MJML Journal

Malikussaleh Journal of Mathematics Learning (MJML)
p-ISSN: 2620-6315 | e-ISSN: 2620-6323
Homepage: https://ojs.unimal.ac.id/mjml/index

Flexibility in partitioning strategies of fourth graders

Yeliz Yazgan
Department of Elementary Education, Bursa Uludag University, Bursa, Turkey, 16059
*Corresponding author: yazgany@uludag.edu.tr | Phone Number: +905386751775

ARTICLE INFO
Received: 22-05-2021
Received in revised: 24-06-2021
Accepted: 16-09-2021
Available online: 30-10-2021

KEYWORDS
Flexibility;
Fractions;
Partitioning;
Partitioning strategies;

ABSTRACT

This study combines the concepts of flexibility and partitioning, and aims to probe fourth grade students' flexibility in partitioning strategies. Seven students participated in this descriptive case study. Students were given three partitioning tasks. Forty-eight answers produced by students were evaluated and classified based on the strategies defined in the taxonomy developed by Charles and Nason (2000). Results showed that students could easily change their strategies both within and across tasks. Namely, they displayed both inter- and intra-task strategy flexibility to a large extent even though they did not have any intervention on partitioning. Another point that findings have implicated was that the fourth graders' flexibility in partitioning strategies may be utilized to introduce concepts of equivalent fractions and mixed numbers. Results are discussed in terms of their implications related to mathematics education, and some recommendations aimed at learning environments and future studies are presented.

INTRODUCTION

The partitioning process includes dividing an object or objects into nonoverlapping and exhaustive parts. Concerning fractions, another stipulation is added: These parts should be of the same size (Lamon, 1999)[1]. The ability to use, internalize and reason about partitioning is present in children at an early age (Pitkethly & Hunting, 1996), and many researchers or educators in the mathematics education domain have attached great importance to partitioning activities due to its key role in establishing initial fractional knowledge (e.g. Empson, 1999; Norton & McCloskey, 2008; Pothier & Sawada, 1990; Siemon, 2003; Streefland, 1991). Furthermore, partitioning is the foundation of other important concepts such as division and multiplication, ratio, and rate (Confrey *et al.*, 2014). On the other hand, in educational sense, flexibility can be described as "the ability to easily adapt or adjust to changing circumstances" (Star, 2018, p.15).

Especially in the current era in which technological developments are very rapid, flexibility is very important in keeping up with changes and dealing with uncertainty. Flexibility is one of key components of creativity (Leikin, 2009), and it is characterized by variety in approaches taken while trying to arrive at a goal (Leikin *et al.*, 2009). This study combines the concepts of flexibility and partitioning, and aims to probe fourth grade students' flexibility in partitioning strategies. Therefore, the next two sections encapsulate the theoretical framework and related literature on partitioning strategies and flexibility.

Partitioning strategies

A remarkable number of mathematics education researchers have elaborated on partitioning strategies of students. Studying with children from kindergarten through third grade, Pothier and Sawada (1983) determined four levels concerning the development of the partitioning process: sharing, algorithmic halving, evenness and oddness, and composition. In the first level, the children are able to use halving, while they move easily to algorithmic halving to obtain fourths, eighths, and so on at the second level. At the evenness and oddness levels, the

[1]Partitioning in a fraction context means producing equal-sized groups or parts as fair shares (Cutting, 2019), and this is called equipartitioning. For simplicity, the term partitioning will be used instead of equipartitioning throughout the study.

children proceed from partitioning involving even numbers that are powers of two to other numbers such as odd numbers and even numbers with odd number factors. At the last level, the children can use multiplicative partitioning strategies (Petit *et al*., 2015).

In his seminal study in which sixteen fourth graders participated, Streefland (1991) used sharing situations as a starting point to help students explore fractions. In these situations, the number of shared objects was sometimes less and sometimes more than the number of people sharing. As a result, Streefland (1991) distinguished five levels of resistance to IN-distractors in the long-term individual learning processes of the students in his study: Absence of cognitive conflict, cognitive conflict takes place, spontaneous refutation of IN-distractor errors, free of IN-distractors, and resistant to IN-distractors. In another study, Lamon (1996) analyzed the partitioning strategies of children from grades four through eight in terms of economy in the number or size of pieces and sophistication in unitizing. She defined three general strategies used by students in partitioning situations in which the number of objects to be shared is more than the number of sharers: Preserved-pieces strategy (only the units that require cutting are marked and cut, and the others are left unmarked and intact), mark-all strategy (all of the units are marked, but only the unit(s) that require cutting will be cut), distribution strategy (All units are marked and cut, and the smaller pieces are distributed). Her findings also revealed that a greater percentage of students preferred economical partitioning strategies rather than less economical cut-and-distribute strategies, and used more composite units as the grade level increased.

Charles and Nason (2000) went beyond previous studies by aiming to reveal new partitioning strategies not mentioned in the literature before and to develop taxonomy for classifying all of previously reported and newly found strategies. In their study, each of twelve third grade students worked on a set of partitioning problems chosen from a bank of 30 tasks. Pursuing their goal, Charles and Nason (2000) accomplished to establish a taxonomy that is based on strategies' potential to facilitate the abstraction of fractions from the activity of partitioning. They sorted all strategies into four classes based on three criteria: fair sharing, accurate quantification of shares, and conceptual mapping (see Figure 1). As will be explained later, this taxonomy is the backbone of the data analysis of the present study.

In 2006, Empson et al. carried out a study involving a large sample consisting of first, third, fourth and fifth graders. Unlike previous studies, the authors aimed to analyze to what extent coordination between number of people sharing and number of things being shared was multiplicative. As a result, Empson et al. (2006) defined two main groups of strategies as Parts Quantities and Ratio Quantities. Strategies in the first group "involved children's partitions of continuous units", while the strategies in the second group "involved children's creation of associated sets of discrete quantities" (p.1). Figure 2 represents all subcategories of Parts and Ratio strategies. Lastly, Steffe and Olive (2010) made recent and detailed analysis of partitioning strategies in their book. As a result, they put forward six partitioning schemes upon which children construct their fractional schemes: The *Equipartitioning Scheme, The Simultaneous Partitioning Scheme, The Splitting Scheme, The Equipartitioning Scheme for Connected Numbers, The Splitting Scheme for Connected Numbers, The Distributive Partitioning Scheme.*

Classes of strategies

Class	Strategies in each class	Characteristics of each class
Class 1	• Partitive quotient foundational strategy • Proceduralised partitive quotient strategy	• Generates fair shares • Accurate quantification of shares • Conceptual mapping
Class 2	• Regrouping strategy • People by objects strategy • Half to each person then quarter to each person strategy	• Generates fair shares • Accurate quantification of shares • No conceptual mapping
Class 3	• Partition and quantify by part-whole notion strategy • Halving the objects between half the people strategy • Whole to each person then half the remaining objects between half the people strategy	• Generates fair shares • Little or no accurate quantification of shares • No conceptual mapping
Class 4	• Horizontal partitioning strategy • Repeated sizing strategy • Repeated halving/repeated sizing strategy	• Does not generate fair shares • Little or no accurate quantification of shares • No conceptual mapping

Figure 1. Partitioning strategies defined by Charles and Nason (2000, p. 211)

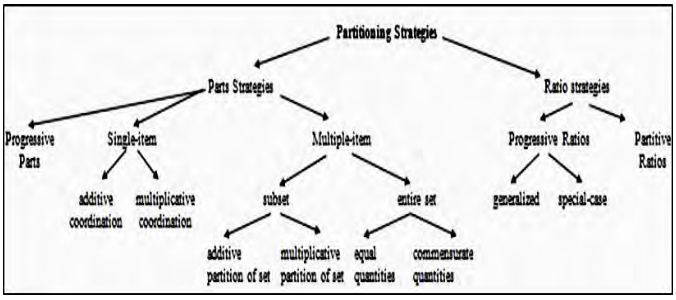

Figure 2. Partitioning strategies defined by Empson et al. (2006)

Apart from the studies outlined above, there are other studies dealing with directly or indirectly students' partitioning strategies (e.g. Norton & Wilkins, 2010). However, in line with the purpose of the study, studies that classify or taxonomize allocation strategies are mainly included here. In general, it can be said that the classifications of partitioning strategies are hierarchical and based on different criteria such as economy and suitability for abstraction. Although the studied grade levels vary from preschool to eighth grade, it is noteworthy that almost every study includes third and/or fourth grade students.

Strategy flexibility

In a cognitive sense, flexibility means changing one's perspective or approach towards a problem, and switching between the answers, the characteristics of the stimuli, the strategies or the problems in a flexible way (Liu *et al*., 2018). One important aim of contemporary education is the development of flexible problem-solving skills (Kalyuga *et al*., 2010). Documents on mathematics education have long emphasized that students should have the ability to use multiple strategies and switch between strategies in line with the characteristics of the problem, personal factors, and environmental effects (Low & Chew, 2019; Nguyen *et al*., 2020). In this context, strategy flexibility can be described as a combination of choosing the most appropriate strategy for a given problem, using multiple strategies, and switching between strategies (Star & Rittle-Johnson, 2008; Star & Seifert, 2006). Some researchers (e.g. Verschaffel *et al*., 2009) term the "choosing the most appropriate strategy" as "strategy adaptivity". Since there is not consistency regarding the use of the terms "flexibility" and "adaptivity" within the literature (Selter, 2009), the author of the present study prefers to use the term "strategy flexibility" in a broader sense including "strategy adaptivity".

In mathematics education, strategy flexibility has usually been elaborated within the concept of a specific subject area. Algebraic and linear equation solving (e.g., Star & Rittle-Johnson, 2008; Wang *et al*., 2019), computational estimation (e.g., Star *et al*., 2009), strategic flexibility in addition and subtraction (e.g., Selter, 2001), mental computation (e.g., Torbeyns *et al*., 2009), problem solving (e.g., Elia *et al*., 2009; Jausovec, 1991), geometrical knowledge (Levav-Waynberg, & Leikin, 2012) are examples of these subject areas. Overall results of these studies show that (i) strategy flexibility can be developed through education and curriculum, (ii) students have the potential to employ different and appropriate strategies without any training, (iii) easiness, accuracy, and fluency of strategies are key factors for the development of strategy flexibility, (iv) students display low strategy flexibility in solving non-routine problems, and (v) gifted students can employ various strategies while solving a problem and apply different strategies for different problems. Additionally, Star (2018) claims that the results of some national/international studies on strategy flexibility indicate the following main points: (i) although they appreciate the value of flexibility, experts do not always choose the best strategy when solving problems, (ii) the value given to flexibility as an educational goal varies from teacher to teacher, (iii) students generally appreciate the emphasis on flexibility.

In two separate studies, strategy flexibility has been examined by being divided into two different types. In one of them, Xu *et al*. (2017) made a distinction between potential and practical flexibility. The authors defined potential flexibility as "knowledge of multiple (standard and innovative) strategies for solving mathematics problems" and practical flexibility as "the ability to implement innovative strategies for a given problem" (p.2). In the other work conducted by Elia *et al*. (2009), strategy flexibility was classified as intra-task and inter-task. Intra-task flexibility means being able to change strategy while solving a problem. Inter-task flexibility means being able to switch to a different strategy when faced with a new problem situation. In other words, the first one implies changing strategies within problems, while the second one implies changing strategies across problems. This study also draws on inter- and intra-task classification to delve deeper into the strategy flexibility of students.

Importance and aim of the study

Despite the abundance of studies on children's partitioning strategies and on flexibility, none of these studies deal with these two domains in conjunction. Hence, distinctively from the above-mentioned studies, this study intends to elaborate on fourth graders' strategy flexibility in solving partitioning tasks. In connection with this aim, answers were sought to two specific research questions:

- Do the fourth graders exhibit inter-task strategy flexibility while working on partitioning problems?.
- Do the fourth graders exhibit intra-task strategy flexibility while working on partitioning problems?.

METHOD

Participants and Sampling Technique

Seven fourth graders (ages 9-10) participated in the study. They came from three different fourth grade classes of an elementary school in Bursa/Turkey. Since the classroom teachers knew the students, they were consulted in the selection of the participants. The classroom teachers expressed that they had chosen students with more self-confidence in expressing themselves and higher mathematical perception. This method applies to purposive sampling as "researchers handpick the cases to be included in the sample on the basis of their judgement of their typicality or possession of the particular characteristics being sought" (Cohen *et al*., 2007, p.114-115) in this sampling technique. Once being given explanations about the purpose and process of the study all participants voluntarily took part in the study with informed consent of their parents and teachers.

Since partitioning is not stated as a learning goal in the Turkish math curriculum at the elementary school level, participants of this study had not come across these kinds of activities in their textbooks or their learning environment before. Also, mixed and improper numbers had not been taught to them at the time the current study was carried out.

Research design

This study was built upon the basic principles of the descriptive case study for several reasons. First, this research design aims at obtaining an overall analysis of a bounded system (Merriam, 2009). In this study, units of analysis are limited to a few fourth graders. Second, a descriptive case study elaborates on a phenomenon within its context (Yin, 2003). In this sense, the current research handled flexibility in partitioning strategies as a phenomenon. Besides, the researcher took into account the factors such as students' background, lack or rareness of partitioning activities in learning environments in Turkey, and did not intervene in any way. Lastly, one of the difficulties of descriptive case study is that the boundaries between context and phenomenon are unclear (Yin, 2003). Therefore, the results found in descriptive case studies cannot be generalized since each situation is different from the other (Creswell, 2007), as it is in this study.

Information about the tasks

Three partitioning tasks were presented to the participants within the context of a cartoon family (Sizinkiler) consisting of a mother (Çıt Çıt), a father (Babişko) and two kids (Zeytin and Limon). The first task was about sharing three pizzas among four people; sharing six pizzas among nine people was required in the second one. The third task was related to sharing five construction papers among three people (See **Table 1**). As these three partitioning tasks based on prototypes by Streefland (1991) were used in a previous study by Yazgan (2010), the researcher did not need to perform any other validation studies.

Table 1. Tasks given in the study

Problem 1	Sizinkiler family goes to a pizzeria for dinner. But the pizzas are quite big for them so they order 3 pizzas instead of 4. In your opinion, how can they share the 3 pizzas equally? How much pizza does each person get?
Problem 2	At the table next to Sizinkiler, there is another group consisting of 9 people. They order 6 pizzas. Now, again, show your answer by drawing and express each person's share as a fraction.
Problem 3	One day Zeytin constructs a picture by cutting and gluing construction papers at school. He needs a green piece of paper, but two of his friends also need that at the same time. The teacher says: "I have 5 pieces of green paper. Share them equally among you." Can you help them with sharing?

As seen in **Table 1**, the difficulty level of the sharing process increases in each task. For example, in the second problem, the variety of fractions that can be employed by the students to present the result of sharing is more than that of the first problem. Moreover, in the third problem, the number of objects being shared is more than the number of people so that students instinctively use improper or mixed fractions. The first two questions especially prompt to use circular region models (as pizza was given in those contexts), while the last question is favorable to use rectangular region models *(as construction paper was given in that context)*.

Procedure

The researcher had a semi-structured interview with each student in a separate room. At the beginning of each interview, the researchers showed the picture of Sizinkiler family and had a chat with the students. The students were asked whether they know the family, name of each member of the family etc.

Then, each task was presented to the students one-by-one on separate sheets. When a student completed a solution, the researcher asked whether there was any other way to share. The next task was presented only after the student believed that all solutions for the task he/she was working on were revealed. During interviews, the researcher encouraged the students to think aloud by asking questions such as "What are you thinking?" or "Could you explain what you did?" In addition, students were asked not to erase, only to retry when they believed their solution was wrong. The researcher interviewed four students on the first day and three students on the second day. Interviews lasted between 26 and 45 minutes, and all of them were audio-recorded. All sheets collected from the students, audio files, and field notes taken by the researcher constituted the data of the study.

Analysis of data

The researcher evaluated and classified all answers based on the strategies defined in the taxonomy developed by Charles & Nason (2000). However, the scope of two strategies was changed. One of them was the *regrouping* strategy. In Charles & Nason's (2000) study, the application of *regrouping* strategy follows these stages: Determining the number of people sharing as a unit fraction, dividing each whole into parts by the number of people, dividing the total number of pieces obtained by the number of people. Within the scope of the *regrouping* strategy dealt with in this study, another number can be chosen as unit fraction such that the total number of pieces obtained can be divided by the number of people sharing. The other change was in the *whole to each person then-half the remaining objects between half the people* strategy. In line with numbers used in the third task, this strategy was renamed *whole to each person then two-thirds to each person*.

Two students had difficulty in drawing the solutions of four answers (two questions each), since the denominator of the chosen unit fraction was large (like 12). These students preferred to explain their thoughts verbally or in writing rather than drawing figures. In such cases, the researcher used these detailed descriptions to determine the strategy.

What has been done in this study is to adapt the inter-task and intra-task flexibility defined by Elia *et al*. (2009) to the partitioning strategies. In this sense, the meaning of inter-task flexibility stayed untouched. However, the scope of intra-task flexibility was extended. It consisted of three components in this study: approaching the same problem with different strategies, changing the strategy when one does not work, and using the combination of several strategies for the solution of one task.

RESULTS AND DISCUSSION

General overview

Students produced 48 solutions in total. On average, each student came up with almost seven solutions for three tasks. The number of solutions produced by a student varied between six and nine. Students referred mainly to five partitioning strategies*: People by objects* (1), *partitive quotient foundational* (4), *partition and quantify by part-whole notion* (5), *half to each person then a quarter to each person* (6), *whole to each person then two-thirds to each person* (6), *regrouping* (26). In 20 solutions, each share was correctly quantified by the student.

The maximum and minimum numbers of different strategies used by one student were four and two, respectively. On the task basis, maximum strategy variety was observed in the first task (five different strategies). In the second task, the number of diverse strategies used by students was the lowest (merely two strategies). In terms of the strategy classes determined by Charles & Nason (2000), students used Class 1 strategies in eight solutions, Class 2 strategies in 33 solutions, and Class 3 strategies in seven solutions. Detailed information about strategies used by students for each task and quantifications of shares can be seen in Table 2.

Indicators of inter-task flexibility

Within the scope of inter-task flexibility, the researcher observed that students could easily change their ways of sharing based on task characteristics. For example, in the third task, almost all students were able to apply a completely new strategy (*whole to each person then two-thirds to each person*) which they did not use for the first two tasks, since it was the only task in which each share was more than a whole. An instance of inter-task flexibility was demonstrated by S3 who could readily switch to different strategies as tasks were changed. One of the strategies he utilized to solve the first task was *half to each person then a quarter to each person*, while he employed *regrouping* for the second task. In the third task, *whole to each person then two-thirds to each person* was another strategy he implemented (Figure 3).

Table 2. Strategies used by students for each task

	Task 1		Task 2		Task 3	
	Strategy	QES*	Strategy	QES	Strategy	QES
S1	Half to each person then a quarter to each person	+	Regrouping	+	Partitive quotient foundational	+
	Partitive quotient foundational	+	Regrouping	+	Whole to each person then two-thirds to each person	+
S2	Regrouping	+	Regrouping	+	Whole to each person then two-thirds to each person	+
	Half to each person then a quarter to each person	+			Regrouping	-
	Regrouping	+				
S3	Partitive quotient foundational	+	Regrouping	+	Partitive quotient foundational	+
	Half to each person then a quarter to each person	+		+	Whole to each person then two-thirds to each person	+
	Regrouping	-				
S4	Regrouping	+	Regrouping	-	Regrouping	-
	People by object	-	Regrouping	-	Regrouping	-
			Regrouping	-	Whole to each person then two-thirds to each person	-
S5	Regrouping	-	Regrouping	-	Partition and quantify by part-whole notion	-
	Regrouping	-			Regrouping	-
	Regrouping	-				
S6	Half to each person then a quarter to each person	+	Partition and quantify by part-whole notion	-	Partition and quantify by part-whole notion	-
	Partition and quantify by part-whole notion	+			Whole to each person then two-thirds to each person	-
S7	Regrouping	-	Regrouping	-	Regrouping	-
	Partition and quantify by part-whole notion	-	Regrouping	-	Regrouping	-
	Half to each person then a quarter to each person	-	Regrouping	-	Whole to each person then two-thirds to each person	-

*Quantification of Each Share

Indicators of intra-task flexibility

One of the indicators of intra-task flexibility was the variety of strategies employed for one task. For instance, S7 employed three different partitioning strategies (*regrouping, partition and quantify by part-whole notion*, and *half to each person then a quarter to each person*) for the first task. However, her quantification of each share was wrong. First, she correctly identified each person's share (3/4), but then she crossed it out and replaced it with an incorrect one (3/12). Finally she inserted this incorrect fraction in her other solutions (see Figure 4).

Although almost only one strategy was used for the second task, the participants used sharing methods that can be expressed in four different fractions (2/3, 4/6, 6/9, 8/12). Three students used only one of these methods. Two students solved the second question by using two different methods. Lastly, two students employed three different methods while working on the second task (see Figure 5 for a sample). After one of them, S4, had finished his first two drawings, he and the researcher had a conversation as follows:

R: How did you decide on the number of pieces to divide each pizza?

S4: Well, I considered whether the number I chose was appropriate. For example, I divided each pizza into four slices in my mind first, but it did not work. Then I tried three and six, and I was able to distribute all the pieces exactly.

R: Got it. Is there any other number that happens to come into your mind?

S4: (paused for a few seconds) Hmm, I think 12 would also be OK. Can I write it down instead of drawing it?

R: Sure (Following the conversation, S4 wrote down his thoughts as seen in the bottom part of Figure 5).

Some students changed the sharing procedure when it did not work for the solution, which was another significant indicator of intra-task flexibility. For example, in his first try to solve the second task, S2 divided each pizza into four pieces; he then gave three pieces to each person (see Figure 6a). At this point, he recognized that the total number of pieces was not enough for nine people in this situation. Thereupon he divided each pizza into six pieces, which led him to the correct solution (see Figure 6b). Some students utilized several strategies simultaneously while solving a task. For example, in his solution to the third task, S3 used a combination of Charles and Nason's (2000) *whole to each person then two-thirds to each person* and Lamon's (1996) *preserved-pieces* strategies (See last part of Figure 3). Although the answers of the participating students in this study were not classified on the basis of the strategies described by Lamon (1996), the above-mentioned situation may be regarded as an indicator of intra-task flexibility.

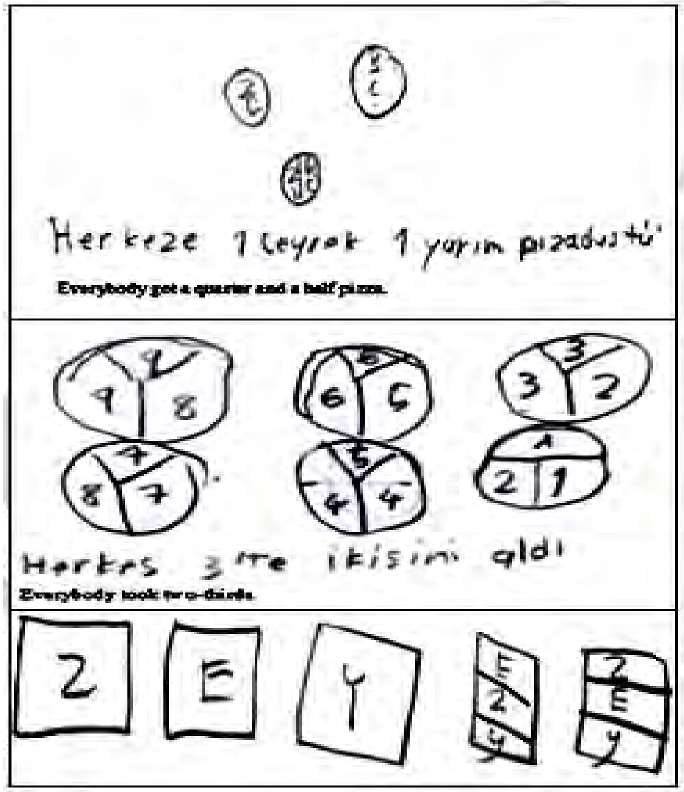

Figure 3. Different strategies used by S3 for the first, second, and third tasks

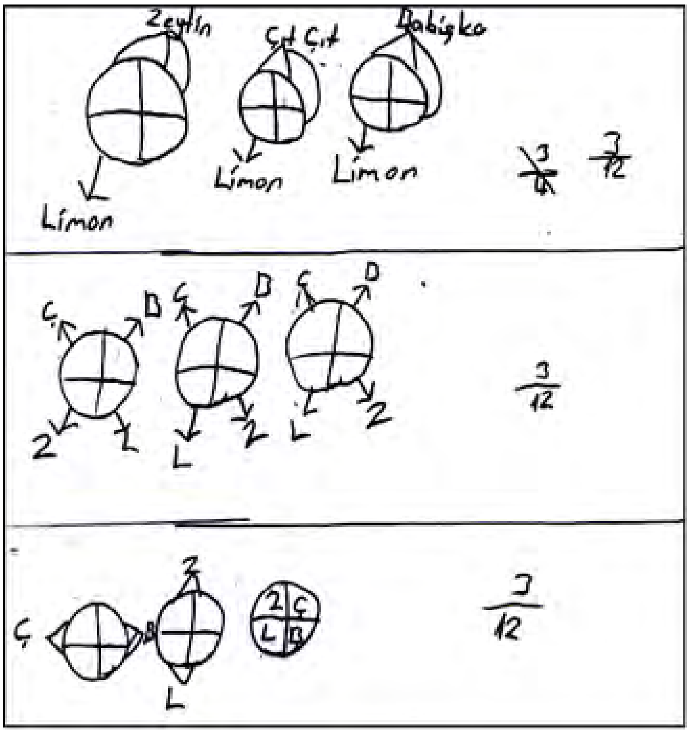

Figure 4. Different partitioning strategies employed by S7 for the first task

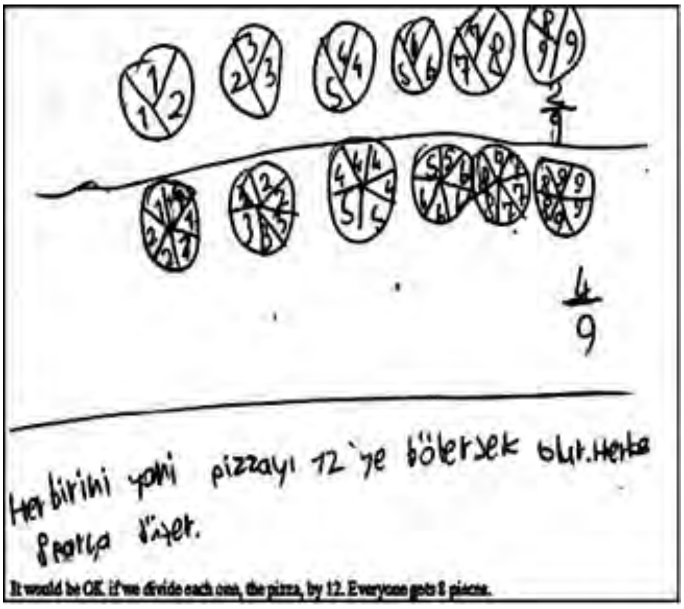

Figure 5. S4's three different sharing methods for the second question

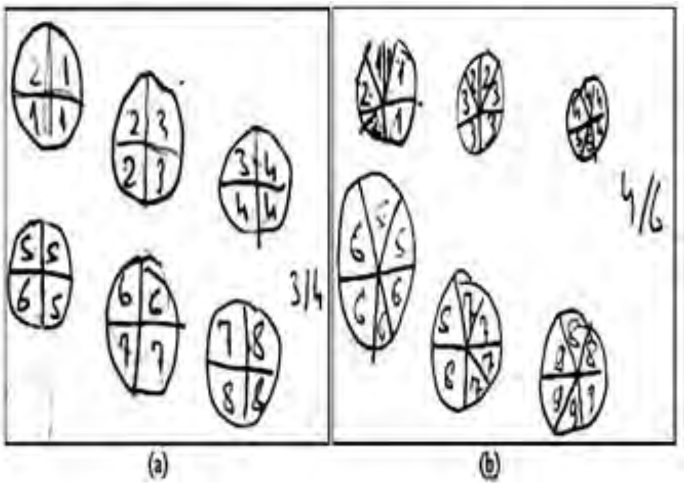

Figure 6. S2 changed his strategy for the second task when it did not work

CONCLUSION

The present study investigated strategy flexibility of fourth graders while solving partitioning tasks. This study differs from other studies on partitioning strategies in that it specifically focuses on students shifting their strategies. Generally, students displayed inter- as well as intra-task strategy flexibility to a large extent even though they did not have any intervention on partitioning. As Pitkethly and Hunting (1996) stated, students inherently have partitioning sense. This study proffers that they also have a natural ability to switch strategies both within and across partitioning tasks.

As Steffe and Olive (2010) assert, partitioning schemes have a vital role in students' construction of fraction schemes. However, it appears that students in this study have not regularly experienced partitioning activities during their school life. For example, although they used partitioning strategies flexibly, students were not always successful in expressing the quantification of each share as fractions. The success rate in this respect was about 42%. In addition, frequency of the use of Class 1 strategies was not very high. Most probably, the students echoed the influence of the emphasis that is put on the part-whole relationship in the traditional education system.

Some findings have implicated that the fourth graders' potential flexibility may be utilized to introduce concepts of equivalent fractions and mixed numbers. Students' solutions showed that they could use the *regrouping* strategy in various ways by selecting different common multiples or divisors of the numbers of objects shared and the people sharing. This situation was especially evident in the answers to the second task. At this point, students may question whether everyone has the same amount despite being expressed in different fractions. Hence, they can comfortably build up the concept of equivalent fractions (Toluk, 1999). Additionally, students could generate strategies for the third task by using their informal knowledge even though they had not learned mixed and improper numbers at that time. Two students stated that each person gets a whole and two-thirds at the end of the sharing, indicating that they were ready to encounter the formal notation of mixed numbers.

Limitations and suggestions

The number of students and the tasks was limited in this study. Although this is not a problem in the sense of the principles of the descriptive case study, a succeeding study incorporating more students and questions may help to clarify some other points that are not addressed much in this study. For example, whether there is a link between the diversity of partitioning strategies used by students and their accurate quantification for each share can be discussed in another more comprehensive study.

In this study, tasks were presented to students and they were asked to work on them. In later studies, as Star and Rittle-Johnson (2008) did in their research, students can be shown the ready-made solutions and asked what kind of a distribution was made in each solution and which one they would prefer. Thus, strategy adaptivity component of the strategy flexibility can be examined in more depth in terms of partitioning. Additionally, a well-designed longitudinal experimental study centered on partitioning activities can answer the question of whether flexibility in partitioning strategies can be developed through instruction. By this way, the influence of such an experimental intervention on the development of students' understanding of fractions can also be observed. The sample of the current study was limited to fourth graders. Replicating this study with different grade levels beginning from lower ones may provide more extensive information concerning the development of strategy flexibility in partitioning.

Star (2018) states that flexibility should be thoroughly examined in mathematical domains other than well-studied ones. The current study tries to do so, albeit partially. It also yields new directions for further research. The author hopes that the results obtained in this study trigger future studies in this domain and contribute to preparing more effective learning environments for students.

Acknowledgement

The authors would like to thank all parties who have assisted in the research and writing of this manuscript so that it deserves to be widely published to the academic community.

REFERENCES

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics, 43*(2), 191-221. https://doi.org/10.2307/3483090

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education*. Routledge.

Confrey, J., Maloney, A. P., Nguyen, K. H., & Rupp, A. A. (2014). Equipartitioning, a foundation for rational number reasoning. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 61-96). Information Age.

Creswell, J. W. (2007). Qualitative inquiry and research design: Choosing among five approaches. Sage Publications.

Cutting, C. (2019). Re-thinking fraction instruction in primary school: The case for an alternative approach in the early years. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 204–211). Perth: MERGA.

Elia, I., van den Heuvel-Panhuizen, M., & Kolovou, A. (2009). Exploring strategy use and strategy flexibility in nonroutine problem solving by primary school high achievers in mathematics, *ZDM, 41*(5), 605-618. https://doi.org/10.1007/s11858-009-0184-6

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction, 17*(3), 283-342. https://doi.org/10.1207/S1532690XCI1703_3

Empson, S. B., Junk, D., Dominguez, H. and Turner, E. (2006) Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63* (1), 1-28. https://doi.org/10.1007/s10649-005-9000-6

Jausovec, N. (1991). Flexible strategy use: A characteristic of gifted problem solving. *Creativity Research Journal, 4*(4), 349-366. https://doi.org/10.1080/10400419109534411

Kalyuga, S., Renkl, A., & Paas, F. (2010). Facilitating flexible problem solving: A cognitive load perspective. *Educational Psychology Review*, 22(2), 175-186. https://doi.org/10.1007/s10648-010-9132-9

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27* (2), 170-193. https://doi.org/10.2307/749599

Lamon, S. J. (1999). Teaching fractions and ratios for understanding: Essential content knowledge and instruction strategies for teachers. Lawrence Erlb.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Sense Publishers.

Leikin, R., Berman, A., & Koichu, B. (2009). *Creativity in mathematics and the education of gifted students*. Sense Publishers.

Levav-Waynberg, A., & Leikin, R. (2012). Using multiple solution tasks for the evaluation of students' problem-solving performance in geometry. *Canadian Journal of Science, Mathematics and Technology Education*, *12*(4), 311-333. https://doi.org/10.1080/14926156.2012.732191

Liu, R. D., Wang, J., Star, J. R., Zhen, R., Jiang, R. H., & Fu, X. C. (2018). Turning potential flexibility into flexible performance: Moderating effect of self-efficacy and use of flexible cognition. *Frontiers in Psychology*, *9*, 1–10. https://doi.org/10.3389/fpsyg.2018.00646

Low, C. S., & Chew, C. M. (2019). Strategy flexibility in mathematics education. In C. S. Lim, C. M. Chew, & B. Sriraman (Eds.), *Mathematics Education from an Asian Perspective* (pp. 85-105). USM Press.

Merriam, S. B. (2009). Qualitative research: A guide to design and implementation (3rd ed.). Jossey-Bass.

Nguyen, H. A., Guo, Y., Stamper, J., & McLaren, B. M. (2020). Improving students' problem-solving flexibility in non-routine mathematics. In I. Bittencourt, M. Cukurova, K. Muldner, R. Luckin, & E. Millán (Eds), *Artificial Intelligence in Education* (pp. 409-413). Springer.

Norton, A. H., & McCloskey, A. V. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics*, *15*(1), 48-54.

Norton, A., & Wilkins, J. L. (2010). Students' partitive reasoning. *The Journal of Mathematical Behavior, 29*(4), 181-194. https://doi.org/10.1016/j.jmathb.2010.10.001

Petit, M. M., Laird, R. E., Marsden, E. L., & Ebby, C. B. (2015). *A focus on fractions: Bringing research to the classroom*. Routledge.

Pitkethly, A., & Hunting, R. P. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*(1), 5–38. https://doi.org/10.1007/bf00163751

Pothier, Y., & Sawada, D. (1990). Partitioning: An approach to fractions. *The Arithmetic Teacher*, *38*(4), 12-16.

Pothier, Y., & Sawada, D. (1983) Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14* (5), 307-317. https://doi.org/10.2307/748675

Selter, C. (2001). Addition and subtraction of three-digit numbers: German elementary children's success, methods and strategies. *Educational Studies in Mathematics*, *47*(2), 145-173. https://doi.org/10.1023/a:1014521221809

Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM, 41*(5), 619-625. https://doi.org/10.1007/s11858-009-0203-7

Siemon, D. (2003). Partitioning: the missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher*, *59*(3), 22-24.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. Springer.

Star, J. (2018). Flexibility in mathematical problem solving: The state of the field. In Hsieh, F. J. (Ed.), *Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education* (Vol. 1, pp.15-25). EARCOME.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*, 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018

Star, J. R., Rittle-Johnson, B., Lynch, K., & Perova, N. (2009). The role of prior knowledge in the development of strategy flexibility: The case of computational estimation. *ZDM, 41*(5), 569-579. https://doi.org/10.1007/s11858-009-0181-9

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*, 280–300. https://doi.org/ 10.1016/j.cedpsych.2005.08.001

Streefland, L., (1991). *Fractions in Realistic Mathematics Education*. Kluwer Academic, Dordrecht

Toluk, Z. (1999). *Children's conceptualizations of the quotient subconstruct of rational numbers* [Doctoral dissertation, Arizona State University]. ProQuest Dissertations & Theses Global. https://www.proquest.com/pqdtglobal/docview/304494304/742314C2FBBB4464PQ/1?accountid=17219

Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Jump or compensate? Strategy flexibility in the number domain up to 100. *ZDM, 41*(5), 581-590. https://doi.org/10.1007/s11858-009-0187-3

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualising, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335–359. https://doi.org/ 10.1007/bf03174765

Wang, J., Liu, R. D., Star, J., Liu, Y., & Zhen, R. (2019). The moderating effect of regulatory focus in the relationship between potential flexibility and practical flexibility. *Contemporary Educational Psychology*, *56*, 218-227. https://doi.org/10.1016/j.cedpsych.2019.01.013

Xu, L., Liu, R., Star, J.R., Wang, J., Liu, Y., & Zhen, R. (2017). Measures of potential flexibility and practical flexibility in equation solving. *Frontiers in Psychology, 8*, 1368. https://doi.org/10.3389/fpsyg.2017.01368

Yazgan, Y (2010), Partitioning strategies of fourth and fifth grade students. *E-Journal of New World Sciences Academy, 5* (4), 1628-1642.

Yin, R. (2003). Applications of case study research (2nd ed.). Sage