Developing Students' Understanding of Percentage: The Role of Spatial Representation

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Abstract: The main goal of the current study is to develop intervention of learning to promote students' understanding of percentage. The design of the interventions employed the spatial representation of percentage in the form of a bar model and was designed based on the pedagogical concepts of Realistic Mathematics Education (RME). The participants were taken from year-four primary students (around 10 to 11 years old). The data were collected from classroom observation during the implementation of the interventions. The findings show that the design of the learning interventions supports the students in developing their understanding of several fundamental ideas of percentage. The students could make sense of the proportional relationship underlying percentage in applying counting strategies involving proportional relationship, such as doubling, halving, multiplying, and dividing. They could add or subtract two percentages and treat a percentage as an operator. The spatial representation of percentage in the form of a bar plays a critical role during learning. First, the bar helps the students in visualizing the proportional relationship underlying the two magnitudes of percentage. Second, the bar aids the student in noticing the part-whole relationship underlying the percentage. Third, the bar model triggers students to perform flexible counting or computation strategies. Fourth, the bar helps the student to perform a mental computation. Fifth, the bar facilitates the students in keeping track of their computation process. Sixth, the bar model helps students to switch their thinking easily and mentally between the two magnitudes. Seventh, the bar triggers the students in estimating their counting.

INTRODUCTION

As percentage is often employed in many practical applications therefore it is a substantial topic in the core curriculum for science and social subjects at schools (Baroody, Baroody, & Coslick, 1998; Parker & Leinhardt, 1995; Schwartz & Riedesel, 1994; van Galen & van Eerde, 2013). It is a mathematical idea for communicating proportional relationships in a hundredth. However, many studies report that percentage is highly challenging to make sense for many students for its complex mathematical relationships underlying the concepts (van Galen & van Eerde, 2013; Cole & Weissenfluh, 1974; Jannah & Prahmana, 2019; Parker & Leinhardt, 1995). For example, students are still facing challenges in defining the meaning of percentage and how a percentage relates to another percentage. For example, what does 10% means? How 10% means can be related to 50%?

By employing the power of spatial representation (Putrawangsa, 2021; van Galen & van Eerde, 2013), this study aims to investigate the characteristics of instructional tasks that support early-grade students in constructing their conception of percentage. The instructional activity employs the use of the spatial representation of the percentage in the form of a bar model to trigger or facilitate students to do mathematical exploration. The main learning goal of the instructional activity is to support students in developing their understanding of the proportional relationship of percentage and percentage as an operator.

Developing Understanding

In this study, we view developing understanding as to the process of establishing a rich and meaningful connection among the mental representations of mathematical ideas. Mathematical understanding is constructed by making connections between the new knowledge and the existing knowledge (Barmby, et al. 2007; Piaget, 1976; Skemp, 1982). The new knowledge is assimilated into a proper existing knowledge constructing an ability to identify the new knowledge (Piaget, 1976; Slavin, 2019). If there is sufficient existing knowledge to assimilate the new knowledge, it will create associations between them that produce an understanding. Nickerson (1985) defined this relationship as “the more one knows about a subject, the better one understands it, and the richer the conceptual context in which one can embed a new fact, the more one can be said to understand the fact.” (p. 235-236). This indicates that developing an understanding of a subject involves forming as many as possible links between the subject and the existing knowledge, for example, by connecting between two different ideas which have not been related before.

Therefore, in this study, developing understanding is defined as the progress of making the connection between the new knowledge and the relevant existing knowledge such that the new knowledge can be employed as a way of thinking or reasoning.

The Percentage

The fundamental notion of percentage is the idea of proportionality where it explains the proportional relationship between two magnitudes or ratios, namely the percentage and its reference (Parker & Leinhardt, 1995; van Galen & van Eerde, 2013). The proportional relationships involve one-hundred part-whole relationships that give a non-absolute measure but a relative measure (Fosnot & Dolk, 2002). It implies that the percentage as part-whole relationship expresses the relative value of the part compared to the whole. For example, the value of 20% is not absolute. It is always relative depending on the whole that the 20% refers to. Here, students are not required to explain the relation in such a formal manner, but they have to exhibit an awareness that percentages are always associated with something (the referent magnitude) and, therefore, they cannot be associated without taking into account what they refer to (Marja Van Den Heuvel-Panhuizen, 1994). As it represents part-whole relationships, percentage involves mathematical ideas of ratio (Parker & Leinhardt, 1995). In addition, percentages offer a uniform operator or fractional comparison of distinct amounts (Fosnot & Dolk, 2002). For example, in discounts and interest rates context, percentages act as an operator, e.g., 50% off whatever price. In the context of 50% of $120, for example, the percentage acts as an operator of looking at or finding half of the $120.

The recent study aims to develop students' understanding of percentage as a proportional relationship (e.g., ratio and part-whole relationship) and the percentage as an operator since these notions are among the most critical ideas in learning percentage for early grades.

Spatial Representation

In general, spatial representation in mathematics can be regarded as the representation of mathematical ideas which trigger the use of spatial reasoning to think and reason about the ideas. For example, the spatial representation of numbers, such as number lines, is used to trigger students to see and explore the structure and the relation underlying the magnitude of numbers (Fosnot & Dolk, 2001b). The array representation of multiplication fosters students to think about the multiplicative structure of rectangular surface area (Putrawangsa, 2013). Many studies highlight that the use of spatial representations of mathematical ideas in mathematics learning, such as number line, bar, and array, influence the way students think and the reason that fosters mathematical understanding (Barmby, Harries, Higgins, & Suggate, 2009; Fosnot & Dolk, 2001a, 2001b; Putrawangsa, 2013; Putrawangsa & Hasanah, 2020a, 2020b; Hendroanto, et al. 2018; Marja Van Den Heuvel-Panhuizen, 2003; van Galen & van Eerde, 2013).

In the context of percentage, it is suggested that the spatial representation of percentage in the form of a bar model effectively facilitates students in making sense of mathematical ideas underpinning percentage (Marja Van Den Heuvel-Panhuizen, 2003; van Galen & van Eerde, 2013). The bar model allows students to imagine, reason, and communicate the proportional or the part-whole relationship represented in percentage (van Galen & van Eerde, 2013). Moreover, the representation supplies a stronghold for estimating the percentage and the relative value represented by the percentage, especially for the problems involving numbers that are not simply converted to a simple fraction or percentage (Marja Van Den Heuvel-Panhuizen, 2003). The bar model, furthermore, supports the students with more opportunities to make progress.

Since a percentage represents complex proportional relationships between two magnitudes, it is required an external representation can be used to show the relationships. Therefore, the bar model is advised as it indorses several benefits (Fosnot & Dolk, 2002; van Galen & van Eerde, 2013). First, the bar model has a surface area that makes it easier to speak in the terms of the part and the whole area. Next, the bar model can be used to record the track in making estimations involving percentages. Third, the bar model supports students to progress their understanding. Moreover, the bar model is an effective visualization used for teaching percentage as it lets the student think and reason flexibly through the visualization. It helps students to easily look at the relationships between the magnitude of the percentage and the magnitude where the percentage belongs to. It assists students to flexibly switch over their thinking from one magnitude to another which prompts students to develop various computation strategies, such as doubling, halving, and preserving ratios.

The Instructional Design

The instructional activities in this study are designed based on the view of Realistic Mathematics Education (RME). RME suggests that students will learn mathematics effectively when they are allowed to investigate phenomena that are meaningful (realistic) for students. RME calls this heuristic as didactical phenomenology (Freudenthal, 1986; Gravemeijer, 1999; Larsen, 2018; M. Van den Heuvel-Panhuizen & Drijvers, 2020). The didactical phenomenology is the core idea of RME (Larsen, 2018) suggesting that learning mathematics should begin from phenomena that are meaningful for the student, that request to be organized, and that promote the progression of learning processes. According to Gravemeijer (1999), the goal of a phenomenological exploration is to encounter "situations for which specific approaches can be generalized and to find situations that can arouse paradigmatic solution procedures that can be taken as the basis for vertical mathematization". The didactical phenomenology principle suggests that the instructional designer should provide students with contextual problems in the form of phenomena that are meaningful for students. The phenomena should trigger students to experience reinventing mathematical ideas and the emergent model of mathematical thinking and reasoning under the teacher's support, facilitation, and guidance (Gravemeijer, 1999).

Considering the didactical phenomenology principle, the recent study uses the power indicator of a computer or laptop as the learning context to actualize the didactical phenomenology principle of RME. The choice of such a context is based on several considerations: First, the computer power indicator is a familiar context among students due to the massive use of computer-like devices recently. Second, a computer power indicator is usually presented in form of a bar representing a percentage and of the remaining power represented by the percentage. This feature makes it possible to connect the context to the notion of percentage and the bar model to represent the percentage. Moreover, the bar representation of the power indicator can be an effective context to connect students not only to the bar model but also to the idea about double number line model where this model allows students to think and reason flexibly about percentage and the relative value represented by the percentage (Fosnot & Dolk, 2002).

RESEARCH METHOD

This study aims to not only understand how the students can be supported to learn percentage but also to understand how each characteristic or element of the intervention impact students' thinking and responses. Therefore, to acquire such a deep understanding, this study is conducted in a small-scale study involving two year-four primary students (around 10 to 11 years old). The selection of the participants was conducted randomly and administrated by the school.

This study was administrated according to the Design Research framework involving three main phases: design preparation, design experiments, and retrospective analysis (Gravemeijer & Cobb, 2013; Plomp & Nieveen, 2013; Putrawangsa, 2019). During the preparation, the literature review was conducted to clarify the essential mathematical ideas underlying percentage and research findings regarding the teaching and learning of percentage. The findings from the literature review were then used to inspire the formulating of a learning intervention to support students in learning percentage. The learning intervention was articulated in the form of a hypothetical learning trajectory (HLT) depicting the learning goals, the sequence and the form of the learning activities to achieve the learning goals, and the conjectures of students' responses (thinking and reasoning) toward the learning activities (Simon, 1995). The overview of the intervention is elaborated in the result section.

In the next phase, the design experiments, the design of the learning intervention has experimented with the targeted students in a classroom setting. The purpose of the experiment is to critically observe and understand how the interventions work (or do not work) in shaping students' thinking or reasoning to the conjectured responses. In the experiment, one of the researchers acted as the teacher while the other researchers observed the learning. As the source of the data, the whole classroom activities, including students' actions and conversations, were video recorded, and students' works both on paper and onboard were collected and documented. Researchers' findings and impressions on the learning were discussed and documented right after the learning as additional supporting data to clarify the learning.

Finally, in the retrospective analysis, the whole set of the data generated by the classroom experiment were analyzed to identify how the interventions shape students' understanding (thinking and reasoning) of percentage. The analysis was guided by three reflective questions formulated in what, how, and why questions, namely: What learning activity in HLT does work to promote the emergent of the conjectures of students' responses elaborate in the HLT? How does it work (or not work)? Why does it work (or not work)? To answer the questions, the data from classroom experiments, including student works and other learning artifacts, were analyzed comprehensively and critically to acquire understanding and explanation on what, how, and why the interventions shape students' cognitive development. To answer the first question (what), students' actual responses during the learning were compared with the conjectures of students' responses in the HLT episode by episode. Then, the data relating to the episode were investigated to answer the second question (how). Finally, the last question (why) was answered by explaining or discussing the findings from the previous two questions (what and how) through the eye of the relevant theories to gain a better understanding of the phenomena.

RESULTS

The presentation of the result is elaborated into two sections. The first section elaborates the overview of the learning intervention, and the second section focuses on presenting students' responses regarding the intervention.

Overview of the Learning Intervention

Based on the literature review on percentage, the learning interventions were designed to develop students' understanding of the part-whole relationship underlying percentage and the percentage as an operator.

The computer power indicator was used as the context of the learning (see Figure 1). There were two percentage problems proposed to the students through the context. The first problem was about finding the time if the percentage of the remaining time is given and the whole time is known, for example, "*If the power indicator shows 100% or fully charged, the laptop lasts about 4 hours 40 minutes or 280 minutes, so how long the laptop last if the percentage of the power indicator is shown 60%?"* This problem is questioning about the part if the whole and the percentage of the part are known. Through the problem, it is estimated that students will have the opportunity to explore the part-whole relationship and the relative value of percentage. The students could likely see the part-whole relation between the percentage and the value represented through the percentage. For example, they understand that halving the percentage will change the value of the percentage in the same ratio. For example, halving the percentage means halving the value represented by the percentage. As the students investigate the part-whole relationship of percentage, the students will be aware of the use of percentage as an operator. For example, defining 50% of 280 as multiplying 280 by ½ or dividing 280 by 2.

While the first problem asks for "What is P if P is X% of the whole W?", the second percentage problem involves "What is the percentage of P from the whole W?", for example, "*If the power lasts for 280 minutes indicated by 100%, what is the percentage of the power lasts for 112 minutes?"*. The problem is about obtaining the percentage of the remaining time if the remaining time and the full time are identified.

In each problem, some follow-up problems were given to progress students' understanding to a higher level. For example, the students were asked to determine the remaining time if the laptop is charged for 20%, 45%, or 80%. Meanwhile, the follow-up questions for the second problem questioned the percentage for 56 minutes, 84 minutes, or 112 minutes.

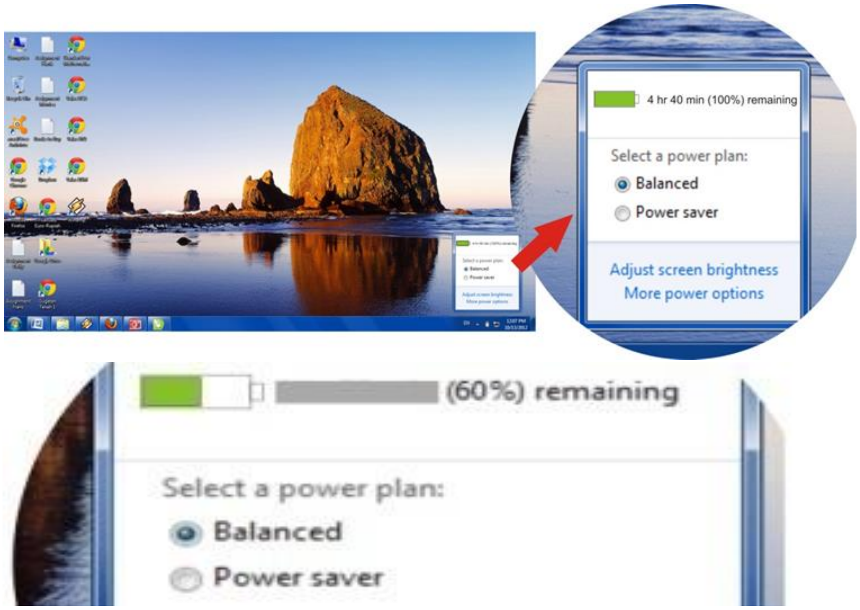

Figure 1: Computer power indicator

To foster the RME's principle of progressive mathematization (Gravemeijer, 1999), the students were given non-contextual problems of percentage to check students' ability in generalizing their understanding beyond the context. For example, the students were asked "What is 70% of 400?" It is expected that the students can generalize the use of the bar model to think about mathematics problems involving percentages.

Students' Responses and Reasoning

To start the learning, the teacher introduced the learning context to the students by showing a laptop where the power indicator of the laptop shows 85% indicating that the laptop will hold up for 2 hours and 20 minutes. First of all, the students transformed the time into minutes. They found taha 2 hours and 20 minutes is 140 minutes. As the teacher asked them about the 85%, the students knew that the number stands for the percentage of the remaining power of the laptop. The students also knew that 100% implies that the laptop is charged fully. It is noticed that the discussion on the context successfully motivated students to explore more about the percentage in the context of the power indicator.

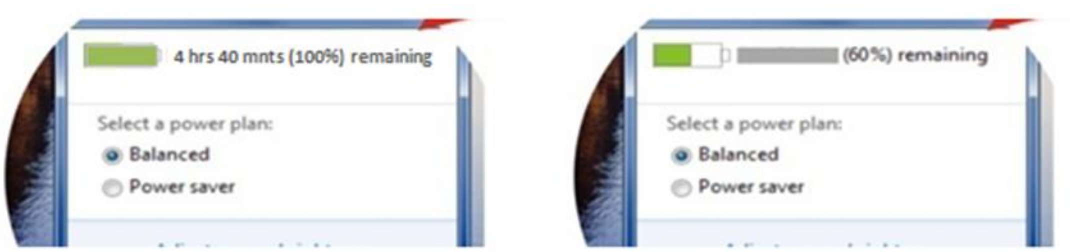

Figure 2: The first percentage problem

As the students were getting motivated, the teacher then gave students the first percentage problem. The problem questioned the student to figure out the time if the laptop had been charged for 60% and if it was fully charging it lasted for 4 hours and 40 minutes (see Figure 2). The students knew that 4 hours and 40 minutes refer to 280 minutes. Students' first response to the problem shows that the student fooled by the variable of the percentage and the time (i.e., adding 1 hour and 20 minutes with 10%) as it is shown in the following students' talk: "*Since 100% is 2 hours and 40 minutes, So half of it is 1 hour and 20 minutes which is 50%, but I need 10 (10%) more to get 60%, So, it is 1 hour and 20 minutes plus 10 which is 1 hour and 40 minutes*".

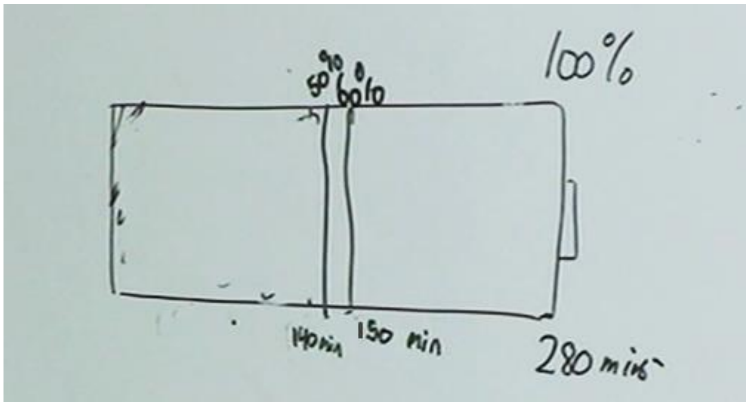

Figure 3: Students' first strategy to determine the minutes for 60%

To prompt the students to reflect on their thinking, the teacher drew the replication of the power indicator in the form of a bar-like representation on the classroom board. The teacher then asked the student to write down what they knew from the given problem on the representation (see Figure 3). To express their initial solution, the student drew the line in the middle of the bar to indicate 50% and wrote 140 minutes representing the time for the 50%. They recognized that 50% is one-half of 100%, therefore, they need to halve 280 minutes as well resulting in 140 minutes intended for 50%. Since they intended to get 60%, they added 10% to both the percentage and the time magnitudes, therefore, they concluded that 150 minutes is for 60% (see Figure 3). Since the finding was considered incorrect, the teacher asked students to reflect on their findings by looking at the bar carefully. After a while, they realized that they made a mistake, but they were still in difficulties in finding the minutes for 60%.

To help the students, the teacher proposed finding the minutes for other familiar percentages, such as 25%. It was easy for the students to get the minutes for 25% by just looking at the bar. Students' reasoning is shown by the following students (S) and teacher (T) discussion:

Student: *"It is 70 minutes".*

Teacher: *"How do you know that?"*

Student: *"Since it is half of the half, you divide 140 by 2."*

Half of the half indicates here is the half of 140 (50%) which is 70 since 140 is the one-half of 280 (100%) (see Figure 4).

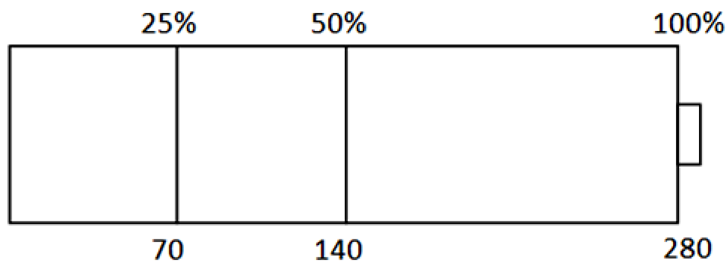

Figure 4: Students' solution to get the minutes for 25%

The teacher then questioned whether 25% would aid them to get the minutes for 60%. They said that 20% would not help them. Instead, they thought that 20% or 10% might help them to get the minutes for 60% as they looked at the bar. Interestingly, the student could think that 25% would not help them to get 60%, instead, they realized that 10% or 20% could help.

Then, the students opted to work with 10%. Here is their strategy to get the minutes for 10%. they sketched a line on the left side of the bar in proportion and wrote 10% on the percentage magnitude (see Figure 5). The following record shows their thinking.

Teacher: *"So, how many minutes for 10%? Probably, you can find the relation between 10% and 50%."*

Then, the students looked at the bar and said:

Student: *"I divided this by 5 (pointing 140)."*

Teacher: *"How do you know that you have to divide it by 5?"*

Student: *"Because the distance between 10% and 50% is 5. So, if I divide this (pointing 140) by 5, I will get 10%."*

Teacher: *"That will be interesting. Why don't you find it out for 10%?"*

After a while, they said:

Student: *"It's 28 minutes"*

Teacher: *"28 minutes. How do know you that?"*

Student: *"Cos I divide this by 5 (pointing 140)."*

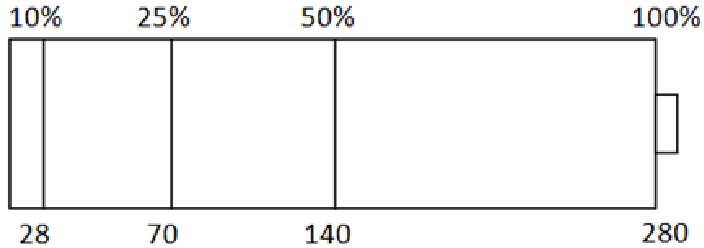

Figure 5: Students' strategy to get the minutes for 10%

Then, the teacher incited the student to think about the connection between the minutes for 50% and 10% to get the minutes for 60%. However, the students still used their earlier strategy which adds the same amount on both magnitudes without realizing that each magnitude represents a different whole. Here, the students were still confused in understanding the difference between the two magnitudes (the percentage and the minutes) since each of them refer to a different unit and whole.

After a while, instead of looking at the relationships between 10% and 50%, the students invented a different strategy. They multiplied both magnitudes by 6. They claimed that 6×10% leads to 60%, and consequently, 6×28 minutes produces 168 minutes. Therefore, they claimed that168 minutes are for 60%. Students' reasoning can be seen in the following excerpt. The word *'distance'* in the conversation refers to the difference between 10 and 60.

Student: *"I do 28 times 6"*

Teacher: *"28 times 6? How do you know why you have to time it by 6?"*

Student: *"mmm… the distance between 10 and 60 is 6. So, this is the lowest we can go (pointing the minutes for 10% which is 28 minutes). So, I do 28 times 6 (he got 168 for 28 times 6)".*

Teacher: *"That is interesting."*

Next, the teacher suggested the students see the relationship between 10%, 50%, and 60%. By looking at the bar, the students were encouraged to look at the minutes for 10% (28 minutes) and 50% (140 minutes) and how the information can be connected to the minutes for 60% (168 minutes). Finally, the students could see that if 10% is added to 50% it will be 60%. Therefore, they need to add the minutes of the 10% (28 minutes) to the minutes for the 50% (140) to get the minutes for 60% which is 28+140 = 168 minutes.

Student: *"Since we have 10% and 10% is 28, we do 140 (50%) plus 28 (10%) which is 168 which is the minutes for 60%."*

Before resuming the activity, the teacher engaged the student to do a reflection of what they had accomplished to find the minutes for 60%. They said that they had two approaches to get the minutes for 60%. The first is multiplying the percentage and the minutes by 6, and the second is combining (adding) two combine percentages and consequently their respective minutes.

To enhance student understanding, the teacher offered the student a follow-up problem which was figuring out the minutes for 45%. The students knew that 45% represents 126 minutes. The following excerpt shows students' thinking toward the problem.

Student: *"I subtracted 50% by 10% to get 40% that is 140 (50%) minus 28 (10%) which is 112. Then, I halved 28 (10%) to get 5% (the minutes for 5%) that is 14. Then, I added 14 (5%) to 112 (40%) to get 45% which is 126 minutes (see Figure 6)."*

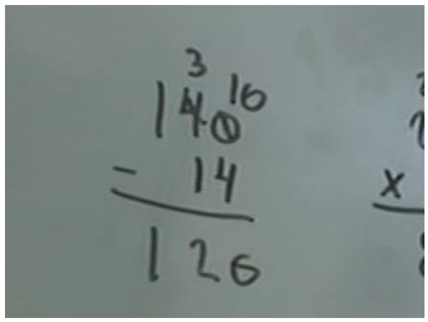

Figure 6: Students' computation to get the minutes for 45%

Another follow-up problem was given to the students. They are asked to determine the minutes for 30%. The students looked at the bar for a while and they found it in 84 minutes. They drew the line for 30% on the bar (see Figure 7) to check their findings.

Student: *"It is 84 (minutes)."*

Teacher: *"How do you know that?"*

Student: *"It is from 28 times 3. Here is 10 (10%). 10 times 3 is 30 (30%). 10% is 28. So, 28 times 3, which is 84 (minutes)."*

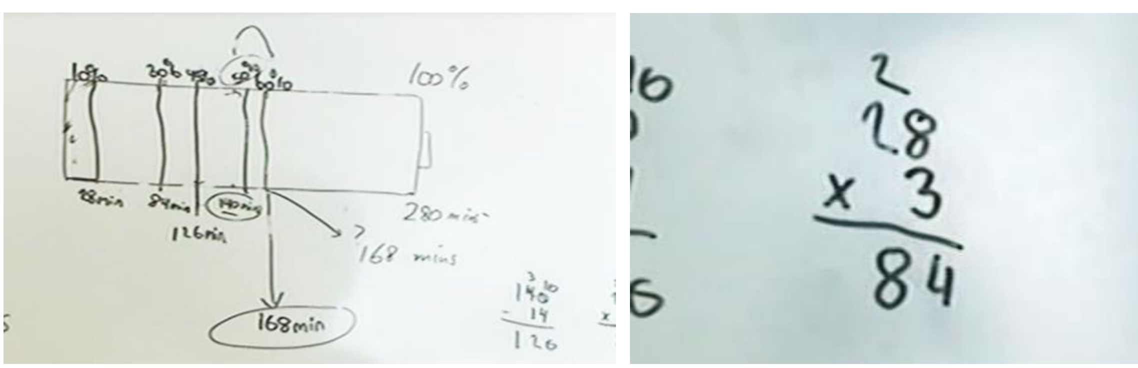

Figure 7: Students' solution to get the minutes for 30%

Afterward, the students were asked to select their percentages and were asked to determine the minutes of the selected percentages. The students selected 20%. By looking at the bar, they calculated the minutes for 20% mentally and said it is 56 minutes for 20%. They multiplied 10% by 2 to get 20%. Consequently, they the minutes for 10% (28 minutes) by 2, and they got 56 minutes (see Figure 8). It seems that the student could conserve the ratio on both proportions.

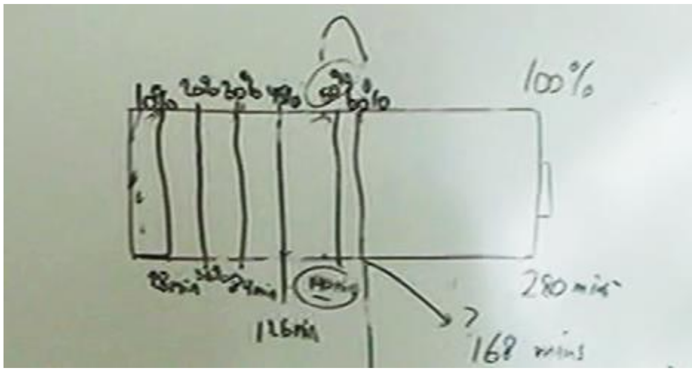

Figure 8: Students' solution to get the minutes for 20%

The student then tested whether they could find the minutes for 1% by asking the students to find out the minutes for 21%. However, the students would not be able to solve the problem although they had been suggested to see the connection among the known percentages and to use his previous counting strategy. The teacher then transformed the problem into finding the minutes for 85%. To figure out the minutes for 85%, they multiplied 28 minutes for the 10% by 8 to get 80% resulting in 204 minutes. Then, they added 14 (5%) to 204 minutes to get the minutes for 85% which is 218 minutes (see Figure 9).

Student: *"It is 28 times 8 because 28 is 10(%), so 28 times 8 which is 204. I need 5% more, which is half of 10(%). It will be 5%, and 5% will be 14 minutes. So, I added 204 plus 14, which is 218".*

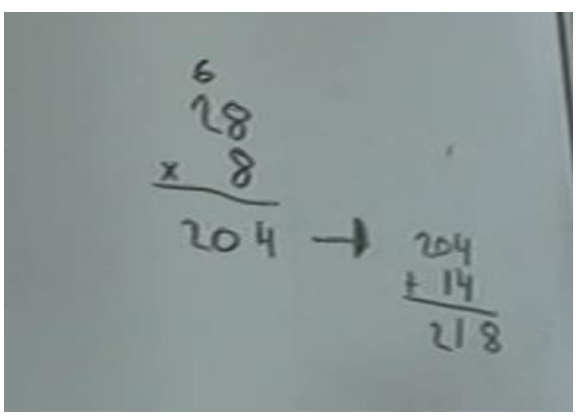

Figure 9: Students' counting strategies to get the minutes for 85%

As the students could make sense of the problems regarding '*finding the minutes if the percentage and the whole minutes are known'*, the teacher then pointed to the second percentage problem as follows: *"If the power indicator shows that the laptop lasts for 280 minutes indicated by 100%, what is the percentage of the power lasts for 112 minutes?"* It is identified that it was not difficult for the student to deal with the problem as they just looked at the bar for a while and said: *"56 (20%) times 2 which are 112 minutes. So, it for 40%"* Then, they said that if the laptop lasts for 112 minutes it has been charged for 40%. It seems their previous experience of working with the bar helps them in reasoning and visualizing their thinking about the problem. As the follow-up problems, the students were asked to determine the percentage for 42 minutes and 210 minutes. Figure 10 shows students' solutions to the problems. Their solutions indicate that they could reason and calculate through the bar by seeing the change on the ratio on both magnitudes to produce the value of other percentages or to figure out the percentage of a given value.

"What is its percentage if the laptop lasts for 42 minutes?"
"I added 28 (10%) plus 14 (5%) which is equal to 42 minutes and 42 is for 15%."

"What is its percentage if the laptop lasts for 210 minutes?"
"I did 28 times 8 which is 224. It (224) is also (for) 80%. Then, I halved this (pointing 28 minutes for 10%) I got 14 (5%). So, I did 224 minus 14 which is 210. So, that is (for) 75%."

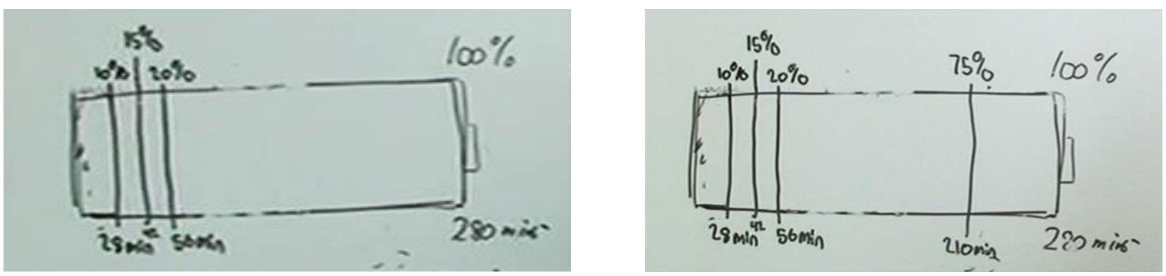

Figure 10: Students' solution in finding the percentage for 42 minutes and 210 minutes

To facilitate the process of progressive mathematization (the process of abstracting mathematical ideas out of the context, such that the ideas become mental mathematics which finally can be utilized as a tool to think or reason mathematically), the students were asked the following problem: *"What is 70% of 400?"* Interestingly, they could answer the problem by developing counting strategies on the bar model. Firstly, they found 40 for 10% since 10×10% is 100% and 10×40 is 400. Then, they multiplied 10% by 7 to obtain 70% and also for 40 and they got 280 minutes for 70% (see Figure 11).

Here, they could maintain the ratio of the change on both magnitudes (the percentage magnitude and the value where the percentage refers to). Here, they could recognize the link among percentages, such as dividing 100% by 10 to obtain 10% and multiplying 10% by 7 to produce 70%.

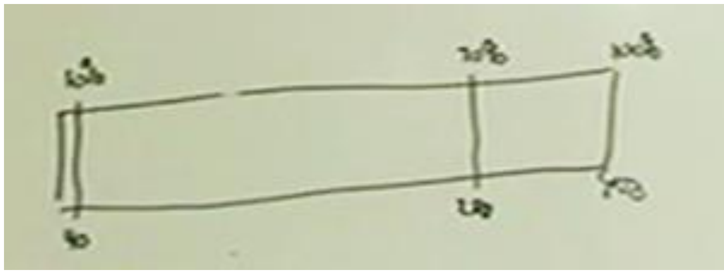

Figure 11: Students' reasoning in finding 70% of 400

DISCUSSION

Regarding the findings, it is claimed that the learning context together with the bar representation of the percentage plays a significant role in developing students' understanding of the percentage. The students were able to recognize the part-whole relationship (e. g. 50% means one-half), recognize the transformation on the percentage magnitudes proportionally (e. g. multiplying or dividing both magnitudes by the same number), and consider a percentage as an operator (e. g. seeing 50% of 400 as halving 400). The next paragraphs will discuss in detail students' cognitive development and the role of spatial representation in learning.

Student's Development of Understanding

In the learning, the students showed their understanding of the proportional relationships underlying percentage and treated percentage as an operator. Such understanding can be observed through the use of proportional-based strategies in solving percentage problems, such as doubling and halving, multiplying or dividing percentages, adding or subtracting percentages, and working with ratios.

Doubling and Halving

The doubling and halving are mostly applied simultaneously as doubling is the inverted operation of halving. The idea of doubling and halving as a counting method has a long root in human-life history, especially among Egyptians and Russians (Fosnot & Dolk, 2002). To get 4 x 12, Egyptians will begin counting from $1\times12 = 12$, $2\times12 = 24$, $4\times12 = 48$, meanwhile Russians calculate as the following: $4\times12 = 2\times24 = 1\times48 = 48$.

The findings of the current study display that the students were able to utilize doubling and halving in dealing with percentage problems. In figuring out the minutes for 20%, for example, they doubled the minutes for 10% to obtain the minutes for 20% and again doubled it to identify the minutes for 40%. Compared to doubling, it is identified that halving is utilized more frequently. It is probably because the student usually starts solving percentage problems by looking at or thinking of 100% first, then, halved the percentage to get the smaller percentage, such as having 100% to get 50% and halving 50% to get 25%. They even were able to determine 2.5% by halving

10% to obtain 5% and then halving 5% to get 2.5%. They knew that if the percentage magnitude is halved or doubled, they also need to do the same on the minute magnitude. Here, they are aware that they need to maintain the ratio on both magnitudes once doing doubling or halving.

The spatial representation of the bar facilitates them to come with and apply the doubling or halving strategy. Such spatial representation stimulates the students to construct the meaning and the impact of the manipulation. When halving, for instance, the student could see that the bar is being divided into two parts equally; meanwhile, the bar is being extended in doubling. The visual image of the bar triggers the student to recognize the connections between division and multiplication where doubling means multiplying by 2, and halving implies division of 2. For example, they recognized that to get half of 140 minutes, they divided 140 by 2. To make the minutes for 20% from the minutes for 10%, the student multiplied 10% by 2; and consequently, they doubled 28 (the minutes for 10%) as well to have the time for 20%.

In addition, the bar shows the spatial visualization of the whole and the parts which lead the students to recognize what the whole and what the parts refer to. For example, when halving 100%, they understood that they had to halve the minutes for 100% (the 280 minutes) to have the minutes for 50% that is 140 minutes. Here, they knew that 50% is one part of the 100% (as the whole). Meanwhile, 140 minutes is one part of the whole (the 280 minutes). At the end of the lesson, they could differentiate the whole in each magnitude (the 100% and the 280 minutes).

Working with A Ratio by Multiplying or Dividing

Multiplying and dividing percentages are the other two counting strategies applied by the students in solving percentage problems. The multiplication and the division are based on the doubling and halving since doubling and halving employ the multiplication by 2 (doubling) or the division of 2 (halving) and preserve the ratio on the percentage magnitudes and the magnitude represented by the percentage. The ratio represents the relation between values expressed in numbers to articulate how one is different but related to the other (Walter, 2004). For example, if there are two men and six women in a class, the composition can be expressed in the proportion of 2:6 which is equivalent to the ratio of 1:3. The ratio 1 : 3 is also equivalent to the proportions generated by multiplying or dividing the ratio by a constant number. Such a strategy of establishing equivalent ratios is then called preserving ratio (see Fosnot & Dolk, 2002). The ratio also indicates the quantitative relationship between a magnitude to the whole. For example, the ratio 1:2 indicates that the proportion of 1 to the whole remains 1 over 1+2 (equivalent to 1/3). Meanwhile, the ratio 1:5 shows that the proportion of 1 to the whole is 1 over 1+5 (equivalent to 1/6).

The findings show that the student could identify the idea of ratio and utilize the idea in solving the percentage problem. For instance, in determining the minutes for 10% provided that 100% indicates 280 minutes, they divided both magnitudes (percentage and time magnitude) by 10 (100%: 10 and 280: 10) resulting in that the minutes for the 10% is 28 minutes. Additionally, to determine the percentage for 224 minutes, they multiplied both magnitudes by 8 since 8×28 is 224, therefore, the percentage for 224 is $8\times10\%$ which is 80%. Looking at the students' reasoning, elaborated above, it indicates that the student could recognize the idea of ratio preservation. They understood that, when the percentage magnitude is multiplied or divided by a specific number, the same action is also necessary to be done for the other corresponding magnitude, for example, the minutes, to preserve the ratio of the percentage and the time magnitude.

Working with the ratio in dealing with the percentage problem seems to be inspired by the model used to visualize the problem. The use of the bar model to represent the problem aids in seeing the part-whole relationships of the percentage magnitudes and the referent magnitude. The visualization of the bar model eases the student to see the changes on both magnitudes simultaneously concerning the constant ratio. Figure 11 shows students' solutions in determining the minutes for 50%, 25%, and 10% knowing that 100% is for 280 minutes. To determine the minutes for 50%, they halved the 100%. The bar indicates that 100% is for 280 minutes, therefore, they also halved 280 and obtained 140 minutes for 50%. A similar strategy was also done to get the minutes for 25% by halving the minutes for 50%. The strategies seem to be triggered by the visualization of the problem in the form of a bar representing double number lines, percentage on one side, and the minutes on the other side. This spatial visualization inspires the students to develop logical thinking and reasoning. For example, if one magnitude is halved, so another magnitude is necessary to be halved as well to preserve the ratio constant on both magnitudes. In addition to doubling and halving, the students also utilized the division by 5. To get the minutes for 10%, for instance, they divided 50% by 5. As 140 minutes represent 50%, it is necessary to divide 140 by 5 as well to get the minutes for 10% (see Figure 12).

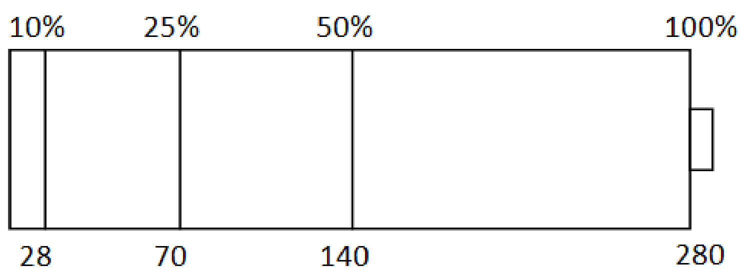

Figure 12: Students' counting strategies to determine the minutes for 50%, 25%, and 10%

Another example shows that the students are aware of the ratio represented in percentages. In solving the percentage of the minutes for 210 minutes while 280 minutes indicated 100%, the students firstly, solve the minutes for 10% by dividing 280 minutes by 10, and then they said, "*I did 28 times 8 which is 224. It (224) is also (for) 80%. Then, I halved this (pointing 28 minutes for 10%) I got 14 (5%). So, I did 224 minus 14 which is 210. So, that is (for) 75%*" (See Figure 13).

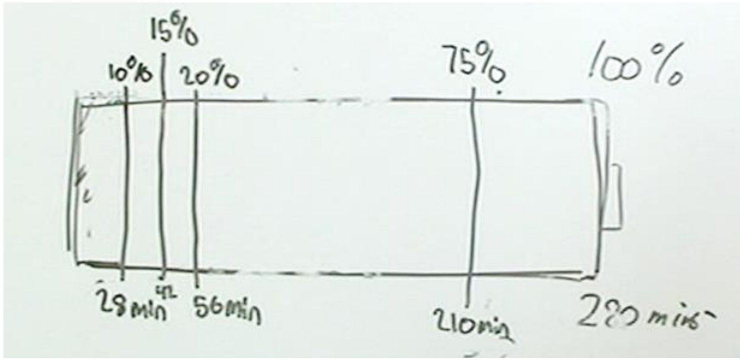

Figure 13: Students' solution in determining the percentage for 210 minutes

This complex counting strategy shows that they could think flexibly and meaningfully in dealing with percentages. They simultaneously employed all known computation principles, such as doubling and halving, multiplying, or dividing the magnitudes, and adding or subtracting known percentages to generate other percentages. Here, they used 10% as the benchmark to get 75% by applying the counting principles. Moreover, working with the ratio shows that they treated the bar model as a ratio table. The counting strategies shown in Figure 13 can be represented in the ratio table shown in Table 1. If they applied an algorithm, they would not be able to generate such a flexible, rich, and complex way of thinking. they might just see percentage as a certain procedure of calculation rather than seeing percentage as relations on relations.

It seems that the spatial representation of the percentage in the form of a bar facilitates the students in developing complex but meaningful counting strategies. The visualization triggers students to easily see the possible relationships among the information presented on the bar. For example, the spatial visualization of 100% on the bar triggers the students to do a series of halving to get 50%, and 25%, adding 50% to 25% to obtain 75%, dividing 50% by 5 to produce 10%, doubling 10% to reach 20%, subtracting 50% by 20% to obtain 30%, or multiplying 30% by 3 to acquire 90%. Such a flexible and meaningful way of thinking and reasoning will not be easily developed or thought if the students learn percentages procedurally through memorizing or applying algorithms.

Percentage	Minutes	Reasoning
100%	280	280 minutes indicate 100%
10%	28	Dividing 100% and 280 by 10
80%	224	Multiplying 10% and 28 by 8
5%	14	Dividing 10% and 28 by 2
75%	210	As 80% is subtracted by 5%, 224 is subtracted by 14.

Table 1: Ratio tables for in determining the percentage for 210 minutes

Adding or Subtracting Percentages

It identified that adding or subtracting percentages are among the counting strategies employed by the students to solve the percentage problem. For instance, they added the minutes for 50% and

10% to determine the minutes for 60%. They also used similar strategies to produce other unknown percentages. The minutes for 45%, for example, is generated by subtracting the minutes for 50% by 10% (to get the time for 40%). Then, they added the minutes for 40% to the minutes for 5% obtaining the minutes for 45%. Moreover, to determine the percentage for 42 minutes, they combined 28 minutes (10%) and 14 minutes (5%) to get 42 minutes. Therefore, the percentage for 42 minutes is 15%. The counting strategy indicates that the student could identify the relationships among the known information. It shows that the student could flexibly switch their thinking between the percentage magnitude and the minute magnitude or the other way around. This shows an understanding of the associations between the two magnitudes (ratio relationship) and an understanding of the relationships among the known information within each magnitude (part-whole relationship).

To support the development of such a complex and flexible way of thinking, it requires a proper model that helps the students to visualize and mentally record their reasoning process. Here, the bar model plays a critical role. The findings show that the bar model helps the student to keep them on track when counting which reduces the cognitive load once counting. Sometimes, the student just looked at the bar and then got a solution. For instance, in figuring out the percentage for 42 minutes, the student just observed the bar and did a mental computation. By looking at the minutes for 10%, they counted mentally that adding the minutes of 10% to the minutes for 5% leads to the minutes for 15% which is 42 minutes.

In addition, the spatial representation of the percentage supports the students to see the relationships among the known information to generate new information. The simultaneous visualization of numbers on percentage magnitude and the time magnitude aids the student to see the relationship among the numbers. In findings the minutes for 40% and 60%, for example, the students added 10% to 50% to get 60% and subtracted 50% by 10% to obtain 40%.

The Role of the Spatial Representation

Spatial representation in mathematics is the representation of mathematical ideas into spatial constructs or affairs which triggers the use of spatial reasoning to think and reason about the ideas. For example, the spatial representation of numbers in the form of number line foster students to associate the meaning of the magnitude of numbers as the distance from zero (Fosnot & Dolk, 2001b). Meanwhile, the spatial representation of multiplication in the form of array foster students to think of multiplication as surface area and use the properties of area to explore the multiplication or another way around (Putrawangsa, 2013).

Many studies highlight that the use of spatial representations of mathematical ideas in mathematics learning, such as number line, bar, and array, influence the way students think and the reason that fosters mathematical understanding (Barmby et al., 2009; Fosnot & Dolk, 2001a, 2001b; Putrawangsa, 2013, 2021; Putrawangsa & Hasanah, 2020a, 2020b; Marja Van Den Heuvel-Panhuizen, 2003; van Galen & van Eerde, 2013). In the context of percentage, for example, the spatial representation of percentage in the form of a bar model effectively facilitates students in making sense of mathematical ideas underpinning percentage (Marja Van Den Heuvel-Panhuizen, 2003; van Galen & van Eerde, 2013; Jannah & Prahmana, 2019). The bar model allows students to imagine, reason, and communicate the proportional or the part-whole relationship represented in percentage (van Galen & van Eerde, 2013). Moreover, the representation supplies a stronghold for estimating the percentage and the relative value represented by the percentage, especially for the problems involving numbers that are not simply converted to a simple fraction or percentage (Marja Van Den Heuvel-Panhuizen, 2003). Therefore, the bar model provides the students with more opportunities to progress.

The bar model which is also considered as a double number line (Fosnot & Dolk, 2002) provides simultaneous information and data of percentage. This model can be used effectively to show the part-whole relationship (proportional relationship) and the ratio relationship underlying percentage. Such characteristics indicate the didactical use of the model to support the students in developing their understanding of percentages.

According to the recent study, the spatial representation of the percentage in the form of bar representation facilitates the students in making sense of the given problems, developing strategies to solve the problems, evaluating their solution, and making connections among the solution. The current study identifies at least seven critical roles of the spatial representation of percentage in the foster mathematical understanding of percentages as part-whole relationships and percentage as an operator, namely:

First, the bar model is an effective model to aid the students in seeing the proportional and ratio relationship underpinning percentage. Recognizing the relationship allows the students to manipulate the percentage by applying multiplication or division on the percentage and its reference. For example, to determine the minutes indicated by 30%, the students multiplied both 10% and 28 minutes (the minutes for 10%) by a constant 3 since 3×10% is 30%. Moreover, understanding the relationships allow the student to add or subtract two percentages. For example, the students subtracted 50% by 10% to generate 40% and at the same time, they subtracted 140 (the minutes for 50%) by 28 (the minutes for 10%) to get 112 (the minutes for 40%).

Second, the bar model helps the student in seeing the part-whole relationship represented through percentage. The visualization of the bar model forming surface area facilitates the students to talk in the terms of part and whole. The surface area allows the student to have a sense that 50% simultaneously is half of 100% and a double of 25%, at the same time, 25% is one-half of 50% and one-quarter of 100%. The students also could see that 10% is one-tenth of 100%, 80% is four times of 20%, or 8 times of 10%.

Third, understanding the part-whole relationship and the proportional relationship foster the students to have a flexibility of thinking and developing various complex counting strategies. For instance, to determine the minutes for 75%, the student divided 100% by 10 obtaining 10%. They then multiplied 10% by 8 to get 80%. Afterward, they halved 10% to get 5%. The last, they subtracted 80% by 5% to get 75%.

Fourth, the visualization of the percentage in the form of a bar promotes the student to do mental computations. It is found that the students looked at the bar while doing mental calculations when dealing with the given percentage problems. For instance, in finding the minutes for 75% elaborated above, they first look at the bar for a while and did calculations verbally.

Fifth, the visualization of the numbers on the bar helps in keeping track of the trajectory of the computation process. The bar provides spaces to write the calculation process. For instance, the students drew a line in the middle of the bar (splitting the bar into two equal parts) to indicate the percentage for 50%. Twenty-five percent was generated by drawing a line splitting the area of the 50% into two equal parts. The track of the splitting allows the students to see the relationship between 25% and 100% where 25% is one-fourth of 100%.

Sixth, As the bar shows two magnitudes of percentages simultaneously, it helps students to switch their thinking, for example, from the percentage magnitude to the minute magnitude. It is found that the student could see that 10% is one-tenth of 100% and, at the same time, see that 28 (the minutes for 10%) is one-tenth of 280 minutes (the minutes for 100%).

Seventh, the bar allows students to make estimations while counting. For example, the students could justify that the minutes for 60% must be greater than the minutes for 50% since not only 60% is greater than 50% but also 60% is closer than 50% to 100% on the bar. Looking at the minutes for 5% (14 minutes), the students could estimate the minutes for 2.5% saying that *"the minutes for 2.5% is around 7 minutes as 5% is 14 minutes"*.

The didactical use of the spatial representation of percentage is in line with the study by van Galen and van Eerde (2013) on the use of bar model to promote percentage where they advise three advantages of using the bar in thinking of percentage, such as: First, the bar model has a surface area that makes it simpler to talk in the terms of "the part" and "the whole". Second, the bar model gives a good track to approximate a percentage, especially in cases where the problems involve numbers that cannot be simply converted into a familiar number. Third, the bar model offers more opportunities to progress students' thinking.

The findings of the current study also support the idea of spatialized instrumentation (Putrawangsa, 2021; Putrawangsa & Hasanah, 2020a) or embodied mathematics (Thom, D'Amour, Preciado, & Davis, 2015) where both ideas highlight the role of spatial reasoning in constructing mathematical understanding. The spatial representation of the percentage stimulates the students to use their spatial reasoning to process the information presented on the bar. For example, the bar representation together with students' spatial reasoning facilitates the students in seeing the relationship between 50% and 100% by halving the area or the length of the bar presented 100%. Moreover, the complex relationship between the percentage and its relative value can be easily explained through the correlated dual magnitudes presented in the bar model, namely the percentage magnitude and the value represented by the percentage. For example, the students can see that halving 100% into 50% will consequently halve the value represented by the 100%. Here, the spatial representation of the bar model supply relatively spatial mathematical perceptions to the students (e.g., percentage as an area which can be split or combined) where the perceptions trigger the construction of the intended students' thinking and reasoning which is articulated in the form of spatial mathematical actions through the spatial representation (e.g., halving or doubling percentage through the bar). The mathematical actions action then stimulates other spatial mathematical perceptions and actions. This reciprocal relationship between mathematical perception and actions contributes to the construction of mathematical knowledge and understanding (Putrawangsa, 2021; Putrawangsa & Hasanah, 2020a; Shvarts, Alberto, Bakker, Doorman, & Drijvers, 2021).

CONCLUSION

The findings of the current study indicate that the design of the learning employed the spatial representation (the bar model) fosters the development of the participating students' understanding of the concepts of percentage. They could see the meaning of percentage as a proportional relationship (ratio and part-whole relationship) and percentage as an operator. In dealing with percentage problems, the spatial representation of the percentage inspires the students to develop various strategies, such as doubling and halving, working with ratios (multiplying or dividing both magnitudes to preserve the ratio), and adding and subtracting two percentages. However, it is identified that the spatial representation could not be able to help students to make sense of 1%. The smallest percentage the student could go is 2.5% by halving 5%.

It identified that the spatial representation of the percentage problems in the form of a bar is critical in developing students' thinking and reasoning independently. There are at least seven functions of the bar model in developing students' counting strategies, namely: First, the bar model becomes a powerful visual representation that facilitates the students in seeing the ratio relationship of the two magnitudes represented by percentage (the percentage magnitude, and the magnitude where the percentage refers to). Second, the bar model aids the student in noticing the part-whole relationship of each magnitude. Third, understanding the part-whole and ratio relationship supports the students to generate flexible and meaningful counting strategies. Fourth, the bar provides visualization that aids the students to do mental computation. Fifth, the bar helps in keeping track of the trajectory of the counting process. Sixth, as the bar shows two magnitudes simultaneously, it helps students to switch their thinking easily and mentally between the two magnitudes (for example the percentage magnitude and the time magnitude). Seventh, the bar allows students to make estimation of their counting.

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