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Supporting Pre-service Mathematics Teachers' Professional Noticing of Students' Reasoning About Length

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Abstract:
This study examined how the levels-of-sophistication framework supports pre-service mathematics teachers' professional noticing of students' reasoning about length measurement. Three pre-service teachers were asked to analyse students' written solutions in the tasks that reflected different characteristics of students' reasoning before and after participating in an intervention based on the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework The findings indicated that the levels-of-sophistication framework enabled the pre-service teachers to give their full attention to students' mathematical understanding and to provide proper instruction to support students' learning and hence, it had a significant role in the improvement of the professional noticing skills of these pre-service teachers. Thus, it is suggested that the levels-of-sophistication framework can be used to support pre-service teachers' professional noticing skills and to better prepare them to teach length measurement based on students' reasoning.
Keywords: professional noticing, students' reasoning, length measurement, levels-of sophistication framework, pre-service teachers

INTRODUCTION

Understanding the mathematical thinking of students is crucial for selecting and designing instructional tasks, understanding students' reasoning, assessing students' learning progress, and recognizing and remedying students' learning difficulties (Battista, 2017). Therefore, teachers should take students' learning, understanding, explanations, and unexpected responses into consideration while designing lessons and implementing teaching (Goodell, 2006; Yang & Ricks, 2012). Even though teachers prepare lesson plans and design lessons suitable for students before the class, they may deviate from time to time from these plans taking students' needs into consideration during the class. While doing this, teachers should pay attention to the events that occur in the classroom and notice the instances that provide opportunities for student learning, and they should also elicit students' thinking through questioning in order to eliminate students' misconceptions and difficulties. These practices can be signs of teachers' noticing skills (Jacobs et al., 2011). Teachers who are able to notice students' thinking can create tasks that direct students' attention to appropriate learning opportunities (Mason, 2011). Thus, teachers should be able to identify students' thinking through students' explanations, justifications, and questions, and they should also be able to interpret them, and then make instructional moves based on this recognition (Luna et al., 2009).

Unfortunately, research indicates that pre-service teachers experience difficulties in focusing on students' thinking (Jacobs et al., 2010). Professional noticing of student thinking cannot be developed by itself. Therefore, we cannot expect pre-service teachers to acquire this skill at the very beginning of their teaching profession. In addition, although gaining teaching experience throughout the years facilitates the development of attending and interpreting skills, it alone is not sufficient for the development of the responding skill (Jacobs et al., 2010). In other words, among the three components of professional noticing, the skill to decide how to respond seems to be more difficult to develop than the attending and interpreting skills because this skill necessitates attending to students' strategies, interpreting their understanding, as well as having knowledge of students' mathematical development in order to determine the appropriate next step (Jacobs et al., 2010). As instructional actions, pre-service teachers generally offer re-teaching the concept with a procedural focus (Cooper, 2009) and telling students how to solve given problems (Son, 2013).

Fortunately, research reveals that pre-service teachers' responding to students' mathematical understanding skill can be improved through providing opportunities to analyse students' strategies in the solutions for given problems and discuss possible responses to students' understanding (Schack et al., 2013). Thus, there is a need for pre-service teachers to confront different kinds of students' misconceptions and errors and consider a variety of response alternatives (Son, 2013). Research indicates that practice-based opportunities can help pre-service teachers learn to notice students' mathematical thinking in educational settings (Stockero et al., 2017a).

With these ideas in mind, in this research study, we aimed to explore the professional noticing skills of pre-service elementary mathematics teachers in the context of length measurement. Since we were inspired by the work of Battista (2006), we designed and used tasks about length measurement including different students' solutions prepared by taking into account the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework. The reason for focusing on length measurement is that elementary and middle school students experience difficulties in understanding and learning length measurement (Barrett et al., 2006; Battista, 2006). In addition, as there is a limited number of research studies that investigated pre-service teachers' professional noticing on measurement in the related literature, it is important to shed light on how pre-service teachers notice students' thinking in the context of measurement. The following research questions guided the present study:

(1) How do pre-service teachers attend to and interpret students' understanding of length measurement?

(2) What course of action do pre-service teachers offer based on students' understanding of length measurement?

THEORETICAL BACKGROUND

Professional Noticing of Students' Mathematical Thinking. In the literature, there are various conceptualizations regarding noticing. For example, van Es and Sherin (2002) focus on what important events that teachers notice, how teachers interpret them and how teachers relate these specific situations to general principles of teaching and learning. Santagata et al. (2007) focus on how pre-service teachers analyse lessons and how they reflect on what they notice using a lesson analysis framework. However, since many events take place in classrooms simultaneously, classrooms are considered complex environments. Hence, even though noticing various instances is useful, paying attention to each instance in the classroom can be difficult. Therefore, Jacobs et al. (2010) focus on teachers' noticing of students' mathematical thinking in which teachers attend to students' strategies, interpret their understanding, and make in-the-moment decisions in class about how to proceed. This special form of teacher noticing is called professional noticing of children's mathematical thinking.

Their framework consists of three skills, which are attending to children's strategies, interpreting children's understanding and deciding how to respond on the basis of children's understanding. The first skill in the framework is identifying mathematically significant details in the strategies of students. This skill is important because it enables the teacher to gain insight into students' understanding. The second skill is the ability to interpret students' mathematical understanding by using the details in the students' strategies. The third skill is deciding on instructional responses on the basis of students' understandings. Although there is no single correct response, teachers possessing this skill are expected to have the ability to utilize what they find out about students' understanding from a particular situation, and that teachers' responses should be consistent with research on students' mathematical development.

In the professional noticing framework, the focus is on in-the-moment decision making that appears when a student presents a mathematical explanation and before the teacher responds to it. While making in-the-moment decisions, rather than long-term planning, teachers constantly analyse children's mathematical thinking and associate particular situations with what they already know about the mathematical development of children (Franke et al., 2007; Lampert, 2001).

Students' Reasoning about Length Measurement. The length concept is crucial in both everyday life and formal geometry. Lengths are frequently used by people in daily life to explain the size of objects and the distance covered. In addition, the measurement of length has an important position in geometric measurement because it includes the main concepts of measurement. Therefore, the lack of students' understanding of length measurement hinders students from learning basic concepts in measurement (Martin, 2007). Children initially learn the concept of length, and then the concepts of area and volume since to understand the area (a two-dimensional measure) and volume (a three-dimensional measure), the understanding of length (a one-dimensional measure) is essential. Hence, gaining understanding of the concepts in length measurement is necessary for understanding area and volume measurements and advanced concepts of measurement in the subsequent years (Outhred et al., 2003).

When students are not able to make sense of what they are doing while measuring length, they might experience difficulties in and have some misconceptions about length measurement. For instance, students can have misconceptions about the usage of units. Some students iterate a unit by leaving gaps between units or overlapping units while measuring (Lehrer, 2003). Many students regard iterating as merely putting units end to end, but not as covering the length without gaps (Clements & Stephan, 2004). Furthermore, many students use different units, such as both pencils and paper clips, or they use the same units in different sizes, such as big and small paper clips, at the same time while measuring because they think that the overall length is covered in any case (Lehrer, 2003). In addition, students count numbers next to marks on a ruler rather than focusing on spaces between marks (Bragg & Outhred, 2004). Similarly, Tan-Sisman and Aksu (2016) note that sixth grade (12 years old) students had a variety of misconceptions and errors regarding linear measurement and perimeter. For instance, in their study, students stated that the ruler must be longer than the object being measured and that the centimeter (cm) is not a proper unit to measure an object in meters. Moreover, research shows that students usually confuse perimeter and area when these concepts are taught just by using a set of procedures or formulas (Moyer, 2001). They think that figures with the same perimeter must have the same area (Tsamir & Mandel, 2000), and if the area of a figure increases or decreases, the perimeter of the figure also increases or decreases and vice versa (Tirosh & Stavy, 1999). In order to eliminate these misconceptions and errors and enhance students' understanding, teachers should be able to notice students' reasoning in length measurement.

Non-measurement reasoning and measurement reasoning are two types of students' reasoning about length measurement offered by Battista (2006) (**Table 1**). Non-measurement reasoning does not involve numbers. It is based on comparing lengths according to the holistic appearance of shapes, comparing lengths directly by reorganizing parts of shapes, comparing lengths by pairing up parts of shapes, and comparing lengths of shapes by using transformations (sliding, turning, and flipping parts of the shape).

Table 1. "Levels of sophistication in students' reasoning about length" Conceptual framework (Battista, 2006, p. 141)

Non-measurement Reasoning	
N0	Holistic visual comparison
N1	Comparison by decomposing or recomposing
1.1 Rearranging parts for direct comparison	
1.2 One-to-one matching of pieces	
N2	Comparison by property-based transformations
Measurement Reasoning	
M0	Use of numbers unconnected to unit iteration
M1	Incorrect unit iteration
M2	Correct unit iteration
M3	Operating on iterations
M4	Operating on numerical measurements

On the other hand, measurement reasoning involves unit iteration, i.e., repeatedly locating the length of a small unit along the length of an object, numerical or logical operations on iterations and numerical or inferential operations on length measurements.

Students generally begin to develop non-measurement reasoning prior to measurement reasoning. However, non-measurement reasoning continues to improve even after measurement reasoning emerges. In addition, M4 level indicated in **Table 1** is specified as the most sophisticated level because at this level, students can incorporate non-measurement reasoning by using the strategies in N2 into measurement reasoning. These levels can support teachers' understanding of students' difficulties and misconceptions while students are engaged in length measurement activity. Hence, the levels of sophistication framework is a useful tool not only to improve teaching but also to identify and deal with students' difficulties and misconceptions (Battista, 2006).

RATIONALE OF THE STUDY

The importance of pre-service teachers' ability to recognize significant details in students' actions in order to attend, interpret and respond to students' thinking has been emphasized by teacher educators in recent years. Hence, in the literature, there are various studies in which pre-service teachers' professional noticing skills were explored in different mathematical contents such as pattern generalization (Callejo & Zapatera, 2017); solving equations (Monson et al., 2020); derivative (Sánchez-Matamoros et al., 2015); fractions (Ivars et al., 2020); geometry (Baldinger, 2019; Ulusoy & Çakıroğlu, 2020); statistics (Shin, 2020; 2021); exponents (Ulusoy, 2020) and arithmetic (Fisher et al., 2018; Jacobs et al., 2010; Schack et al., 2013; Warshauer et al., 2021). In these studies, researchers asked pre-service teachers to reflect on students' mathematical thinking through students' written work (Baldinger, 2019; Callejo & Zapatera, 2017; Ivars et al., 2020; Monson et al., 2020; Sánchez-Matamoros et al., 2015; Shin, 2020), class videos (Shin, 2021; Ulusoy, 2020; Warshauer et al., 2021) and video excerpts of clinical interviews (Fisher et al., 2018; Schack et al., 2013; Ulusoy & Çakıroğlu, 2020; Warshauer et al., 2021), or both student work and video clips (Jacobs et al., 2010). Consequently, pre-service teachers' noticing of student thinking was explored in specific content domains of mathematics. As there is a limited number of research studies that investigated pre-service teachers' professional noticing on measurement in the related literature, there is limited understanding of how pre-service teachers notice students' thinking in this content. Thus, it is believed that examining pre-service teachers' noticing of student thinking in measurement can elaborate on previous research and contribute to the literature.

In order to promote learning, teachers should focus on students' mathematical thinking (Simpson & Haltiwanger, 2017). However, pre-service teachers' professional noticing skills cannot develop naturally. Rather, the development of this skill requires a process and extended support (Stockero et al., 2017b). Therefore, how pre-service teachers' noticing skills develop with a certain intervention in a certain process is worth investigating. Research on teaching and learning length measurement documented that students experience misconceptions and difficulties (Barrett et al., 2017; Bragg & Outhred, 2004; Clements & Stephan, 2004; Curry et al., 2006; Grant & Kline, 2003; Martin & Strutchens, 2000; Tan-Sisman & Aksu, 2016). Teachers should understand students' thinking in length measurement, what difficulties and misconceptions students have, and the reasons underlying them (Lehrer, 2003) because if teachers are able to identify these misconceptions and know how to overcome them during the lesson, they can enhance students' understanding (Jaworski, 2004). In addition, it is suggested that pre-service teachers should practice noticing before entering the teaching profession (Tasdan et al., 2015). When teachers learn how to attend to, interpret and respond to student thinking before they actually start teaching, their future instruction will be more likely to enhance student learning. Therefore, more opportunities should be provided for pre-service teachers to examine students' mathematical thinking, misconceptions, and errors in a particular content (Lee, 2021). Thus, in this study, we investigate how the professional noticing skills of pre-service elementary mathematics teachers in the context of length measurement develop through discussions based on the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework. It is believed that by understanding research-based frameworks of students' mathematical thinking, pre-service teachers can recognize significant aspects of contents, associate their knowledge with relevant aspects in students' reasoning and interpret it in order to decide what course of action to take, and thus develop professional noticing skills (Moreno et al., 2021).

METHOD

In the present study, a qualitative methodology was employed, and an exploratory case study was utilized to examine how pre-service teachers notice students' reasoning about length measurement. This method was influential in obtaining an in-depth exploration and understanding of pre-service teachers' professional noticing skills. The case was three pre-service elementary mathematics teachers, and their noticing skills of attending to students' strategies, interpreting students' mathematical understanding, and deciding how to respond on the basis of students' understanding in given tasks in the context of length measurement.

Participants

The participants of the study were third-year pre-service teachers enrolled in a four-year elementary mathematics teacher education program in one of the state universities in Turkey. In general, pre-service teachers who graduate from this program gain the right to become mathematics teachers in middle schools, grades from 5 to 8 (ages 11-14). In order to select participants to participate in the study, firstly, cumulative grade point averages (GPA) of thirty third year pre-service teachers were sorted from highest to lowest. Then, they were divided into three groups in such a way that ten pre-service teachers with the highest GPAs comprised the first group, the following ten pre-service teachers comprised the second group, and the last ten pre-service teachers comprised the third group. Finally, one pre-service teacher from the first group (P1), one from the second group (P2), and one from the third group (P3) were selected by using maximum variation sampling. P1's GPA was 3.41, P2's GPA was 3.02 and P3's GPA was 2.58 out of 4. All of the participants were female. Before the study, participants had completed pure mathematics courses (e.g., Algebra, Calculus, Probability) and educational courses (e.g., Educational Sociology, Educational Psychology, Educational Philosophy) and they were taking a Methods of Teaching Mathematics Course when the study was carried out. They did not take a course that includes the teaching and/or learning of geometry and measurement until the data were collected. Furthermore, they had not taken any Teaching Practice courses and they did not have any kind of internship experience in middle schools. In addition, they did not have an opportunity to observe and examine middle school students' mathematical thinking before.

Data Collection Procedure

Pre-interviews at the beginning of the study and post-interviews at the end of the study were conducted with the participants individually (**Table 2**). During the interviews, students' answers to the three tasks

Table 2. Summary of the data collection process

Procedure	Description	Purpose
Pre-interview	Presenting students' written answers in the tasks to the pre-service teachers and asking questions	In order to determine pre-service teachers' initial professional noticing skills
Session 1	Providing information about foundational concepts of length measurement including conservation, transitivity, equal partitioning, unit iteration, accumulation of distance, origin, and relation between number and measurement	In order to develop pre-service teachers' professional noticing skills in length measurement
Session 2	Introducing "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework	
Session 3	Analysis of students' non-measurement reasoning in the tasks adapted from Battista (2006)	
Session 4	Analysis of students' measurement reasoning in the tasks adapted from Battista (2006)	
Post-interview	Presenting students' written answers in the tasks to the pre-service teachers and asking questions	In order to determine pre-service teachers' final professional noticing skills

(which are explained later) were presented to the participants, they were asked to analyse students' written solutions, and they were asked the following questions for each task: (i) *Can you describe what each student did in response to the task?* (ii) *Can you explain what you learned about these students' understandings?* (iii) *If you were the teacher of these students, how would you respond to them? Can you provide suitable activities that would help students overcome their misconceptions/errors?* All interviews were video-recorded and transcribed for analysis.

After the pre-interviews were completed, participants were exposed to four intervention sessions of 2 hours in length. The intervention was based on research about students' reasoning about length measurement and the conceptual framework involved in students' reasoning levels. The aim was to provide the participants with information, based on the previous research, about how students think when they engaged in length measurement tasks in order to promote the improvement of their noticing skills of students' thinking. In the first session, information about foundational concepts of length measurement including "conservation, transitivity, equal partitioning, unit iteration, accumulation of distance, origin, and relation between number and measurement" were provided to participants. Researchers assert that these concepts constitute the basis for children's understanding of length measurement and should be taken into account in length measurement instructions (Sarama & Clements, 2009). In the second session, participants were introduced to and provided with information about the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework. During the third and fourth sessions, first, participants analysed students' answers that represent each level of students' reasoning about length in the given tasks that are involved in the study conducted by Battista (2006) and identified students' strategies and reasoning levels individually. Then, they participated in a group discussion led by the first author and discussed what they noticed about students' strategies and understanding. In this way, group discussions provided them with an opportunity to hear the opinions of other participants about students' strategies and understanding, and to determine whether they agree or disagree with each other's opinions. In addition, they were asked to decide what course of action they would take after detecting students' misconceptions/errors, if any. After thinking for a while, they explained their suggestions and reflected on other group members' instructional actions. Group discussions also enabled them to recognize different alternative instructional activities offered by other members of the group as a response to students' understanding.

Tasks

Task 1 presented in the pre-interview was adapted from a scenario developed by Corcoran (2012) and the other tasks were designed by the researchers (see **Appendix**). The designed tasks were presented to two mathematics education researchers for review in order to ensure content validity and tasks were

Table 3. Students' reasoning levels in the tasks presented in the post-interview

Task 1		Task 2		Task 3	
Sema	Jale	Meral	Poyraz	Ali	Veli
Operating on numerical measurements (M4)	Holistic visual comparison (N0)	Incorrect unit iteration (M1)	Correct unit iteration (M2)	One-to-one matching of pieces (N1.1) and Holistic visual comparison (N0)	Comparison by property-based transformations (N2)

revised according to their comments. In the Turkish middle school mathematics curriculum, there are objectives related to calculating and estimating the perimeters of different shapes (Ministry of National Education, 2018). Therefore, while designing tasks, the focus was on the concept of perimeter. These pre-service teachers are expected to teach this concept when they become teachers.

Each task included the answers of two pairs of hypothetical students. These tasks reflected different characteristics of students' understanding in order to enable pre-service teachers to compare and contrast different answers to the same tasks and differentiate between two students' understandings. The first tasks included students' both non-measurement and measurement reasoning, the second tasks included students' measurement reasoning and the third tasks included students' non-measurement reasoning about length. Moreover, while one student's answer was correct, the other student's answer was incorrect in each task. Students' incorrect answers in the tasks involved specific errors or misconceptions which were ascertained according to the related literature. The aim of including an incorrect answer was to examine how pre-service teachers notice these misconceptions or errors.

In addition, students' answers to the tasks designed for post-interview were determined by taking into account the conceptual framework involved in students' reasoning about length measurement. Each student showed different levels of reasoning (**Table 3**). In this way, it was possible to examine how the intervention helped the pre-service teachers to identify students' reasoning levels that can be inferred from their answers in the given tasks.

Data Analysis

For data analysis, qualitative methods were used. The focus was on how pre-service teachers notice students' reasoning about length measurement in the pre- and post-interviews. Pre-service teachers' responses were grouped and compared according to whether they (i) identified the students' strategies; (ii) interpreted the students' understanding; and (iii) provided suitable activities that help students overcome their misconceptions/errors. For this purpose, in the first cycle of analysis, the interview transcripts of the pre-service teachers were read several times and categories were created based on the meaning that emerges from the raw data. In the second cycle of analysis, additional codes for the categories emerged and the codes were revised and refined (Strauss & Corbin, 1994). In this way, by using an open coding method, the pre-service teachers' responses to the three questions were grouped and the descriptions of the levels of attending, interpreting and responding skills were developed.

Two coders expert in the area coded the data individually. Then, codes were compared and discrepancies were discussed, and a consensus was arrived. Inter-rater reliability was 92% for the interviews. Three levels for each component of professional noticing emerged in the study according to previous research on professional noticing of students' mathematical thinking and to the codes created in the analysis of the participants' responses. In **Table 4**, the categories, description of these categories and examples for each category from the research data were provided. The levels of attending were analysed in terms of the extent to which the participants uncovered significant details in students' answers. The levels of interpretation were analysed in terms of the extent to which the participants provided information about students' understanding and differentiated between the two students' understanding. The levels of deciding how to respond were analysed in terms of whether they made general or specific suggestions and the extent to which the participants provided information about their specific suggestions.

Table 4. The features of the categories in noticing students' reasoning

Categories	Description	Examples
Attending		
Low	Described students' strategies as correct or incorrect	Cenk's strategy is incorrect because she gave wrong answer. Pelin got the correct answer.
Medium	Provided general description of both students' strategies	By matching sides of the shapes, Hale thought that the perimeters were equal to each other. Fidan knew that the straight segment in the 1st shape was smaller than the total of the two segments in the 2nd shape.
Robust	Provided detailed description of both students' strategies	Suzan thought that shape consisted of unit squares and the perimeter of the shape decreased as the piece of unit squares were removed, that is, since the area of the shape decreased, the perimeter of it also decreased. Melih thought that the boundary limiting the closed shape was the perimeter. He made numerical operations by counting unit-lengths which were outward-facing edges of the squares as 1 cm in the shapes and found the perimeters correct.
Interpreting		
Low	Provided general interpretation of both students' understanding	I thought Melih got the point. In a similar example, he could reason in the same way and solve such a problem, but Suzan did not seem to understand much.
Medium	Provided general interpretation of one student's understanding and detailed interpretation of another student's understanding	I thought that Sema had fully understood perimeter and her reasoning was measurement based. She did operations using numerical measurements. On the other hand, Jale's reasoning was non-measurement because she did nothing for measurement reasoning. It was holistic because she compared perimeters by looking at the whole shapes. She had a misconception regarding the relationship between perimeter and area. That is, she thought that since the areas of the shapes decreased by cutting pieces, their perimeters also decreased and cutting a larger piece led to a much bigger decrease in the area as well as in perimeter. However, in this case, even though a piece was cut, the lengths of new side(s) were formed after the cut were considered while calculating the perimeter.
Robust	Provided detailed interpretation of both students' understanding Differentiated two students' understanding as well as explained students' misconceptions/ Errors	Actually, they both knew what perimeter meant. Both had measurement reasoning since they used the number of unit length and counted by repeating it. However, while measuring the perimeter of the shapes, Meral did not take equal units. She did not take a fixed unit. She measured repeatedly in different units. She did not understand that she had to take a single unit. That is why she was measuring perimeters wrong. Hence, she was in the incorrect unit iteration level. On the other hand, Poyraz knew what the unit was. He knew what the unit length was. He understood that while measuring the perimeter, the units had to be equal in length and had to be repeated consecutively along the shapes. He made sense of the fact that equal and same units had to be used in measuring perimeter. Hence, Poyraz was at the level of correct unit iteration.
Responding		
Low	Provided general suggestion	I put emphasis on overcoming this misconception by giving the student some more explanatory instructions.
Medium	Provided specific undetailed suggestion	I could start with a simple example composed of unit squares, but in a one-way without such curves. I asked Cenk to find the perimeter. Then, I progressed from simple to complex. So, I could get rid of his misconception.
Robust	Provided specific detailed suggestion	Meral assumed that for instance, 7 units and 1 unit were equal in length. To help her, I could bring something whose length was 7 cm such as a 7 cm piece of wire, and a 1 cm piece of wire. I asked her "Did you think these two were equal in length?" I expected that she would say no, the wire 7 cm in length was longer. I asked her "If 7 units and 1 unit were not the same, could we think of them as the same unit?" Then, I could ask her "Could you consider again which units you took when calculating the perimeter?" In this way, she could recognize that she considered unequal units as equal.

Ethical Considerations

For the research, ethical permission was gathered from the university Institutional Review Board (IRB). At the beginning of the study, participants were informed of the nature and purpose of the research and

Table 5. Attending students' strategies in the pre- and post-interviews

	Task 1		Task 2		Task 3	
	Pre-interview	Post- interview	Pre- interview	Post- interview	Pre- interview	Post- interview
P1	Robust	Robust	Medium	Robust	Robust	Robust
P2	Medium	Robust	Medium	Robust	Medium	Robust
P3	Medium	Medium	Low	Robust	Medium	Medium

then a consent form was obtained. In addition, pseudonyms were used for the participants rather than their real names.

FINDINGS

In this section, we describe to what extent the pre-service teachers attended to students' strategies, interpreted students' understanding, and provided suitable activities in the pre- and the post-interviews.

Attending to Students' Strategies

We grouped participants' responses into three categories (low, medium, and robust) considering to what extent they attended to students' strategies to describe the students' responses. Their explanations indicated that while they mostly provided a general description of both students' strategies (medium evidence) in the pre-interviews, they mostly could provide a detailed description of both students' strategies (robust evidence) in the given tasks in the post-interviews (**Table 5**).

In the pre-interviews, for Task 1, P2 and P3 provided general descriptions of students' strategies. To illustrate, P2 said "Suzan thought of perimeter in terms of area. Melih counted the sides of the shapes, thus obtaining an accurate result." In this comment, although P2 recognized both students' strategies, she did not provide details about Suzan's thinking and Melih's counting. On the other hand, P1 could provide a detailed description of both students' strategies as follows:

> Researcher: Can you describe what each student did in response to the task?
>
> P1: Suzan thought that shape consisted of unit squares and the perimeter of the shape decreased as the piece of unit squares were removed, that is, since the area of the shape decreased, the perimeter of it also decreased. Melih thought that the boundary limiting the closed shape was the perimeter. He made numerical operations by counting unit-lengths which were outward-facing edges of the squares as 1 cm in the shapes and found the perimeters correct.

In the post-interview, just like in the pre-interview, P3 provided a general description of both students' strategies without providing details about Jale's and Sema's thinking. On the other hand, P1 and P2 could provide a detailed description. P2 increased her attending level by providing robust evidence in the post-interview. P1 and P2 were able to realize that Sema performed numerical operations by counting unit-lengths to find the lengths of three sides and she used the triangle inequality for the other sides while comparing the perimeters of the shapes before and after the cut and they provided step by step detailed description of Sema's solution. They also could notice Jale's intuitive rule regarding the relationship between area and perimeter and could explain this in detail. In addition, For Task 2 in the pre-interviews, P3 just described Cenk's strategy as incorrect and Pelin's strategy as correct without explaining why that was so. P1 and P2 described students' strategies without providing details. For instance, P2 said "Cenk counted squares in the track. Pelin calculated all the lengths to find the perimeter of the track she designed." In this comment, P2 did not mention what students took as a unit in iteration. In the post-interviews, all the participants were able to attend to Poyraz's strategy by providing a detailed description as soon as they read the task. The dialogue between the researcher and P2 is given below.

Table 6. Interpreting students' understanding in the pre- and post-interviews

	Task 1		Task 2		Task 3	
	Pre-interview	Post-interview	Pre-interview	Post-interview	Pre-interview	Post-interview
P1	Medium	Robust	Low	Robust	Robust	Robust
P2	Low	Medium	Low	Robust	Low	Medium
P3	Low	Medium	Low	Medium	Low	Low

> Researcher: Can you explain how Poyraz found the perimeters of the shapes?
>
> P2: Poyraz calculated the sides of the unit squares surrounding the shapes by taking the length of the side of each square as 1 unit. In this way, by iterating this unit correctly, he found the perimeter of the pink figure as 32 units and the perimeter of the green figure as 36 units.

While attending to Meral's strategy, even though P2 and P3 could not find what Meral considered to be a unit length along with the shapes immediately, they could, after thinking for a while, explain it in detail. The following explanation of P3 illustrates this situation:

> Researcher: Can you describe what Meral did in response to the task?
>
> P3: I tried to understand how Meral found the perimeters as 12 units. When I counted all the squares in the figures, there were more than 12. He did not take a square as a unit. If I counted the number of squares on a single side, there was no such side whose length is 12 units… Aha! I found. For instance, Meral took the length of the upper side of the green shape, i.e., 8 units above, as 1 unit. Then, she accepted 2 units bending down as 1 unit in the same way. I mean she considered the length of each side of the shapes as 1 unit. She calculated all of them as 1 unit when bent down, left or right, i.e., at each bend. In this way, she found the perimeters of both shapes as 12 units incorrectly.

These comments indicated development in participants' noticing of students' strategies. Since they explained what students take as a unit and also unit iteration in detail, their comments showed robust evidence in the post-interviews different from the ones in the pre-interviews. For Task 3 in the pre-interviews, the explanations of P2 and P3 were categorized as medium evidence. For example, P3 said "By matching sides of the shapes, Hale thought that the perimeters were equal to each other. Fidan knew that the straight segment in the 1st shape was smaller than the total of the two segments in the 2nd shape." In this comment, although P3 noticed students' strategies in their answers, she attended to the students' strategies with general descriptions. Yet, the explanation of P1 included details about the students' strategies because she emphasized that Fidan used transformation that preserved the length of the line segments in her strategy, and Hale used estimation and visualization while matching dissimilar sides to infer the lengths of the sides. In the post-interview, while P3 was categorized as having a medium level of attention skill similar to that in the pre-interview, P1 maintained her robust level of attending skill. In addition, P2's attending level increased in the post-interview, and she provided robust evidence of attending skill. In her explanation, she mentioned how Ali matched the pieces of the shapes according to their appearance while comparing the perimeters. In addition, she described in detail how Veli used transformation while making the shapes to get rectangles and properties of a rectangle, and she also identified where the segments were left.

Interpreting Students' Understanding

Similar to attending, participants' interpretations were categorized into three categories (low, medium and robust). They mostly interpreted students' understanding by providing low evidence in the pre-interviews. On the other hand, they mostly provided interpretations in robust and medium categories in the post-interviews (**Table 6**).

In the pre-interviews, for Task 1, while the explanation of P1 indicated medium evidence of interpretations, i.e., a general interpretation of Melih's understanding and a detailed interpretation of Suzan's understanding, interpretation of P2 and P3 were categorized as low evidence because they tended to provide general interpretations of both students' understanding. For instance, P3 said "I thought Melih got the point. In a similar example, he could reason in the same way and solve such a problem, but Suzan did not seem to understand much." It was seen that P3 provided her interpretation in broad terms. In the post-interview, P1 increased her interpretation level by providing a detailed interpretation of both students' understanding, differentiating their understanding as well as explaining Jale's misconception. Even though P2 and P3 provided a general interpretation of Sema's understanding, they could interpret Jale's understanding in detail; hence, their interpretation level also increased in the post-interview. As an example, the dialogue between the researcher and P2 is given below.

> Researcher: Can you explain what you learned about these students' understandings?
>
> P2: I thought that Sema had fully understood perimeter and her reasoning was measurement-based. She did operations using numerical measurements. On the other hand, Jale's reasoning was non-measurement because she did nothing for measurement reasoning. It was holistic because she compared perimeters by looking at the whole shapes. She had a misconception regarding the relationship between perimeter and area. That is, she thought that since the areas of the shapes decreased by cutting pieces, their perimeters also decreased, and cutting a larger piece led to a much bigger decrease in the area as well as in perimeter. However, in this case, even though a piece was cut, the lengths of new side(s) which were formed after the cut were considered while calculating the perimeter.

It was seen that while interpreting Jale's understanding, P2 strengthened her interpretation by using details of Jale's answer. For Task 2 in the pre-interviews, all participants provided a general interpretation of both students' understanding. To illustrate, P3 said "Cenk did not understand what perimeter meant. On the other hand, I thought Pelin understood what perimeter meant because she counted it correctly." It was seen that P3 did not question possible reasons underlying Cenk's incorrect strategy and she also explained whether Pelin knew perimeter in broad statements. In the post-interviews, interpretation of P3 showed medium evidence, i.e., the general interpretation of Poyraz's understanding and detailed interpretation of Meral's understanding. Hence, her interpretation level increased in the post-interview. Other participants' (P1's and P2's) interpretation skills increased two levels because they provided robust evidence of interpretation of students' understandings during the post-interviews. The following explanation of P1 illustrates this situation:

> Researcher: Can you explain what you learned about these students' understandings?
>
> P1: Actually, they both knew what perimeter meant. Both had measurement reasoning since they used the number of a unit length and counted by repeating it. However, while measuring the perimeter of the shapes, Meral did not take equal units. She did not take a fixed unit. She measured repeatedly in different units. She did not understand that she had to take a single unit. That is why she was measuring perimeters wrong. Hence, she was in the incorrect unit iteration level. On the other hand, Poyraz knew what the unit was. He knew what the unit length was. He understood that while measuring the perimeter, the units had to be equal in length and had to be repeated consecutively along with the shapes. He made sense of the fact that equal and same units had to be used in measuring perimeter. Hence, Poyraz was at the level of correct unit iteration.

This comment was in alignment with the increase in depth of P1's interpretations of students' understanding of units. For Task 3 in the pre-interviews, P1's comment showed robust evidence of interpretation. On the other hand, P2 and P3's interpretations were categorized as low because they tended to provide general interpretations of both students' understanding, and they did not mention the reasons for the student's error. For instance, P3 stated: "Fidan solved this problem correctly. I thought Fidan better understood the subject, but Hale did not understand the concept of length." In the post-interviews, there was not any change in P1 and P3's levels of interpreting students' understanding. P1 maintained a robust level of interpretation in the post-interview. However, P3 could not increase her low level of interpretation. On the other hand, P2 as shown below increased her interpretation one level and provided a general interpretation of Ali's understanding and detailed interpretation of Veli's understanding.

> Researcher: Can you explain what you learned about these students' understandings?
>
> P2: Ali did not understand the length concept because he just looked at the appearance of pieces while matching them. He used one-to-one matching and visual comparison. Veli, on the other hand, compared the perimeters by sliding segments, that is, comparison by property-based transformations. Their reasoning was non-measurement because they did not use numbers and did not count while comparing perimeters. Veli understood that all segments that formed sides of the shape were considered while finding the perimeter. He knew that the transformation he made preserved the lengths of the segment. In this way, he could make inferences based on rectangle properties. He knew that vertical and horizontal segments could be moved to fit exactly the boundaries of the shapes. He concluded that both shapes could be converted to congruent rectangles, but four segments were left in the 2nd shape and two segments were left in the 1st shape, and hence, the perimeter of the 2nd shape was bigger.

This comment showed that while interpreting Veli's understanding, P2 strengthened her interpretation by using details of Veli's answer.

Deciding How to Respond on the Basis of Students' Understanding

After interpreting students' understanding, the participants proposed an instructional action in order to eliminate students' misconceptions. Participants' instructional actions were grouped into three categories which are general, specific undetailed and specific detailed as explained in **Table 4**. Here, general means low evidence, specific undetailed means medium evidence and specific detailed means robust evidence. While the participants mostly provided instructional actions in the specific-undetailed category in the pre-interviews, they mostly made specific-detailed instructional suggestions in the post-interviews (**Table 7**). They did not propose activities to students who provided a correct solution since they believed that these students had understood the length measurement and perimeter concepts.

In the pre-interviews, for Task 1, only P1 suggested a specific-detailed instructional action by providing details about how and why she decided to use that activity to help Suzan to overcome her misconception. However, P2 and P3's suggestions were categorized as specific-undetailed (medium evidence). For example, when asked how to respond to Suzan's understanding, P2 said: "I could design an activity consisting of problems from simple to complex. First, I explained what perimeter meant through a simpler problem which was made up of unit squares." Moreover, P3 said: "By using a virtual manipulative like GeoGebra, I could show that when a certain piece of a shape was removed, the perimeter increased." In these comments, participants did not mention the details of the activities. In the post-interviews, all participants offered instructional actions in the specific detailed category (robust evidence). Hence, while P1 maintained her robust level of responding in the post-interview, P2 and P3

Table 7. Participants' instructional actions in the pre- and post-interviews

	Task 1		Task 2		Task 3	
	Pre-interview	Post-interview	Pre-interview	Post-interview	Pre-interview	Post-interview
P1	Specific detailed	Specific detailed	Specific undetailed	Specific detailed	Specific detailed	Specific detailed
P2	Specific undetailed	Specific detailed	Specific undetailed	Specific detailed	Specific undetailed	Specific detailed
P3	Specific undetailed	Specific detailed	General	Specific undetailed	Specific undetailed	Specific undetailed

increased their responding levels by providing detailed information about the activities. As an example, the excerpt that shows specific detailed instructional action suggested by P3 is given below.

> Researcher: If you were the teacher of Jale, how would you respond to her? Can you provide suitable activity(ies) that would help Jale overcome her misconception(s)/error(s)?
>
> P3: Jale had a misconception about the fact that when the area of the shape decreased, the perimeter had to decrease. To eliminate this, I could give her a construction paper. I asked her to measure the perimeter of the paper first, with a ruler, and then to note it. Then, I asked her to cut a piece with a scissor. First, she cut a triangular piece and measured the perimeter of the shape with a ruler. Then, I asked her to enlarge the cut and measure the perimeter again. I asked her "How did the perimeter of the shape change with the size of the piece you cut? "When the larger piece was removed, did the perimeter decrease much more? By asking these kinds of questions, I could help her make sense that the perimeter was related to the sides, not the space the shape covers.

In P3's comment, she tried to support Jale's existing thinking and to eliminate her misconception with the help of questioning and providing a concrete experience. For Task 2, instructional actions suggested by P1 and P2 were specific undetailed (medium evidence) and similar to each other in the pre-interviews. The following explanation of P2 illustrates this situation:

> Researcher: If you were the teacher of Cenk, how would you respond to him? Can you provide suitable activity(ies) that would help Cenk overcome his misconception(s)/error(s)?
>
> P2: I could start with a simple example composed of unit squares, but in a one-way without such curves. I asked Cenk to find the perimeter. Then, I progressed from simple to complex. So, I could get rid of his misconception.

In this comment, P2 did not provide details about how such activity would help Cenk eliminate his misconception. In addition, P3 said, "I emphasize overcoming this misconception by giving the student some more explanatory instructions." Since this comment is a general pedagogical decision, which is not specific to mathematics, it was categorized as general (low evidence). In the post-interview, when asked how to respond to Meral's understanding, P3 increased her responding level from general in the pre-interview to specific-undetailed (medium evidence) by saying "I could prepare an activity about unit length and iteration for Meral since she had a difficulty with it." Furthermore, P1 and P2 provided specific detailed instructional actions (robust evidence) considering Meral's incorrect unit iteration. This finding indicated that they were in a better position in the post-interview when it was compared with the pre-interview to provide a suitable activity to eliminate the student's misconception. As an example, the dialogue between the researcher and P1 is given below.

> Researcher: If you were the teacher of Meral, how would you respond to her? Can you provide suitable activity(ies) that would help Meral overcome her misconception(s)/error(s)?

> P1: Meral assumed that for instance, 7 units and 1 unit were equal in length. To help her, I could bring something whose length was 7 cm such as a 7 cm piece of wire, and a 1 cm piece of wire. I asked her "Did you think these two were equal in length?" I expected that she would say no, the wire 7 cm in length was longer. I asked her "If 7 units and 1 unit were not the same, could we think of them as the same unit?" Then, I could ask her "Could you consider again which units you took when calculating the perimeter?" In this way, she could recognize that she considered unequal units as equal.

P1 offered to provide a concrete object, i.e., wire, and posing series of questions to help Meral overcome her misconception. For Task 3 in the pre-interviews, when asked how to respond to Hale's understanding, P2 and P3 suggested specific undetailed activity (medium evidence). For example, P2 said: "I could design an activity by using a virtual manipulative to show that the perimeter of the 2nd shape was bigger." On the other hand, P1 as shown below provided specific detailed instructional suggestions (robust evidence).

> Researcher: If you were the teacher of Hale, how would you respond to her? Can you provide suitable activity(ies) that would help Hale overcome her misconception(s)/error(s)?
>
> P1: I could give two pieces of paper strips and asked Hale to place them on the left part of the 2nd shape to cover two segments in this curved part and asked her to put them next to each other. Then, I wanted her to cover the other straight part (left side of the 1st shape) with a single strip. When she put them under the other, she would realize by herself that the total length of the upper paper strip was much longer and therefore perimeters were not equal.

In the post-interview, when deciding how to respond to Ali's understanding, P3 suggested specific undetailed activity (medium evidence) just like in the pre-interview. In addition, while P1 maintained her robust level of responding, P2 increased her responding level by suggesting specific detailed activity in the post-interview. To illustrate, P2 as shown below suggested that she would want Ali to create the same shapes given in the task on a geoboard and determine and compare the perimeters of the shapes using two equal-length strings.

> Researcher: If you were the teacher of Ali, how would you respond to him? Can you provide suitable activity(ies) that would help Ali overcome his misconception(s)/error(s)?
>
> P2: I could give two strings with equal lengths. I asked him "Could you make these shapes on geoboard with the strings?" The string would be sufficient for the 1st shape, but the other string with the same length would be insufficient in order to make the 2nd shape. When he could not make the 2nd shape with the given string, he would see that their perimeters were not equal, and the perimeter of the 2nd shape was bigger.

P2's comments involved details about how she would use instructional actions to develop Ali's understanding. In this way, by using geoboard, which is a concrete manipulative, P2 thought that she could enable Ali to recognize his mistake himself.

To summarize, the findings revealed that there was an improvement in the participants' noticing for all three components in the post-interviews when they were compared with the pre-interviews (**Table 8**). There was only one low evidence across all three components of the professional noticing framework in the post-interviews while there were 9 such evidences in the participants' responses in the pre-interviews. The participants either increased their existing levels (one level or two levels) or maintained

Table 8. Changes in the participants' noticing in the pre- and post-interviews

	Task 1			Task 2			Task 3		
	AS	IS	RS	AS	IS	RS	AS	IS	RS
P1	R	M→R	SD	M→R	L→R	SUD→SD	R	R	SD
P2	M→R	L→M	SUD→SD	M→R	L→R	SUD→SD	M→R	L→M	SUD→SD
P3	M	L→M	SUD→SD	L→R	L→M	G→SUD	M	L	SUD

(AS: Attending Skill IS: Interpreting Skill RS: Responding Skill)
(R: Robust M: Medium L: Low SD: Specific detailed SUD: Specific undetailed G: General)

them in the post-interviews for each component in each task. It was seen that the greatest growth occurred in interpreting students' understanding. There were 7 low evidence explanations in the pre-interviews for interpreting alone. However, in the post-interviews, only P3's interpretation of students' understanding in Task 3 was low evidence for the interpreting component. While P1 initially had a higher noticing skill in the pre-interview than the other participants, there was still improvement in her noticing skills in the post-interview.

DISCUSSION AND CONCLUSION

Research shows that professional noticing skills of pre-service teachers can be improved with appropriate deliberate scaffolds and interventions such as through a module including video excerpts of diagnostic interviews of students (Schack et al., 2013), using students' learning trajectory as a scaffold (Ivars et al., 2020) and a module based on the development of the understanding of the derivative concept (Sánchez-Matamoros et al., 2015). In the present study, we made an intervention based on the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework. The findings support that the levels-of-sophistication framework helps teachers make sense of students' thinking about length. Moreover, the levels facilitate teachers' understanding of students' difficulties faced while they are learning the length concept and of the steps that students take to master this concept (Battista, 2006). One of the most important contributions of this study to the literature is that even though the pre-service teachers have not taken a course that includes the teaching and/or learning of geometry and measurement before, at the end of the study there was an improvement in the pre-service teachers' professional noticing skills. During the intervention, they became familiar with the levels-of-sophistication framework, they had opportunities to see different students' reasoning levels, to discuss students' reasoning with each other, and to reflect on possible instructional responses in group discussions. Changes in the pre-service teachers' noticing skills in the post-interviews in a positive manner revealed that these opportunities were effective in improving their professional noticing skills.

The pre-service teachers were in a better position in terms of attending to students' strategies in the post-interviews compared to the pre-interviews because they mostly provided robust evidence of attending in the post-interviews. In addition, they were able to identify students' reasoning levels by inferring from students' answers in the given tasks. It can be said that the levels-of-sophistication framework guided the pre-service teachers to focus on details in students' strategies by directing their attention to students' reasoning. The levels show students' reasoning, concepts and strategies, what they can and cannot do, and also the barriers to learning and required mental processes to perform at these levels (Battista, 2003). For this reason, incorporating the levels-of-sophistication framework might have helped the pre-service teachers enhance their knowledge in this regard. In this way, pre-service teachers' increased knowledge of students' reasoning may have promoted a more detailed analysis of students' solutions in the post-interviews. Since noticing of student thinking requires content-specific professional knowledge (Sanchez-Matamoros et al., 2015), courses focus on professional knowledge in particular domains should be incorporated into teacher education programs to help pre-service teachers strengthen their ability to reason about students' mathematical thinking (Shin, 2021).

The pre-service teachers mostly provided medium and robust evidence of interpretation in the post-interviews whereas they generally provided low evidence in the pre-interviews. This situation showed that the pre-service teachers were better at interpreting students' understanding in detail, differentiating students' understanding and also explaining students' misconceptions and errors in the post-interviews. Seeing different students' reasoning, errors and misconceptions might have enabled the pre-service teachers to make different interpretations in-depth. Hence, this study revealed that analysing students' solution strategies in their written work about length measurement was helpful for the pre-service teachers to give attention to students' mathematical understanding similar to other studies (Callejo & Zapatera, 2017; Ivars et al., 2020). Therefore, teacher educators can design and use students' written answers in teacher education programs in order to improve the professional noticing skills of pre-service teachers (Baldinger, 2019; Shin, 2020).

Improvement was also observed in the pre-service teachers' responding skill, which was consistent with previous studies (Schack et al., 2013; Ulusoy & Çakıroğlu, 2020; Warshauer et al., 2021) even though the responding skill has been identified as the most difficult to develop compared to attending and interpreting skills (Jacobs et al., 2010). In the post-interviews, most of the instructional actions suggested by the pre-service teachers were grouped into the specific-detailed category even though few of the suggestions were in this category in the pre-interviews. Furthermore, it was seen that the pre-service teachers provided student-centred responses in which students took charge of activities rather than teacher-centred responses and generally suggested the use of concrete manipulatives or objects in the activities during the post-interviews. This could be attributed to the fact that as the pre-service teachers realized different students' reasoning about the length and their reasoning levels, they focused on specific-detailed actions more. Moreover, the discussion environment enabled them to find out what other group members thought and to realize alternative decisions on the basis of students' mathematical understanding. Thus, providing pre-service teachers opportunities to think about and discuss possible instructional actions in teacher education settings can help them realize effective practices to utilize in classrooms as future teachers (Monson et al., 2020).

In general, all participants' professional noticing skills either improved or remained at the same level in the post-interviews for each component in each task. For some components in the tasks, P1 did not shift in her noticing because she was already at the highest level (robust) in the pre-interview, and she maintained it in the post-interview. For other components in which she was at the low or medium levels in the pre-interviews, she reached the robust level in the post-interviews. Although P3, who had lower noticing skills compared to P1 and P2 in the pre-interview, could not reach a robust level for all components of the noticing framework for all tasks, even four-session interventions led to improvement in her noticing skills to some extent. Thus, there is a need to provide opportunities and more time for pre-service teachers to link their knowledge to practices of attending, interpreting, and responding to students' mathematical thinking early in teaching education programs for further development of noticing skills (Warshauer et al., 2021).

Teachers should understand students' reasoning in length measurement, what difficulties and misconceptions students have, and the reasons underlying them (Lehrer, 2003). In the present study, during the pre- and post-interviews, the pre-service teachers described and compared correct and incorrect answers of different students to each task. This situation enabled the pre-service teachers to notice different characteristics of students' reasoning about length and both students' errors and misconceptions in length measurement as well. It can be said that providing opportunities for identification of errors and misconceptions, analysing underlying reasons for them, and proposing actions to eliminate them can support pre-service and in-service teachers' understanding of students' difficulties (An & Wu, 2012; Sanchez-Matamoros et al., 2015).

In conclusion, the findings of the study suggest that designed tasks and interventions based on students' reasoning in the "Levels of Sophistication in Students' Reasoning about Length" (Battista, 2006) conceptual framework had a significant role in the improvement of professional noticing skills of pre-service teachers. That is, tasks that reflect different students' reasoning including non-measurement and measurement reasoning and different reasoning levels in this study and the levels-of-sophistication framework seemed to be effective tools because they enabled the pre-service teachers to give their full attention to students' mathematical understanding and to provide proper instruction to support students' learning. Students' reasoning levels about length measurement must be understood by pre-service elementary mathematics teachers so that they can develop their professional noticing skills in length measurement. Thus, by using tasks building on the levels-of-sophistication framework, we can support pre-service teachers' professional noticing skills as well as we can better prepare them to teach length measurement based on students' reasoning rather than focusing on procedures. The present study is limited to the hypothetical students' reasoning that was used to test pre-service teachers' professional noticing skills. For further research studies, it is recommended that after finishing the pre-interview, four discussion sessions, and post-interview, pre-service teachers' professional noticing of actual students' reasoning can be tested.

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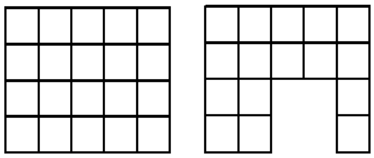
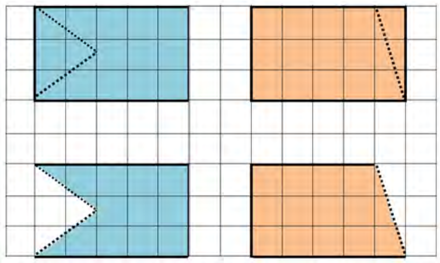
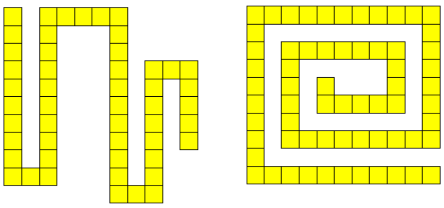
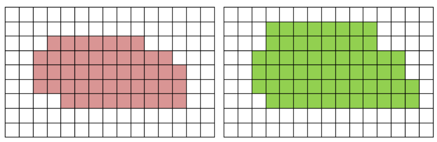
APPENDIX

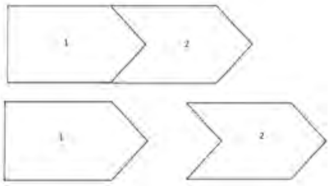
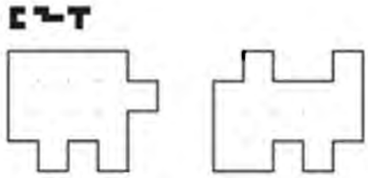
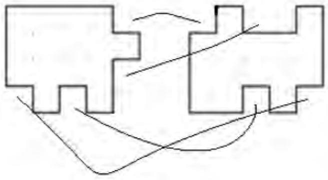
<table>
<tr><th>Students' reasoning</th><th>Tasks presented to the pre-service teachers in the pre-interview</th><th>Tasks presented to the pre-service teachers in the post-interview</th></tr>
<tr><td>One non-measurement reasoning and one measurement reasoning</td><td>Task 1

1st shape 2nd shape
Hakan teacher teaches his students the perimeter of geometric shapes. He asks them to examine the figures on the above, which consist of squares with sides of 1 cm, and to compare the perimeter of these two shapes.

Suzan: In the second shape, the perimeter was reduced because a piece of it was removed from the first.
Melih: The perimeter has increased because it is 18 cm in the first shape and 22 cm in the second shape.</td><td>Task 1

Buse teacher distributed rectangular cardboards to her students and asked them to cut these cardboards along the dotted lines. She then asked the students to compare the perimeters of the shapes formed after the cut with the perimeters before the cut. Answers of Sema and Jale are given below.

Jale: Since the piece is cut, the perimeters of both shapes have decreased compared to the original. However, since the piece cut from the blue rectangle is larger than the piece cut from the orange rectangle, the perimeter of the blue shape has been reduced more.
Sema: $Perimeter\ of\ the\ blue\ shape = 5 + 5 + 3 + (x + y) = 13 + (x + y)$
$Since\ (x + y) > 3$
$Perimeter\ of\ the\ blue\ shape > 16$
$Perimeter\ of\ the\ orange\ shape = 4 + 3 + 5 + z = 12 + z$
$Since\ z < 4$
$Perimeter\ of\ the\ orange\ shape < 16$
Therefore, the perimeter of the blue shape increased after the cut, while the perimeter of the orange shape decreased after the cut.</td></tr>
<tr><td>Measurement reasoning</td><td>Task 2

The track designed by Cenk The track designed by Pelin
Cenk and Pelin are building an amusement park from Legos. Cenk and Pelin, who want to include roller coasters in amusement parks, have designed different roller coaster tracks consisting of unit squares. Then, they wondered the perimeters of the tracks they designed and calculated them. The roller coaster tracks designed by Cenk and Pelin are given above.

Cenk: The perimeter of my track is 50 units.
Pelin: The perimeter of my track is 130 units.</td><td>Task 2

Meral and Poyraz have learned how to find perimeter of geometric shapes in the lesson. At the end of the lesson, the teacher distributed the activity sheet containing the figures given above to the students as a homework. Solutions of Meral and Poyraz are given below.

Meral: Perimeter of both figures is equal in length and 12 units.
Poyraz: Perimeter of the pink figure is 32 units and perimeter of the green figure is 36 units.</td></tr>
</table>

Students' reasoning	Tasks presented to the pre-service teachers in the pre-interview	Tasks presented to the pre-service teachers in the post-interview
Non-measurement reasoning	**Task 3**  **Ayten teacher has students do puzzle activities in her classroom. In the activity, she asked the students to compare the perimeters of the puzzle pieces (1 and 2) given above.** **Hale:** I matched the four sides, which are already same, of the pieces. Also, I matched the two segments at the left part of the 2nd shape with the line segment at the left part of the 1st shape because if I straighten the two line segments, I will get the straight line segment in the 1st shape. So, the shapes have equal perimeters. **Fidan:** If I move the two segments at the left part of the 2nd shape down, they overlap. Therefore, the perimeter of the 2nd shape is larger than the perimeter of the 1st shape.	**Task 3**  1st shape 2nd shape **Ali and Veli play pentomino. They create the shapes given above by putting the three given pieces together in different ways, and then compare the perimeters of the shapes they create.** **Ali:** (by drawing as below) I think these shapes have equal perimeter.  **Veli:** (moving segments to make a rectangle as below) The perimeter of the 2nd shape is larger. Because when I finished, there were more segments left over in the 2nd shape than in the 1st shape.