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DEVELOPING PRE-SERVICE MATHEMATICS TEACHERS' UNDERSTANDING OF METRIC AND TOPOLOGICAL SPACES USING REFLECTIVE QUESTIONS

Research Article

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Abstract

This paper presents the ways that reflective questions were used to develop pre-service mathematics teachers' understanding of metric and topological spaces. In particular, the purpose of developing PSTs' understanding of metric and topological spaces is that understanding these mathematical concepts leads to PSTs' understandings on how to teach and learn geometrical figures in mathematics classes after completion of their degree programs. This paper details the interpretation of the PSTs' reflections on metric and topological spaces. From PSTs' reflections, it was found that reflective questions helped individual PSTs to develop their understanding of matric and topological spaces, including giving the meaning of metric and topological spaces, and relating with real objects available in their daily environment. The findings have implications in teaching and learning metric and topological spaces.

Keywords: Metric, topology, metric and topological spaces, pre-service mathematics teachers

1. Introduction

This paper starts by addressing the concepts of metric and topological spaces as the concepts used to engage pre-service teachers (PSTs) with reflective questions. Then, paper addresses the challenges of implementing reflective questions in Tanzanian university mathematics classes. After that, the paper presents constructivist theory as a theoretical framework used in this study. The paper also addresses the methodology used in this study. Then, the paper presents the results of using reflective questions with PSTs on the concepts of metric and topological spaces. The paper ends by providing concluding thoughts about using reflective questions for teaching and learning the concepts of metric and topological spaces, and other mathematical concepts.

2. Metric and topological spaces

Metric and topological spaces are concepts that are taught to University mathematics pre-service teachers while they are in year three of their degree programs. These concepts are critical in teaching and learning geometrical objects/figures such as a circle, rectangle, square, and polygons. Geometrical objects/figures are concepts that are taught in Tanzanian primary and secondary schools. We expect that these teachers will teach in primary and secondary schools after completion of their degree programs.

Metric is simply defined as a distance function and it is about measurements. For instance, a meter-ruler has metric ranging from 0 to 100 cm. It has a space within this range, and because of that a meter-ruler is a metric space as it obeys the following properties;

- The distance on a ruler is always greater than or equal to zero. The reason is that we measure an object on a ruler by starting at zero as the initial point of reference.
- The distance on a ruler is always equal to zero if and only if the two points, let say A and B on a ruler are equal.
- The distance, d on a ruler from points A to B is equal to that of from points B to A. This obeys the symmetry property. That is to say, $d(A,B) = d(B,A)$.
- If we locate another point between the two points A and B, let say C, we realize that the distance between the points A and B is less than or equal to the summation of the distances of the points A and C and that of points C and B. This obeys triangular inequality property. That is to say, $d\ (A,B) \leq d(A,C) + d(C,B)$

Topology is simply defined as the various shapes that is formed from a single object without deforming its nature such as size and shape. The object can be stretched, compressed or other way around without destroying it. For example, a door is an example of a topology which provides two spaces, entry and exit. These spaces are known as topological spaces because the door can allow someone to go in and out of the house and also can prevent someone to enter the house by closing it.

3. The challenge of implementing reflective questions in university mathematics classes

Reflection is one of the teaching tools for developing learners' understanding of a concept (Ben-Hur, 2006). Through reflections, learners realize what they were familiar before participating in the teaching and learning process. During participation in the process, learners can reflect on errors and inconsistencies during small group and class discussions. At the end of the teaching and learning process, learners can realize what they have understood about the concept (Ben-Hur, 2006; Deogratias, 2020; Alugar, 2021; Nadiahan & Cabauatan, 2021). Because of that teachers need to provide reflective questions before, during and at the end of the teaching and learning process in mathematics classes to foster learning achievements. Despite these associated advantages of using reflective questions, pre-service teachers are not given this opportunity to reflect on their learning of mathematics in the university mathematics classes (Deogratias, 2020). University instructors usually get opportunity to reflect on their teaching and learning process after marking the tests and final examinations. In doing so, the instructors and pre-service teachers have no opportunities to utilize the reflective questions in mathematics classes, including correcting errors and inconsistencies about a mathematical concept.

4. This study was framed under the lens of constructivist theory

Constructivist theory was used in this study as a theoretical framework because individual PSTs reflected on the concepts of metric and topological spaces through constructing their mathematical ideas before, during and at the end of the lesson. The reflections are critical for conceptual development and understanding of a mathematical concept (Ben-Hur, 2006; Deogratias, 2020; Vygotsky, 1978). After individual PSTs' reflections on the concepts of metric and topological spaces, they worked together in a single small group followed by class discussions to elaborate their understandings of the concepts with knowledgeable others. Also, individual PSTs reflected on their constructed mathematical ideas on metric and topological spaces after a single small group discussion followed by class discussions.

5. Methodology

This case study explored the ways that reflective questions were used for developing PSTs' understanding of metric and topological spaces in a daylong research meeting. In particular, this study explored the research question: How do reflective questions can be used to develop PSTs' understanding of metric and topological spaces in a research meeting?

Three PSTs volunteered participated in this study. These PSTs were pursuing a bachelor's degree in education in one of the Tanzanian public universities, and they were in year three of their studies. They were taking both mathematics and education courses because we expect these teachers to work in primary and secondary schools after completion of their degree programs.

Ten reflective questions were designed that guided individual PSTs while constructing mathematical ideas about metric and topological spaces:

- What comes in your mind when you hear the word "metric"? Please explain for me.
- What comes in your mind when you hear the word "space"? Please explain for me.
- What comes in your mind when you hear the word "topology"? Please explain for me.

- What comes in your mind when you hear the term "metric space"? Please explain for me.
- What comes in your mind when you hear the term "topological space"? Please explain for me.
- Do you think metric and topological spaces are important in your daily life environment/situations? Please explain for me with vivid examples.
- How can you develop the meaning of metric using local resources? Please explain for me.
- How can you develop the meaning of metric space using local resources? Please explain for me.
- How can you develop the meaning of topology using local resources? Please explain for me.
- How can you develop the meaning of topological space using local resources? Please explain for me.

At the beginning of the lesson, individual PSTs were asked to respond on each reflective question because I was interested to know the mathematical ideas of each individual PSTs brought in the research meeting. My assumption was that an individual PSTs were not empty headed, but rather they had knowledge about metric and topological spaces. After individual response to the above reflective questions, there was a single small group discussion followed by class discussions for more elaboration of a concept. At the end of the meeting, I provided a final reflection which consisted three item questions provided:

- What have you learned today that you were not familiar before today's lesson? Please explain for me.
- What surprised you today about the lesson? Please explain for me.
- What have you understood today? Please explain for me.

The final reflection was important to understand what each individual PSTs learned, and understood in the research meetings about the concept of metric and topological spaces.

During a single small group discussion followed by class discussions, PSTs got opportunity to reflect on their ongoing learning, including reflecting on errors, and analyzing errors and inconsistencies. The collected data from individual PSTs' responses were analyzed by using (Clarke & Braun, 2006).

6. Results

The results on the reflective questions on metric and topological spaces are presented based on PSTs' responses. The results focus on three aspects: metric as a distance function, topology as a geometric orientation, and space as a field.

6.1 Metric as distance function

Through using reflective questions, PSTs realized that metric is a distance function. A metric is concerned with measurements (Sigh, 2019). This is a result of having two points; initial point which is the starting point in measurement, and the terminal point which is the ending point in measurements. As such the measurement between the initial and terminal points is a distance function. The two points are the defined points, one being the starting point in measurements and the other being the ending point in measurements. For example, one of the PSTs defines: "metric is the length between any two points in a set" (From Individual Reflection). This meaning can be interpreted that metric is the distance function, which involves any two defined points. One point being an initial and the other one as a terminal point. Because of that it involves a set—a collection of two points—initial point and terminal point.

6.2.Topology as a geometric orientation

Through using reflective questions, PSTs realized that topology is a geometric orientation. Topology is an open set because it can be expressed in different ways without losing its generosities (Jiri, 2014; Kelly, 1995; Munkes, 2000). For example, one of the PSTs presents the meaning of a topology. "Suppose A is an open set in which there exists subsets, say W is a collection of finite subsets in A, then W is said to be a topology on A" (Individual Refection). This meaning entails that the subsets W can be arranged in different perspectives without destroying it. This meaning implies that topology is seen in the geometrical orientation of an object or figure. For example, a chair can be designed in a way that it is both a chair and a bed. In this case a chair is a multipurpose tool, can be used as both a chair and a bed without destroying it. What makes the difference is the geometric orientations; when it is a chair it occupies smaller size than when it is a bed; the shape and size might look difference due to geometric orientations, but it retains its originality because it is not destroyed.

6.3.Space as a field

Through using reflective questions, PSTs realized that a space is a field. A space can be interpreted as a field in which something is within (Jain & Ahmad, 1999). As a field, it might be in different dimensions, including one dimension, two dimensions, and three dimensions. When it is in one dimension, the field is a set of real numbers. When in two dimensions, the field is an area. In three dimensions, the field is a volume. For instance, one of the PSTs presents the meaning of a space "Space can be defined as the field occupied by a set A" (Individual Reflection). This shows that a space is non-empty; it has a collection of elements. Viewing in this perspective, we develop the meaning of a space by categorizing it into two parts as "metric space" and "topological space."

Metric space is the space occupied in a distance function (Copson, 1968; Mejlbro, 2009). This space consists elements. It is a non-empty space in such a way that the distance between the points obeys three properties; positivity, symmetry and triangle inequality. For example, one of the PSTs presents the meaning of a metric space. "suppose a and b are any two points in a given set A, and $d(a,b)$ be the distance function between two points. Then (A,d) is referred to as a metric space under the following axioms:

i. $d(a,b) \geq o$ for all $a,b \in A$ (Positivity)
ii. $d(a,b) = 0 \Leftrightarrow a = b$ (Symmetry)
iii. $d(a,b) \leq d(a,c) + d(c,b)$ for all $a,b,c \in A$ (Triangle inequality)"
(From Individual Reflection)

Topological space is a space which can be oriented geometrically in different ways without destroying it (Sigh, 2019; Sutherland, 2009). This means that there should be elements in this space in such a way that a space is non-empty with an empty set, has intersections of subsets, and union of finite subsets. For example, one of the PSTs presents "Let A be a set and W be the collection of subsets of A, then the space (A,W) is called topological space under the following conditions.

i. The empty set, ϕ and A should in W
ii. Intersection of subjects in A should in W
iii. The union of finite subsets in W should be in A" (Individual Reflection)

PSTs went further by realizing that metric and topological spaces can be understood relationally by relating with real objects that we see in our daily life environment. For example, one of the PSTs addresses: "I was surprised by the idea of metric space and topological space; that both can be developed using local resources. For instance, a ruler can be used to explain the concept of metric space and a door can be used to explain the concept of a topological space" (Final Reflection). From this reflection, we can notice that a meter rule is an open set of short defined lengths between 0 and 100. The rule can be used to describe the concept of metric space which is a space occupied between two points. While using a door as a local resource can be used to describe the concept of topological space in which a door can either open or close without destroying it. For instance, when a door is opened, it gives a space which is an entrance and when a door is closed, it prevents any incoming invaders. Thus, the openness and closedness of a door gives a space which is a good example of a topological space.

7. Conclusions

Reflective questions are important in teaching and learning metric and topological spaces, and other mathematical concepts. Using reflective questions, various concepts can be understood, including the meaning of metric and topological spaces. To foster understanding of the meaning of metric and topological spaces, we can relate the concepts with things/real objects available in our local environment.

Using reflective questions in teaching and learning the concepts of metric and topological spaces, PSTs realize the importance of the concepts in mathematics, beyond mathematics and daily life situations (Bryant, 1994). For example, we can use the concept of metric space in reducing electricity. This interpretation is based on distance; the more the distance of power supply, the longer the wires/cables are needed, the more the cabling costs. Because of that understanding the meaning of metric and metric spaces help electricians to make decision through cost analysis. Also, topological space plays a major role in searching engines. For example, a topological space makes possible for many people to use the same search engine to look for certain resources in an internet without disruption.

Using reflective questions for teaching the concepts of metric and topological spaces, PSTs realize that the concepts are not abstract. They can be taught relationally. To avoid the notion of memorizing procedures and facts, PSTs were given a reflective question on how they can develop the meaning of metric and topological spaces using local resources. Before participating in the research meeting, the two PSTs were not aware on how they can develop the meaning of the terms using local resources. But after participating in the research meeting, PSTs realized that we can use a ruler to develop the meaning of metric space and door to develop the meaning of a topological spaces. Because of that, using local resources during teaching and learning mathematics is critical for conceptual development and understanding of mathematics (Deogratias, 2020). The more we engage PSTs with mathematical concepts using various teaching and learning materials, the more they are actively engaged in learning and, hence learning achievements.

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