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Tracing proof schemes: some patterns and new perspectives

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ABSTRACT

The aim of this paper is to review some studies conducted with different learning areas in which the schemes of different participants emerge. Also it is about to show how mathematical proofs are handled in these studies by considering Harel and Sowder's classification of proof schemes with specific examples. As a result, it was seen that the examined studies were addressed in the learning areas of Analysis, Geometry, Algebra, Linear Algebra, Elementary Number Theory, Probability, Combinatorics, and MIX. Students in early grades tend more towards external and empirical proof schemes. On the other hand, the characteristics of the proof schemes become more sophisticated as the participants' profiles change to pre-service teachers or as they become more specializing in mathematics. Some results are as follows: the academic achievement levels, genders, and grade levels of the participants in the studies examined in this paper have indicated that they have similar traces with the schemes they use. In addition, it has been determined that new perspectives such as examining Harel and Sowder's classification with new lenses, revealing the overlooked roles of some dynamics in proof, or improving the framework provide an important research area in terms of revealing students' potentials.

INTRODUCTION

Mathematics, as a discipline, differs from other sciences in that it uses mathematical proof with a particular rigor (Heinze & Reiss, 2002; Jones, 1997; Krantz, 2007; Zacharie, 2009). Indeed, proof is a tool specific to mathematics, which is considered as a chain of reasoning that provides to reach a certain result by following logical rules (Krantz, 2007). Therefore, proof is particularly the process of reasoning (Fitzgerald, 1996), and generally an important part of mathematics (Hanna, 2000) and even its basic element (Jones, 1997). As to proving process, it is seen as a fundamental activity in mathematics (Almeida, 2000). Therefore, this process is accepted as an integral part of the curriculum (Schoenfeld, 1994) because of emphasizing the relationships between learning areas (Waring, 2001).

Proof can be defined as a logical argument made by an individual to defend or justify a claim to convince himself or others (Stylianou et al., 2006). Based on this sense, the idea of positioning proof at the center of school mathematics has continued from past to present. Thus, it has been increasingly accepted that justification and proof should be central to mathematical learning at all grade levels (Komatsu, 2016). In mathematics education, particularly, it is increasingly accepted that proof is seen as a skill that should be acquired by students (Recio & Godino, 2001). It is thought that the main reason for this is that the proof and proving process are the basis for doing and understanding mathematics and they are necessary in creating, developing, and communicating mathematical knowledge (Stylianides, 2007). Moreover, this increasing trend towards proof is

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related to the fact that mathematical knowledge refers to both knowledge of theorems and definitions and processes used to construct them (Fernández-León et al., 2021).

Seeking the scheme in overcoming challenges

The definition of proof has not been characterized in a common view by mathematicians, mathematics educators and researchers. In addition, the literature indicates that undergraduate students do not have a sound understanding of proof (Knapp, 2005), they have serious difficulties they face in handling the proof as intended way (Almeida, 2000), and even they often experience disappointment when transitioning from analysis to abstract mathematics (Campbell et al., 2000). Traces of these can be seen in the past research that show that students progress haphazardly without knowing what they will do in the future within the scope of a certain curriculum (Van Dormolen, 1977).

Although it is emphasized that reasoning and proof have an important role in mathematics, research result indicated that students still have challenges (Williams et al., 2012). An overlapping trace of these challenges with Campbell et al.'s (2000) study is seen in the views of pre-service teachers that they deal with proof intensively in geometry and they have difficulties in bridging abstract algebra and they are inadequately prepared to teach proof (e.g., Sears et al., 2013).

It is noteworthy that students develop strategies to justify the truth of an operation or the reality of a phenomenon on their own (Flores, 2006). At this point, if some of the paths they followed in their attempts to prove and the processes they followed are traced, the approaches adopted by them and the patterns (similarities) in their approaches can be revealed. These patterns can be determined by examining their proof schemes. For instance, a study based on the idea that associating proof construction with problem-solving showed that analyzing students' verbal protocols while solving problems is useful for providing information about their proof schemes (for details of similarities and differences, see in Stylianou et al., 2006). At this point, Harel (2007) asserted that proving and problem solving are mental actions that characterize the individual's "ways of thinking". While problem solving approaches reflect the ways of thinking related to the act of problem solving, proof schemes describe the ways of thinking related to the act of proving. Therefore, the proof scheme is defined as a permanent feature of the proofs constructed (Harel & Rabin, 2010) and so it is thought that various schemes provide insight into mathematical perspectives and ways of thinking (Sears, 2019).

As a part of some research on proof, development of different taxonomies describe the complexity of proof schemes (Wasserman & Rossi, 2015). Some researchers (Balacheff, 1988; Bell, 1976; Harel & Sowder, 1998; Van Dormolen, 1977) have tried to open a way to clarify the students' difficulties by examining their approaches to mathematical proofs. Evidence has also been obtained that these approaches are efficient in understanding students' difficulties (Stylianou et al., 2006). In this paper, while briefly mentioning all the approaches (Table 1), we will lay emphasis on Harel and Sowder's (1998) classification of proof schemes, which is more detailed than the others. In the following section, the proof schemes of Harel and Sowder dealt with were explained, and then some examples were used to help understand the schemes.

Harel and Sowder's Proof Schemes

Proof schemes are categorized into three as external conviction, empirical and analytical, each of which has subcategories (Table 1) (Harel & Sowder, 1998; Sowder & Harel, 1998, 2003).

External conviction proof schemes

In this scheme, what the student will present to convince himself and others is in external sources. The type of this resource may be one of the following subcategories.

Authoritarian: One uses the authoritarian proof scheme if he (or she) both trusts and is convinced by the approval of the teacher, or the more knowledgeable classmate or the textbook to justify the conclusion. For example, a pre-service teacher for the task 1 (taken from Pala and Narlı's (2018)) based his (her) proof on the direct transfer of the knowledge the instructor had previously shared with the pre-service teachers about Hilbert's Infinity Hotel. Thus, someone more expert than themselves became a convincing tool for him (her).

Table 1
Some frameworks for conceptualizing the proof

Bell (1976)	Empirical justification				
	Deductive justification				
Van Dormolen (1977)	Particular example				
	Example as generic embodiment of concept				
	Deductive argument				
Balacheff (1988)	Naive empiricism				
	Crucial experiment				
	Generic example				
	Thought experiment				
Harel & Sowder (1998)	External Conviction	Authoritarian			
		Ritual			
		Symbolic			
	Empirical	Inductive			
		Perceptual			
	Analytical	Transformational	Internalized		
			Interiorized		
			Restrictive	Contextual	Spatial
				Generic	
				Constructive	
		Axiomatic	Intuitive-axiomatic		
			Structural		
			Axiomatizing		

Task 1: "Demonstrate that the combination of countably infinite number of countably infinite sets are countably infinite."
Pre-service teacher's proof*: "We have explained this with an example. In a hotel with an infinite number of rooms and all rooms occupied, an infinite number of customers were coming and we were asked to place them. We were shifting the registered customers towards infinity and placing the incoming customers in the rooms. Likewise, the combination of countably infinite sets becomes countably infinite, as in the example."

Ritual: One who evaluates the correctness of an argument only in terms of its form (instead of the correctness of the relevant reasoning) has a ritual proof scheme (Martin & Harel, 1989). For example, a pre-service teacher who used the ritual scheme in his proof for the task 2 (taken from Oflaz et al.'s (2016) study) in Figure 1 was asked why he drew the triangle like this. He said that he had been shown such a triangle before. This coincides with his reaction when he encounters something out of the format he is accustomed to from his previous learning (Figure 1, in lines *i, ii*).

Symbolic: Students have displayed the symbolic proof scheme if they treat symbols independently of any meaning or any relation to the quantities in which they arise. For instance, the example in Figure 2 contributes to understanding a pre-service teacher's reaction (*the cosine term vanishes the minus*) to the symbolic proof scheme in the study of Pektaş (2017).

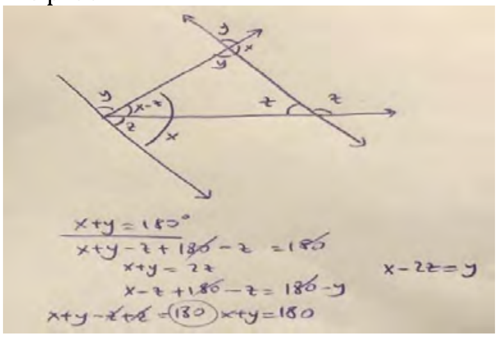
Task 2*: "Prove that the sum of the interior angles measurements of a triangle is 180°"	
His proof: 	His views: (i) *"..Normally triangles are like this..."* (ii) *"..But that is because of high school, because we saw it all the time like that..."*

Figure 1. Geometry; External conviction scheme, ritual

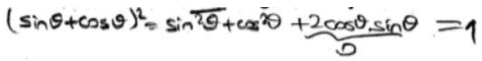
Task 3*: "Why is the following equation true? $\sin^2\theta + \cos^2\theta = 1$. Would this equation be true if the unit circle is not used?"
His proof:


Figure 2. Algebra; External conviction scheme, symbolic

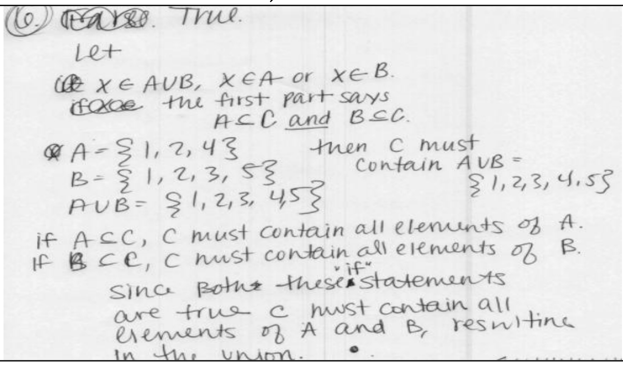
Task 4*: "If A is a subset of C and B is a subset of C, then the union of A and B is a subset of C."


Figure 3. Analysis; Analytical scheme, interiorized

Empirical proof schemes

When the student tries to convince someone else with one or more drawings or what (s)he perceives, it can be said that (s)he is displaying *perceptual proof scheme* (Figure 7). On the other hand, if one tries to convince himself and others by evaluating a conjecture with one or more examples that means he is using *inductive proof scheme.*

Analytical proof schemes

An analytical scheme is a scheme that validates conjectures through logical deductions (Harel & Sowder, 1998, p. 258). It is subdivided into transformational and axiomatic. *Transformational* involves students' justifications dealing with generality aspects of the conjecture and their goal oriented reasoning to resolve the conjecture (Figure 3) (taken from Sears's (2019)). As for the *axiomatic,* in mathematics, knowledge is the cumulative organization of subsequent results as logical consequences of previous ones. This organization includes undefined terms, definitions, assumptions, and theorems. For example, in Euclidean geometry, 'point' and 'line' are not defined, and expressions such as 'a single line goes through two points' are assumed. It can be said that the student operating on such a system has an axiomatic proof scheme.

Focus, research aims and questions

Although proof, by its nature, is frequently addressed primarily in high school geometry (Knapp, 2005), Kleiner's (1991) study also mentions the effects of proofs on different learning areas (analysis, algebra, etc.). Therefore, it is considered important that the proofs should be spread in every learning area of mathematics since the thought about how truth is established in mathematics should not be limited to formal proofs as in geometry (Harel & Rabin, 2010). Considering this suggestion in the literature, it is thought that such an examination (this paper) is needed to see in which different learning areas besides geometry, studies are conducted.

Harel and Sowder introduced the concept of a *student's proof scheme* with the framework they used in their studies (Sowder & Harel, 1998). These researchers stated that while considering the categories of proof schemes, they may be useful for university mathematics students (Sowder & Harel, 2003). However, in their teaching experiments, which included university students and middle school students, they found that the proof schemes of the students allowed for diversity in content areas uttermore.

Moreover, this framework is more widely accepted in the literature (Sarı Uzun, 2020, p. 220). Also, this framework is considered important in terms of screening and evaluating some instructional outcomes. For instance, these schemes allow cognitive sustainability on the three pillars of discovering mathematical relationships, constructing proofs, and formulating conjectures (Rodríguez, 2006, p. 891). Potentially, such a framework can guide teachers and educators in monitoring students' progress and development in a mathematical proof (Lee, 2016, p. 27). This framework overlaps more when compared to other classifications in terms of serving the purpose of our research. Therefore, Harel and Sowder's framework was preferred in order not to overlook the diversity and depth of the participants' ways of thinking, reactions, behaviors in the proving processes. Conducting a literature review taking into account such a framework has the potential to cover all different participants from all grade levels and increase the power of representing as much as possible to view the actions of these participants in the proving processes. In this respect, it is thought that this paper can reveal the similarities or differences in the proof schemes, if any, by allowing the tracing of the participants' schemes to be reviewed from a broad spectrum.

The aim of this paper is to review some studies conducted with learning areas in which the schemes of different participants emerge and with specific examples to show how mathematical proofs are handled in these studies by considering Harel and Sowder's classification of proof schemes. While doing this, it was first aimed to determine in which learning areas (RQ1) and with which participants (RQ2) the studies were conducted. Secondly, it was aimed to reveal what kind of similarities or differences exist in these studies (RQ3). In fulfilling these purposes, the corresponding research questions are accompanied by: 1) In which learning areas are these studies addressed?, 2) What are the proof schemes of the participants?, and 3) What similarities/differences are there in the studies reviewed?

METHODS

The aim of this review study is to examine the studies using Harel and Sowder's framework. More specifically, it is to give an idea of what the proof schemes can be in studies with different learning areas and differentiating participants. Therefore, it is not within the scope of this paper to show a trend between the years. Nevertheless, it is very possible to tell some similarities or differences in the studies examined.

There are, of course, some selection criteria we use when including the studies according to this framework in our review. The studies should: (1) should have been carried out according to the classification of Harel and Sowder, (2) should be after 2000 and even close up to date (as the framework was introduced in 1998), (3) should include different participants such as students, in-service teachers, and pre-service teachers, and (4) should include tasks according to different grade levels and learning areas (e.g., 1-4 grade band; e.g., algebra).

FINDINGS

In this section, the examined studies in this paper were presented under the subheadings in this section as "Learning Areas", "Participants' Proof Schemes" and "Similarities and Differences", respectively. It should be noted here that some of the answers to the research questions can be explained together. For example, some of RQ3 is associated with RQ2. Therefore, some of RQ3 is covered when seeking answers to RQ2, which is about participants' proof schemes.

Learning areas

In the literature, it is seen that there are studies conducted with proof schemes at different grade levels, different participant profiles, different learning areas, and tasks of mathematics. A total of 41 studies (see Table 2), which were positioned in accordance with the main focus of the paper, were reviewed. These studies were organized by the authors of this paper in the learning areas to which they were related in terms of the tasks involved. For example, tasks involving concepts such as "countability", "set", "infinity", "function" were examined in the area of "Analysis (or calculus)". According to this organization, it is seen that the studies examined are in the learning areas of "Analysis", "Geometry", "Algebra", "Linear Algebra", "Elementary Number Theory", "Combinatorics", and "Probability". While some studies include tasks from one of these areas only, some studies involve more than one learning area together. An overview of all of them is given in Table 2. For simplicity, an abbreviation has been used to indicate which learning area these studies cover. These abbreviations are labeled as *AN, GEO, ALG, LALG, ENT, COM* and *PR* for analysis, geometry, algebra, linear algebra, elementary number theory, combinatorics, and probability, respectively. It is also labeled as *MIX* when referring to any study involving more than one learning area. These labels are used in the following sections.

Participants' proof schemes

The proof schemes of the participants are presented in four categories here. These are students' proof schemes, in-service teachers' proof schemes, pre-service teachers' proof schemes, and proof schemes of students specializing in mathematics.

Students' proof schemes

Although the nature of the task varies, it is seen that the students at different grade levels and in the earlier school period generally have similar proof schemes. While the students were able to use most of the proof schemes from different categories and sub-categories in different ways in the tasks they were asked to prove, they tended more towards *external* and *empirical proof schemes.* These tendencies are seen in the studies of Aydoğdu İskenderoğlu (2003), Ören (2007), Kanellos and Nardi (2009), Sen and Guler (2015), Donisan (2020), Ercan (2020), Sevgi and Orman (2020). For example, Donisan (2020) found that most of the students used the ritual proof schemes while about half of them used the empirical proof schemes when justifying the solution of equations. Also, there is no trace of any student using justification involving deductive processes in their first attempts. Somewhat different from the above studies, in her study, Pesen (2018) found that the students mostly used the empirical proof scheme while constructing proofs in algebra tasks and evaluated them as the most convincing. However, in geometric task, the majority of them could not construct any proof, but they found the analytical scheme more convincing in the context of their preferences.

In-service teachers' proof schemes

Knuth's study (2002) [MIX] showed that teachers view visually represented arguments or those based on examples as most convincing. He also found that teachers did not behave in the same way when distinguishing arguments that constituted proof and those of non-proof. He found that they were proficient in the former, but they had difficulty in the latter, and even in relation to the latter that some teachers tended to characterize the empirical as proof. Also, for many teachers in his study, empirical evidence has been a powerful conviction for the truth of the proof. In another study, Şengül and Yılmaz (2021) share similar findings [but in PR] with Knuth: in-service and pre-service teachers mostly use empirical proof schemes.

High school mathematics teachers' proof schemes varied when posing and solving the problems with dynamic geometry software CABRI in Rodriguez's (2006) study [GEO]. He found that teachers mostly use perceptual and inductive proof schemes. However, in a case of the existence of "intuitive-axiomatic" proof schemes has left the importance of these empirical schemes aside. Also, in some cases, such dynamic software has revealed the use of "transformational" and "constructive" proof schemes.

Table 2
Distribution of studies in different learning areas

Targeted context	Learning areas	Studies	Embedded examples in asterisks
RQ1	Analysis	Sarı et al. (2007) İskenderoğlu (2010) Pala & Narlı (2018) Sevimli (2018) Cihan & Akkoç (2019)	Pala & Narlı (2018)
	Geometry	Rodríguez (2006) Ören (2007) Sears (2012) Uygan et al. (2014) Oflaz et al. (2016) Byun & Chang (2017) Nagel et al. (2018) Oflaz et al. (2019) Ercan (2020) Sevgi & Orman (2020)	Oflaz et al. (2016) Ercan (2020)
	Algebra	Harel & Rabin (2010) Pektaş (2017) Donisan (2020)	Pektaş (2017)
	Linear Algebra	Bobos (2004) Kaneko & Takato (2011)	Kaneko & Takato(2011)
	Elementary Number Theory	Stylianou et al. (2006) Cusi & Malara (2007) Stylianides & Stylianides (2009) Çontay (2017)	
	Probability	Plaxco (2012) Şengül & Yılmaz (2021)	
	Combinatorics	Erickson & Lockwood (2021)	
	Mix	Knuth (2002) Aydoğdu İskenderoğlu (2003) Housman & Porter (2003) Kanellos & Nardi (2009) Tossavainen (2009) Williams et al. (2012) Liu & Manouchehri (2013) Şengül & Güner (2014) Sen & Guler (2015) Lauzon (2016) Akin et al. (2017) Pesen (2018) Sears (2019) Grundmeier et al. (2022)	Williams et al. (2012) Sears (2019)

Harel and Rabin (2010) [ALG], one of the studies conducted with teachers, tried to characterize the practices of algebra teachers in the classroom through the *authoritarian proof scheme*, in which the teacher is the sole determinant of mathematical truth in the classroom. Differently from other studies, this study suggests that teaching practices (i.e., TP1-TP9) can support the development of authoritarian proof schemes in students. It was thought that these practices could point to positive accelerations regarding the effects of student learning and teacher change. In other words, these can guide researchers to look at the parameters of change (i.e., teachers' knowledge, beliefs, and general classroom organization), and how teachers change them and their teaching changes in a particular category (p. 18). In her study investigating the impact of textbooks that geometry teachers use to teach concepts and skills related to proof in their lessons, Sears (2012) [GEO] explains the proof schemes that students display in the classroom. She revealed that there is a relationship between the cognitive demand level of a task and the proof scheme used. It is seen in her study that students' *external conviction proof schemes* are more apparent when lower-level proof tasks are presented. Otherwise, it might be concluded that *analytical proof schemes* are more common in proving tasks that require higher cognitive demand.

Pre-service teachers' proof schemes

There is no clear pattern in pre-service teachers as in students. More detailed examinations in the smaller sample group of the studies or the nature of the task may have significantly affected the differentiation in pre-service teachers. An interesting point for these participants compared to students is the existence of pre-service teachers who can predominantly use *analytical schemes*. Examples of these are found in the studies of Cusi and Malara (2007), İskenderoğlu (2010) and Pektaş (2017). Compared to the proof schemes of the students, the use of more than one proof scheme in the same pre-service teacher is seen more prominent here. When more than one scheme is used, the common scheme is sometimes *empirical* at Sarı et al.'s (2007) [AN] and Sears's (2019) [MIX] studies, but *external* at Çontay's (2017) [ENT]. In their studies, in which pre-service teachers examined proof schemes from a global perspective, Oflaz et al.'s (2019) findings are as follows: the schemes common in two different countries are external convictions while the schemes that differ are empirical and deductive.

A remarkable point here is that one of the pre-service teachers in Sears's (2019) study has traces of the "interiorized" proof scheme. Sears stated that for the task 4 given in Figure 3, a pre-service teacher read his own proof, noted the flaws in his proof, and tried to suggest a revised version of it. While developing his argument, he first started with an example (empirical attempt), then noticed a pattern here, and finally came to a conclusion on a generalizable claim through this pattern. Sears stated that he was aware of his scheme and thus reflected an *interiorized proof scheme.*

As in the students, other schemes and sub-schemes were used in some way and more frequently in some of the studies conducted with pre-service teachers. However, unlike the findings in the students, only axiomatic schemes were not used (e.g., Çontay, 2017; Pala & Narlı, 2018) [ENT, AN]. Also, neither the transformational nor the axiomatic scheme was used by different pre-service teachers except mathematics and physics (e.g., Cusi & Malara, 2007) [ENT].

Proof schemes of students specializing in mathematics

Compared to the other participants (students, in-service teachers, or pre-service teachers), this group of participants mostly consists of students who have taken domain-specific mathematics courses (e.g. calculus, analysis) or majored in mathematics. They are distinctly different from the other participants in terms of their background and experiences. For this reason, this participant profile is described as "students specializing in mathematics".

What is striking here is the existence of a pattern in the scheme followed by the students to use a valid argument or generate an acceptable proof, and mathematical arguments they find convincing among them or strategies they use to learn mathematical concepts. In other words, while the arguments of students who dominantly use external conviction schemes are closer to the "invalid" indicator at the point of constructing a proof, those who use the analytic scheme lesser seem to be more successful in giving the proof a "validity" status. An example of this kind can be found the study of Sevimli (2018) [AN]. A similar pattern is closely associated with the terms "mathematically rigorous" and "logically valid" in another study (i.e., Erickson & Lockwood, 2021). The students whose reasoning is accessible for the transformational proof scheme stated that they recognized the argumentation used in a correct combinatorial proof as valid and they were aware that the combinatorial arguments were valid as mathematically rigorous proofs (This also made "contextual restrictive" visible between students' schemes). Others used external (ritual) proof schemes in explaining their reasoning and stated that these schemes do not qualify as rigorous mathematical proofs. Similarly, Bobos (2004) found that students were able to deftly write mathematically correct proofs as if they were using analytical proof schemes [LALG].

On the other hand, the schemes followed in terms of the strategies mathematics students find convincing among the mathematical arguments and the strategies they use to learn the mathematical concepts differ. This situation may result in students being "unsuccessful" in the learning strategy if it is realized by being convinced by an external argument. Otherwise, students who use deductive arguments and are convinced by them to be "successful". An example of this is encountered in the study of Housman and Porter (2003) [MIX].

In Plaxco's (2012) [PR] study, examining the relationships between mathematical proof and definition, students were asked to evaluate proofs in terms of mathematical and logical correctness and get an idea about their proof schemes by identifying aspects of mathematical proof that they consider important and necessary or conversely, unimportant and unnecessary. One has an *intuitive-axiomatic proof scheme,* as the characteristics of the student's proof schemes include examining each line of the argument and looking for a justification, considering the system in which a proof is discussed, and its basic axioms. A similar situation was encountered in another study (i.e., Grundmeier et al., 2022) on the ability to use definitions and assumptions correctly in proofs. The emergence of analytical schemes with high percentages in students' attempts to prove has been an indicator related to formal development. This coincided with their attempt to construct formal proofs and make logical deduction.

Similarities and differences

Here, the similarities and differences in the studies are presented in two categories: some patterns and new perspectives.

Some patterns

Among the frameworks for conceptualizing proof, Harel and Sowder's (1998) proof schemes framework seems to be more comprehensive than Bell (1976), Van Dormolen (1977), Balacheff's (1988) classifications. In addition, Harel and Sowder's schemes in the lower rows (see Table 1) are considered to be more sophisticated and advanced than those in the upper ones (Kaneko & Takato, 2011). However, Harel and Sowder stated that they do not believe that external proof schemes are necessary for the development of analytical proof schemes but that external authoritarian schemes and empirical schemes should fill only the roles of confirmation and assumption at some stages for mathematics students. In the study of Jankvist and Niss (2018), a student who uses the authoritarian sub-scheme of external conviction exemplifies this situation by constantly asking both the teacher and classmates for approval in everything she did in mathematics. In their study, an attempt was made to move this student from at least this scheme to a more empirical one, thus showing her why and how it is appropriate to some extent to "experiment" as part of executing mathematical processes. It is therefore critical threshold to balance about what degree of external and empirical schemes should be allowed. Harel and Rabin (2010) stated that supporting the authoritarian proof scheme is important for teachers to better develop their classroom practices. However, this scheme needs to be reinforced to some extent. Otherwise, students may experience some difficulties where external conviction schemes are more visible. The study of Sears (2012) alerts that when procedural and memorizing tasks are given without establishing a relationship, students will copy whatever the teacher is doing in proof construction process and choose the reasons randomly without evaluating the appropriateness of their choices. Thus, pointing out that the emphasis on without connections and memorized reasons from a literal way may impair creativity and originality. From this viewpoint, ways can be sought to gradually move the external and empirical proof scheme tendencies, which are also seen in the common finding of the studies in the "students' proof schemes" subsection, to analytical proof schemes.

When we look at the studies examined in general, they share some similarities in their findings. Since analytical proof schemes shape the basis of mathematical reasoning, external and empirical schemes should not be overused by students (Dede & Karakuş, 2014). It is seen that there are some attempts in this direction in the studies reviewed to break the external and empirical schemes tendency. These studies are Rodríguez (2006), Stylianides and Stylianides (2009), Lauzon (2016), Cihan and Akkoç (2019), and Grundmeier et al. (2022). For example, in an experimental study (Cihan & Akkoç, 2019), the difficulty experienced in determining transformational and axiomatic proof schemes before the intervention resulted in overcoming these difficulties after the intervention.

The academic achievement levels, genders, and grade levels of the participants (students and pre-service teachers) in the studies examined in this paper indicate that they have similar traces with the schemes they use. The studies of Sarı et al. (2007), Byun and Chang (2017), and Çontay (2017) are in different learning areas such as AN, GEO, and ENT. In terms of *academic achievement levels,* it has seen in these studies that successful participants have the same pattern in using the transformational proof scheme. In terms of *gender*, the studies of Aydoğdu İskenderoğlu (2003), Ören (2007), Şengül and Güner (2014) point out that although they include tasks from different areas, females are in the same pattern in terms of having that they use empirical schemes more dominantly than males. Considering the differentiation of the schemes used according to the *grade levels*, it is seen in the studies of İskenderoğlu (2010) and Pektaş (2017) that the contribution to the differentiation comes from the empirical proof schemes. Also these are studies in which the participants are pre-service teachers from all grade levels, but include tasks from different areas such as AN and ALG.

Regardless of whether the participants are mathematics students, from different disciplines, or pre-service teachers, it is seen that their proving skills in different learning areas (AN, GEO, ENT, MIX) are either on average or they are not at a sufficient level or so on (e.g., Cusi & Malara, 2007; Nagel et al., 2018; Oflaz et al., 2019; Pala & Narlı, 2018; Sen & Guler, 2015; Sevgi & Orman, 2020; Sevimli, 2018; Tossavainen, 2009). It is seen that students in terms of verifying a mathematical result (e.g., Aydoğdu Iskenderoğlu, 2003), and pre-service teachers in terms of how the validity of proofs are evaluated (e.g., Uygan et al., 2014) generally have reasoning close to the empirical proof scheme in the ALG-GEO and GEO, respectively. On the other hand, a study (e.g., Akin et al., 2017) revealed that pre-service teachers enrolled in ALG course tend to use more empirical or external, those of enrolled in GEO course tend to use analytical proof schemes in the same problem set.

Predominantly in studies involving more than one learning area (MIX), traces driven in the way of being able to successfully use a valid argument result in participants pointing to the analytic scheme. Similarly, the same scheme is used in creating acceptable proof, evaluating the proofs in terms of mathematical and logical correctness, finding convincing arguments among them or when it comes to participants' preferences. In the study of Liu and Manouchehri (2013), the reason behind this kind of thinking is related to the fact that students find analytical arguments more convincing or mathematically complete.

New perspectives

Proof schemes of Harel and Sowder were frequently examined with different participants and certain tasks from different learning areas. However, there are also studies suggesting that these schemes have been examined and tested from different perspectives, enabling various dynamics, or the need for some improvements in the schemes. These studies are detailed below.

Perspective 1: K_ETpic graphics

In a perspective about schemes, some improvement (or rearrangement) is suggested to Harel and Sowder's framework in terms of the use of graphics (Kaneko & Takato, 2011). These researches stated that this is because of the fact that the results of a few studies conducted in Japan cause many teachers to worry that using particular figures. Especially in proving process, these figures might catch students' deductive reasoning about mathematical concepts and prevents students from appreciating the generality. Therefore, graphics in printed classroom materials and textbooks that its use tended to be reserved. However, they think that it will provide appropriate environments that encourage students to make sense of the spherical shape of surfaces and to use a graphic tool such as K_ETpic, which allows precise drawings (Figure 4). The researchers analyzed the possible impact of such use of graphics tool on the progress of students' proof schemes. One of the tested tasks is given in Figure 5.

In the proof of this proposition, according to Kaneko and Takato, the high-dimensional R^n model is used as students cannot perceive real objects to support their reasoning. Therefore, deductions are based on symbolic transformations (not geometric intuition). The proof of this proposition is constructed with some properties in the general inner product space. A student who can prove in this way can be considered to be in a "structural" proof scheme. However, according to the researchers, this proof may not allow students to make sense the relationship between transformations and core meaning of structure. Also, it is not easy to understand why the proposition is true by means of this proof. Therefore, this proof can particularly trigger the symbolic proof scheme in students with poor math skills. Based on this idea, with a geometric proof in which graphics play an important role through K_ETpic, students will understand the importance of considering fundamental change and its effect on representation as they transfer an invisible problem to the visible. In summary, the use of graphics can be expected to activate students' interiorized proof scheme (Kaneko & Takato, 2011).

Perspective 2: Gestures and action

The study of Williams et al. (2012) refers to action and gesture, which are important aspects of mathematical communication in proof activities. They contribute to the omnipresent but overlooked role of action and gesture, adding a new lens to revealing invisible proofs. An example of their study is presented in Figure 6. In Figure 6 , the participant's accompanying gestures for persuading (convince to others) highlight the axiomatic scheme as they are based on structural role. In addition, he has a transformational scheme since his actions in (3) and (4) show the reasons why this cannot happen.

Perspective 3: A-didactic environment

In regards to proving skills, researchers need to adopt new lenses beyond the traditional and explore students' potential (Liu & Manouchehri, 2013). Such a perspective dealt with relating students' proof schemes to didactic situations theory. This theory, pioneered by Brousseau (1997), states that the student learns by adapting to an environment (milieu) that produces contradictions, difficulties, and imbalances, just as an individual who is a member of society does. According to this theory, students' interaction with the environment is "didactic" if it occurs in a situation that can be guided by the teacher (or his/her intentions and expectations). If it occurs as a result of students' genuine interest in mathematics that the environment encourages, it is called "adidactic" (p. 30-31).

Under the a-didactic situation of this approach, Ercan (2020) revealed that the environment prepared in this way reflects positively on the students' proof-making processes in geometry subject matter, as a result of his work with middle school seventh-grade students. A problem statement used by the researcher in his study and an example of the answers from the participant groups are given in Figure 7 [GEO]. The researcher stated that the students in this group only gave the proof in Figure 7 that the measures of the corresponding angles are equal. On the other hand, they did not attempt to prove the equality of the measures of the other angles. Also, they used the perceptual proof scheme in their drawings because they had an intuitive understanding.

To sum up briefly, Kaneko and Takato's (2011) study, namely first perspective, showed the importance of students' appreciation of the generality of the proof method. One way to make a leap beyond the empirical schemes may be with the use of a different graphical tool like K_ETpic in their study. Second perspective (Williams et al., 2012) showed that understanding how to produce proof may require students to engage in more than verbal and written work (action and gesture). This new field of study can bridge the gap between deductive and inductive reasoning and help support students who have difficulties especially in abstract concepts that come with proof. Third perspective (Ercan, 2020) has shown that the a-didactic milieu prepared for students positively reflects on students' proving processes. Designing these environments in an earlier stage can be a facilitating interface in learning process of proving.

In addition, in the study of Williams et al. (2012), the context associated with the gesture and action of the participant in GEO learning area allows him to display the analytic (axiomatic) proof scheme that overlaps with the structural role. However, it is thought that referring proofs in every area and examining proofs with tasks from different learning areas is also necessary in order to remove the students' perception that proofs are limited to geometry only. For this, diversifying the tasks to be given to students according to learning areas can be seen as a way to change this limited perception. At this point, the studies in the areas of linear algebra by Kaneko and Takato (2011), probability by Plaxco (2012), the study tasked with analysis of Sears (2019), and combinatorics by Erickson and Lockwood (2021), provide students with appropriate contexts in the use of "interiorized", "intuitive-axiomatic", and “contextual restrictive” proof schemes. In this regard, Bobos (2004) reported that they are planning research that is fully concerned with the design of tasks in linear algebra context, which will better utilize students' theoretical thinking potential and incorporate them into "structural axiomatic" and "axiomatizing" proof schemes.

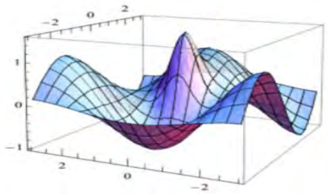


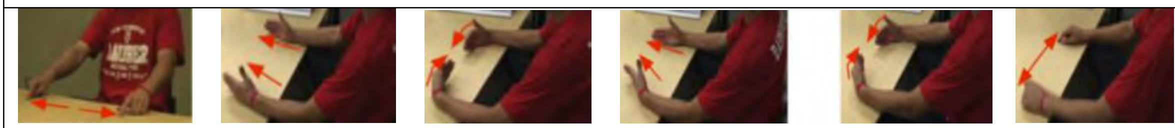
Figure 4. KETpic graphic example [taken from the study of Kaneko and Takato (2011)]

Task 5*: "Presuming that A is a positive definite symmetric matrix and k is a positive number, then the closed domain $D = \{x \in R^n \,|(x, Ax) \leq k\}$ is convex".

Figure 5. Convexity task [LALG]

Task 6*: "For any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side."

Gestures



(1) (2) (3) (4) (5)

Participant begins his answer by taking the side length of any triangle by 5 units (1). Considering the case that the other sides of the triangle are less than or equal to five (2), he shows that these lengths will remain short so that they can come together (3). Then, he states that the other two sides must be longer than the first side (in 4 he indicates impossibility). He returns to the conjecture and comes to the conclusion (5).

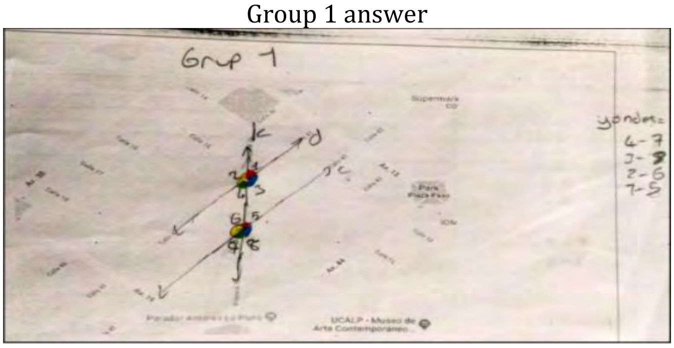
Figure 6. Geometry; Analytical scheme, transformational and axiomatic

Task 7*

"The picture above is a map of part of the La Plata region, which is the center of the province of Buenos Aires, the capital city of Argentina, a South American country. Below is a zoomed-in Google Maps image of a section in the La Plata region."

"In this picture, Calle 41 and Calle 42 streets are parallel to each other, and Diagonal 76 is a street that intercepts these streets. Your task is to find the angles that Diagonal 76 street forms with Calle 41 and Calle 42 and show them which are equal to each other and prove why they are equal."

Group 1 answer



Corresponding angles: 4-7, 3-8, 2-6, 1-5

Figure 7. Geometry; Empirical scheme, perceptual

CONCLUSIONS

This paper provides information about some patterns and new perspectives in the studies that have been considering Harel and Sowder's classification of proof schemes. The proof schemes used by the participants give an idea about their attempts to prove and the similarities in their reasoning while proving. These can set a direction for determining where participants have difficulties while proving and what needs to be done to eliminate them. Indeed, the steps taken to help the participants transform their proof schemes into better versions to the nature of mathematical proof gave positive turns.

In the literature, it is underlined that the proving process is an integral part of the curriculum in terms of emphasizing the relationships between the learning areas. Nevertheless, in the opinions of the participants in some studies, several difficulties related to the proof are still attributed to the intense handling of proof in geometry. Therefore, in order to eliminate the perception that proofs are limited to geometry only, it is necessary to make proof activities or proof-related tasks more visible in different learning areas by referring to proofs in every area. Following the demands in the literature, one of the focuses of this paper is to reduce the pressure placed on geometry in mathematical proof and to spread the proving process to other learning areas. Therefore this paper has provided diversification to meet these needs. As seen in the perspectives, the diversification according to the learning areas made the transformational and axiomatic subcategories of analytical schemes, even sub-subcategories of these proof schemes visible within the framework of Harel and Sowder. For this reason, it is thought that the contexts that can lead to the emergence of analytical schemes, which are expected to be used more by students, should also be taken into account in future studies.

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