identified. In conjunction with that, four constructs, mental image, representation, meaning, and problem solving are used to identify the year five pupils' conceptions of ratio and proportion. Their concepts in the effort of constructing the meaning of ratio and proportion will be identified specifically on how they give meaning to their experiences through involving their reflective abstractions on ratio and proportion.

This study aims to identify the conceptions of ratio and proportion. The participants consisted of seven-year five primary pupils in an inner-city school. Their conceptions will be identified through their abstractions or generalizations from their experiences.

In identifying the objectives, the following research questions will be addressed.

1. What is year five pupils' mental image of ratio and proportion?
2. How do year five pupils represent ratio and proportion?
3. What is year five pupils' meaning of ratio and proportion?

METHODOLOGY

Research Design

This qualitative research utilizes the case study as the research design, whereas the clinical interview is used as the technique of data collection. Cobb & Steffe (1983) also claims that the clinical interview helps to investigate the sequence procedures taken by students in building their mathematics concept. Nik Azis (2014) and Sharifah Norul Akmar (1997) claims that clinical interview helps in answering research questions on students' conceptions because the depth detailed information can be gathered from the children's thinking and their ability to build their schemes through interaction with the surrounding and environment. The data analysis involved four stages, namely transcription of the interview recording into the writing, the establishment of case studies involving descriptions of the

behavior of the research participants about ideas of the ratio and proportion, analysis and cross-case analysis of the participants through triangulation method using 'verbal and non-verbal responses, and the identification of the year five pupils' conceptions on ratio and proportion.

Participants

The samples chosen in this study are three boys and four girls of year five pupils from the same school and the samples are purposefully selected cases. The purposive sampling is being utilized in this study to discover and gain information in depth and rich information (Patton, 2007). Furthermore, in purposeful sampling researcher select individuals intentionally and learn or understand the central phenomenon. This purposive sampling helped in gathering the information of the conceptions of ratio and proportion among the year five pupils.

Instrumentation

The first session was on the mental image, and it involves 10 tasks. The second and third session is on representation and meaning respectively with five tasks each. The clinical interview protocols were adapted from the clinical interview instrument of Nik Azis (1987), Goh (1998) and Lamon (2005). The interview protocols were validated through the evaluation by expert panels, supervisor's review for clarity and peer debriefing by local expert teachers. The expert panels were a professor from foreign university who is an expert in the constructivist theory, lecturers from both local and foreign universities and mathematics expert teachers. The professor checked on the reliability and gave suggestions in the decoding of the items. Two local lecturers gave comments on the content validity and suggested on the constructs of the interview tasks. Further two local expert teachers discussed on the learning standards of ratio and proportion in the curriculum specifications.

The following section is on the summary of the research results and the discussion.

FINDINGS AND DISCUSSION

The findings identified are based on the research questions below:
RQ 1: What is year five pupils' mental image of ratio and proportion?
RQ 2: How do year five pupils represent ratio and proportion?
RQ 3: What is year five pupils' meaning of ratio and proportion?

RQ 1: Year five pupils' mental image of ratio and proportion

Almost all the year five pupils provided the mental picture of ratio as comparison of quantities. In the symbolic category, the respondents used concrete drawings and the colon notation. In the linguistic category, they explained that ratios are comparison of numbers without units and compared them using words like 'partition', 'separation', and 'to'.

In the procedural category, they compared ratios using similar and different measures. The respondents mentioned that quantities are the number of items. They referred to the quantities using discrete sets. The comparison of similar measures is through imagining things (balls, bottles, flags, and tins of paint) with different colours. Further, they compared the numbers and concrete drawings to identify part-to-part ratios. They also imagined ratios in terms of comparison of part-to-whole and whole to part ratios. The respondents also compared quantities from different measure space like comparing two rambutans with three oranges, one motorcycle with two cars, one fan with two lamps, one chair to one fan, one board with two dusters, and three stars with four moons. Diagram 1 below shows the idea of ratios as comparisons between similar measures and Diagram 2 on different measures.

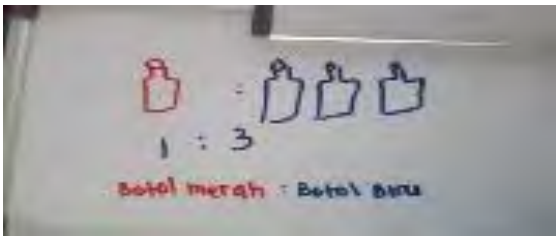


Diagram 1. Tasni's mental image of ratio comparing similar measures.

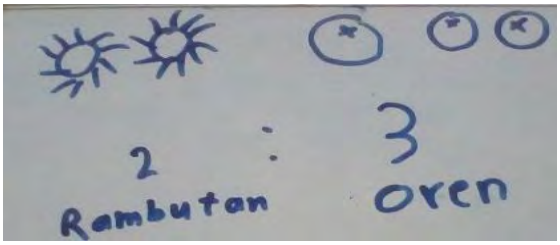


Diagram 2. Faziq's mental image of ratio comparing different measures.

While stating part-to-part ratios, they explained parts and quantities. They concluded that parts are made of quantities and their focus was on counting actions-operations. The comparison of parts involved a mixture of similar and different measures, one to many correspondences and many to one correspondence. Almost all the year five pupils used the words like the parts are interchanged, switched places, reversely said, when comparing the ratios. They gave the idea that the numbers in the ratio change places or change locations (a: b becomes b: a).

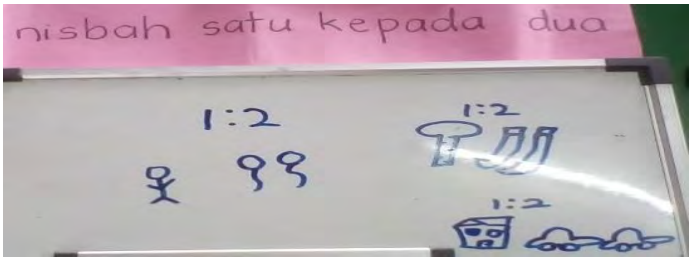
While imagining part to part ratios, the respondents' mental pictures of ratio 1:2 and ratio 2:1 ware as one to many and many to one correspondence. Furthermore, ratios 2:5,5:2, 2:7, and 5:7 were compared as many to many correspondences, and they described the items of the parts based on position or location. Only one respondent clearly explained the idea that the total number of items of both parts will never change even though the position of the parts changes.

In the conceptual category, the year five pupils mentally pictured ratio as measure by using the word 'for', ordered pairs, sharing, dual, interiorized ratio, rate, internalized ratio (building up) using repeated addition, stating each ratio is different, stretcher shrinkers or scaling and multiplicative thinking. One respondent explained on stretchers and shrinkers. The examples given were on scaling up the situation, a ratio of a short building to a tall building, and comparison on the diameters of football and tennis ball.

Another respondent imagined and explained by saying that 'for every' one room there is one chair and one fan (associated sets). This is a relationship between two elements or two sets that have no commonly known connection as stated by (Lamon, 1993) or can be said as an ill-defined connection. While imagining ratios one to two and two to one the description of the respondents is on the internalized ratio (for every) e.g., Excerpt 1 shows for every tree there are two bunches of bananas, every one person has two balloons, two cars for every house. Other examples include for every two chairs there is one table, every one bicycle there are two tyres, for every two girls there are three boys and for every pineapple there are two mangosteens.

Excerpt 1: Shidi's mental image of ratio (for every)

S: I can compare one person and two balloons. Ratio one to two. One person has two balloons. Some more…I can compare one tree and two bananas. For one tree there are two bananas. Ratio one to two.



I: What else?
S: Hmm…One house to two cars. There are two cars for every house. Ratio one to two.

In the practical category**,** one respondent imagined ratios as comparison of parts of the body and the others gave examples on daily life experiences. Another respondent imagined ratio as what is stated in the textbook*,* as one to one comparison. The respondents compared quantities on real-life situations like comparing one bicycle with two tyres, one head with two eyes, one packet of salt to two packets of sugar, one umbrella to two people, one football to two goalposts. In the figurative category, one of the respondents imagined the ratio as two dots and used the movement of fingers.

She compared the numbers in the ratio by showing fingers in the left and right hand. One is on the left and two is on the right. The following excerpt and Diagram 3 illustrate examples of real-life situations and parts of the body.

Excerpt 2: Amira's mental image of ratio
I: Can you explain further?
S: Err. They are for the quantity. For example, if there are one football and eleven football players, then the ratio will be one to eleven. That means one football is for eleven players. The ratio will be one to eleven.
Err…All the eleven players only play with one ball and that one ball is for the eleven players.
I: What else?
S: I write one for the ball and eleven for the players. So, the ratio will be one to eleven.
I: There will be two numbers separated by a colon.
I: What else?
S: I also can compare one person with two ears?
I: How is that?
S: One person has two ears. I compare the person and the number of ears. The ratio will be ratio one to two.
I: Any other examples?
S: When I cook rice, I use one cup of rice and for every cup, I two cups of water.
I: What else?
S: Yes…there is. A child below six-year-old has twenty teeth. Ratio one to twenty. Two children will have forty teeth. One multiply twenty is twenty and twenty times two is forty. Then the ratio will be two to forty.
I: What else crosses your mind about ratio?
S:(thinking for a long time). One car has four tyres. Every car needs four tyres. Ratio one to four.

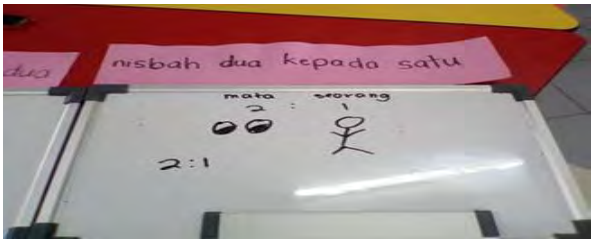


Diagram 3. Amira's mental image of ratio using parts of the body.

The mental picture of proportion involves two categories. They are procedural and conceptual. The respondent's additive thinking falls in the procedural category whereas the conceptual category consists of multiplicative relationship, division or halving, the relationship between four quantities, equivalent ratios involving doubling, tripling, or halving, the covariance of quantities, and misconceptions.

Apart from that, the respondents' mental picture of *comparison of ratios* includes part-to-part as one to many comparisons, part-to- whole and whole-to-part as many to many comparisons. Further, they explained the comparison of ratios as ratios with different positions, location, switched places, interchanged, and reversely said. The year five pupils imagined that the total for both the ratios is the same or the whole remains invariant. However, a few respondents explained that both the ratios are not the same. Diagram 4 shows the examples of comparison of the part-to-part and part-to-whole ratios.

Diagram 4. Mental images of comparison of the part-to-part and part-to-whole ratios.

In the procedural category, one of the respondents used repeated addition as the mental image of proportion as in Excerpt 3 and Diagram 5.

Excerpt 3: Shidi's mental image of proportion
I: What crosses your mind when you hear the word "proportion"?
S: Adding.
I: Can you explain.
S: Thinking for a while. Hmmm. Smiling. Knocking the pen on the mini whiteboard. Closing the eyes. Knocking the marker pen on the forehead.
I: You got any ideas.
S: One house add one car is two.
So, two houses add two cars become four.
I: Why do you add?
S: That is a proportion
I: What else?
S: No more.

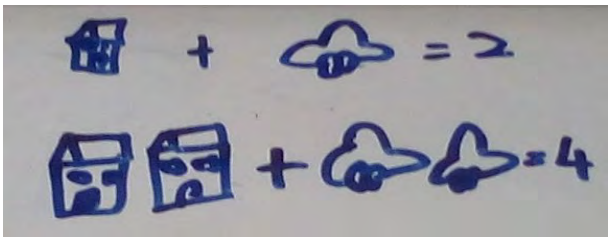


Diagram 5. Shidi's mental image of proportion

In the conceptual category, four of the respondents stated that proportion involves a multiplicative relationship. The multiplicative relationship is between two ratios as in Excerpt 4 and Diagram 6.

Excerpt 4: Nadira's mental image of proportion
S: I have one black ball and two blue balls. So, if I have three black balls then I must multiply two blue balls with three. I will get six blue balls. (Draws six blue balls)
I: Can you explain further?
S: For one black ball there are two blue balls. So, for three black balls then there will be two blue balls multiply by three. One black ball multiply with three will be three black balls. Then the two blue balls also multiply by three. Two times three will be six. So, for three black balls, there will be six blue balls.
I: Is there anything else?
S: No, nothing else.

Diagram 6. Nadira's mental image of proportion

RQ 2: Year five pupils' representation of ratio and proportion

The findings show that many modes of representation were utilized in representing their ideas of ratio. In the symbolic category, the respondents represented with manipulative, visual, and colon representation. In the procedural category, the respondents represented part-to-part comparison using similar and different measures, one to many, many to many correspondences, two parts become a whole and dual ratio. In the conceptual category, the phrase 'for every' was used while representing and stating the ratios.

Two of the respondents surprisingly could represent part-to-part and part-to-whole at a single situation. The respondents used the word all or total while explaining the concept of whole (a: a + b). While comparing part-to-whole ratios, the respondents represented the comparison between a unit/item that has a quantity to the total of the whole quantity/items. While representing the respondents arranged the chips in the part-to-whole correspondence as (a: a +b or b: a + b) and whole to part correspondence (a + b: a and a + b: b). Diagram 7 shows examples of the representation of part-to-part and part-to-whole and whole-to- part ratios.

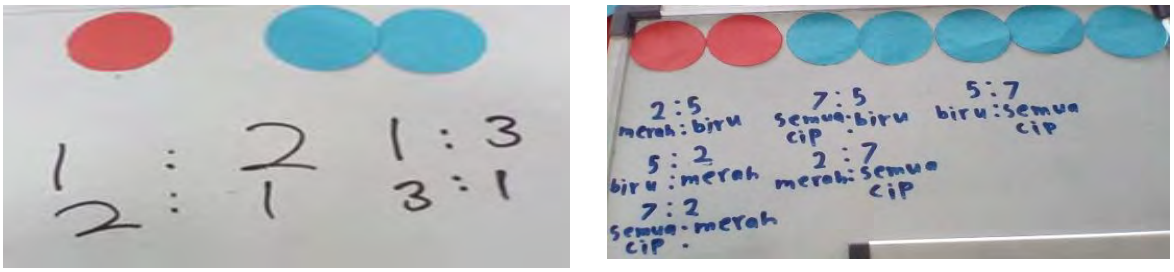


Diagram 7. Representation of part-to-part and part-to-whole and whole-to- part ratios.

Overall, the usage of manipulative items like coloured chips, sponge balls, coloured clips, and paper clips in this study had helped the respondents to represent ratio and proportion in better meaning-making. Therefore, the usage of these kinds of materials, as manipulative representations help the pupils to express their ideas. They were able to make connections between the numerical operation and the physical activity of arranging the chips. In the conceptual category, the idea of doubling was also

noticed when the respondent went on saying two to five, four to ten, and six to fifteen while arranging the chips. Tripling was observed when the respondent multiplied ratio two to five with three to get ratio six to fifteen in a way of recognizing the multiplicative relationship between the ratios. This is also called as abbreviated building up method or a sophisticated building up method (Karagoz Akar, 2007).

The respondents also used figurative representations. For instance, after arranging the chips, the respondents stated the ratio verbally together with their hand and head gestures. They moved their hands and fingers and drew patterns and finally stated that ratios formed have patterns and they turn into proportions as shown in Diagram 8. In representing proportions**,** the respondents arranged the chips accordingly to form equivalent ratios and later wrote them as equivalent fractions.

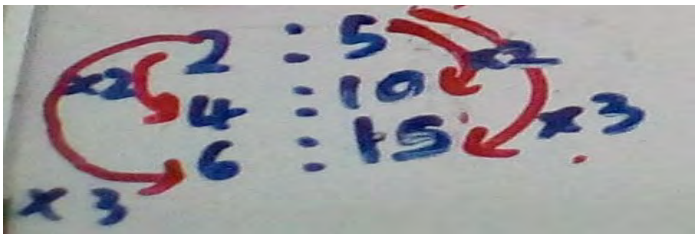


Diagram 8. Drawing arrows to represent proportions.

RQ 3: Year five pupils' meaning of ratio and proportion.

The year five pupils gave meaning to the ratio by explaining based on their prior knowledge**.** The comparisons varied according to comparison between numbers, items, things, same units, parts, similar measures (pure ratio), and different measures. By comparing numbers a few respondents stated that for ratio one to two, the number one is in front and number two is at the back, one thing to two things and the total is three things. However, a few respondents had the idea that ratio one to two has different numbers. They compared two black circles and three white circles as a difference according to the number of chips and their colours. In this situation, the respondents have the conception that ratios are comparisons of two numbers*.*

Even though the comparison of ratios deals with comparison of two quantities most of the time, however in my findings the year five pupils could compare three types of fruits in a single situation. The ratio was compared between three items as in Diagram 9.

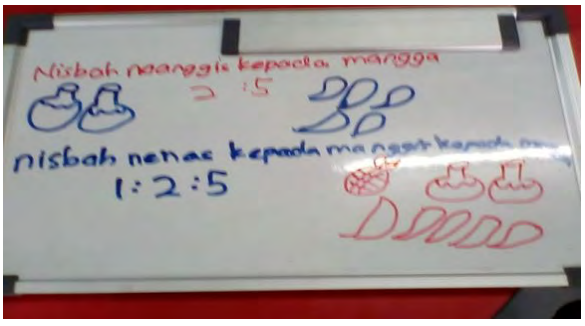


Diagram 9. Nadira's meaning of ratio.

In the procedural category, one of the year five pupils gave the meaning of ratio one to two as one item being shared by another two items. The idea of partition here is the sharing, one umbrella shared by two people. In comparing three boys and two girls, the respondents partitioned the genders and used the terms separate to form the ratios. All the respondents stated ratio of two black chips to three white chips as part-to-part ratios (a: b and b: a) as in Diagram 10.

Diagram 10. Shidi's meaning of ratio as part-to-part ratio.

One pupil used the idea of '*in every*' to give the meaning of part to a whole by stating in every five chips there are two black chips and in every five chips, there are three white chips as in Excerpt 5.

Excerpt 5: Faziq's meaning of part to whole ratio
I: What else?
S: There are two black chips and three white chips. The total is five chips. The whole thing is five, but in every five chips, there are two black chips and three white chips.
I: Can you explain further?
S: In every five chips there are two black chips. In every five chips, there are three white chips.
I: Ooh. What else?
S: No more

Furthermore, while giving meaning to proportion two respondents could see the relationship between the ratios and extended the second ratio by addition. Here the respondents had advanced thinking of proportion because they used repeated addition or building up strategy to form new ratios that are proportional to the initial ratio as in the Diagram 11.

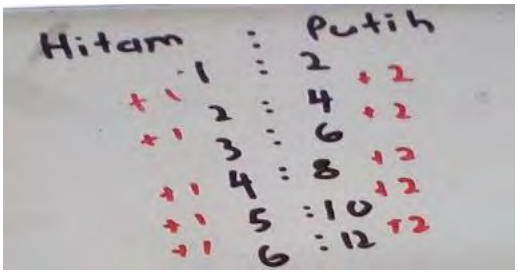


Diagram 11. Daniel's meaning of proportion using repeated addition.

In the conceptual category, one of the respondents explained on equivalent ratio and had the idea of saying a ratio and its equivalence. Three of the respondents stated the ratios formed as equivalent ratios and equivalent fractions. Later they related the ratios as proportions. The meaning given can be algebraically stated as a/b =c/d. Students should know that fractions always compare part to the whole while ratios may compare part-to-part. They must be aware of the importance of multiplication comparisons involved in ratios and proportions. Additionally, the respondents stated that proportion involves a multiplication relationship. For example, the ratio of two girls to three boys will become ratio four to six.

Only one respondent explained using the term simplifying for the ratios formed. However, three of the respondents used the term division while simplifying the ratios. The respondents used division to simplify the ratios (halving) or to unitize the proportion to get back the initial ratio which has the algebraic relationship as 1(k):2(k) =2:4, with k is the integer. They formed proportion through equivalent ratios and extended their explanation by saying that the proportions can be simplified to the initial ratio.

In the case of Amira, she gave the meaning to the ratio of two black chips to three white chips by writing it as ratio four over ten to six over ten. Then she converted them as ratio forty percent to sixty percent. Further, she converted the fractions to decimals by stating the ratio as zero point four to zero point six as in the diagrams below.

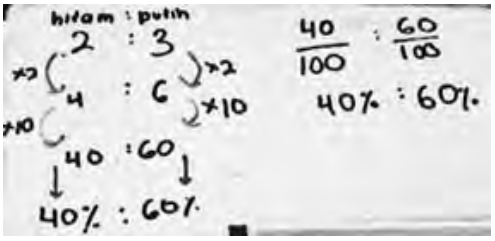


Diagram 12. Amira's idea relating ratio and proportion with fraction and percentage.

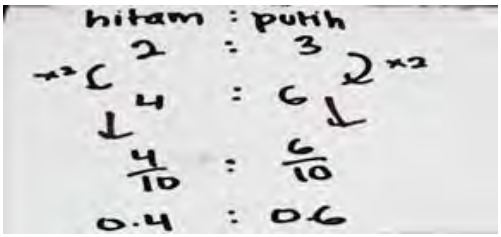


Diagram 13. Amira's idea relating ratio and proportion with fraction and decimal.

In another situation, the ratio of mangosteen to all the fruits was ratio two to eight and was simplified to ratio one to four. Furthermore, one respondent used the idea of multiplication and division simultaneously while forming ratios and explained on proportions. The meaning given for the ratio of three boys to two girls was in the form of a percentage 3:2 as 60 % to 40%. It was surprising that in this study two pupils were able to relate ratio, decimal, and percentage in a single situation. They were able to interrelate the topics.

Additionally, one respondent compared the ratios and gave meaning to proportion using scalar relationship which shows the relationship between ratios. Another respondent compared ratios and gave meaning to proportion using functional relationship which shows the relationship of within ratios. The diagrams below illustrate on these ideas.

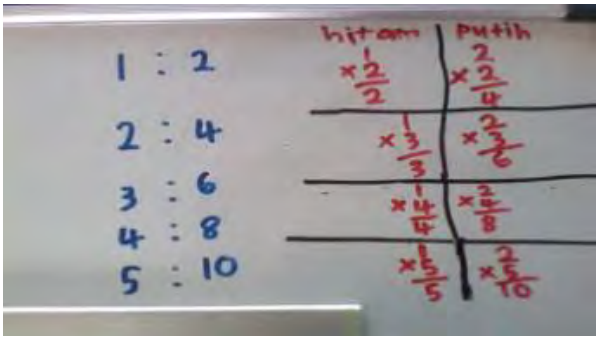


Diagram 14. Faziq's meaning of proportion

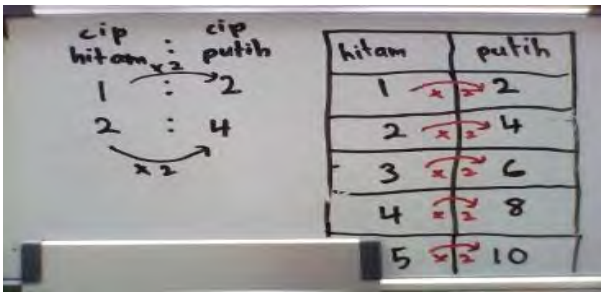


Diagram 15. Maiz's meaning of proportion.

Further, in this study, subitizing was used to form the ratios. (e.g., one of the pupils said two- three then four-six). Their mental action was accompanied by hand and eye movements and gestures as subitizing is an activity involving reflections and mental activity. One respondent equaled the ratios to perform proportions, and another gave meaning to proportion by drawing columns and rows. Arrows were drawn which show the within ratio, connection, and the invariant relationship and in between the ratios which eventually the series of ratios were concluded as proportional.

DISCUSSION AND CONCLUSION

The results showed that the year five pupils were able to construct knowledge that makes sense and gave the meaning of ratio as comparison of quantities, part-to-part comparisons, useful in daily activities, belongings, cooking and parts of the body. Some of them had the idea of relating ratio to fractions which implies the idea of part-to-whole. Meanwhile, proportions are meaningful to them as two equal ratios that are related multiplicatively. Therefore, the findings are consistent and related to past literature. Ben-Chaim, Fey, Fitzgerald, Benedetto & Miller (1998) stated that proportions are two equal ratios. Further, proportions are with the multiplicative relationship between two quantities (Karplus, Pulos & Stage, 1983; Dole, 2008; Petit et al., 2020).

Further, the findings of this research show that the year five pupils used real-life situations in imagining and giving meaning to ratios and proportions. The current teaching methods on mathematics should focus and emphasize more on this matter and need to serve the purpose of learning by life experiential instead of teaching it as theoretical subjects. There should be a paradigm shift from procedural skills to conceptual understanding. Conceptual knowledge is characterized most clearly as the knowledge that is rich in relationships. Educators can opt for constructivist teaching and learning using prior knowledge and real-life situations rather than applying the drill and practice in the mathematics classroom. In addition, the textbook should emphasize that the ratio can be represented in variety of ways.

However, some limitations arise throughout the study. The sample of this study is limited to year five primary students and precisely only seven students are being interviewed. Additionally, because I used seven participants, my results are limited. It would be beneficial for future studies to examine the conceptions of ratio and proportion of a greater number of participants using other methods and research designs. Furthermore, special focus on future studies should be allocated to the difference of ratio and rate as observed in this study particularly in comparing ratio with similar and different measures. In addition, the findings show that pupils were able to use relative thinking to make sense of other mathematical concepts, such as relating ratio and proportion to fractions, decimals, and percentages Therefore, future research can focus on this area.

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